Ferhat Abbas University, Setif 1
Faculty of Sciences
Department of Mathematics

Dr. Aziza Bachmar

Course reminder and solved exercises in Analysis 2

intended for first year Mathematics and Computer Science students

- Limited development
- Reiman's integral
- Ordinary differential equations

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Course reminder and solved exercises in Analysis 2

Teacher : Aziza Bachmar [1]

[1]Applied Mathematics Laboratory. Faculty of Science. Department of Mathematics. University Ferhat Abbas, Sétif1. Sétif 19000, Algeria. e-mail: aziza.bachmar@univ-setif.dz

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Introduction

The aim of this course is to bridge the gap between the knowledge of analysis accumulated in high school and the fundamentals that will form one of the pillars of mathematical analysis training in the bachelor's degree. Given that recruitment to the first year of analysis is fairly heterogeneous, it seems a good idea to start by recalling the basic notions that will be used throughout this course, so as not to lose anyone along the way. Where necessary, at the start of each chapter, we'll remind you of what you're supposed to know by the end of the course. As far as possible, we'll also try to provide the essential results of each chapter on a single page, so as to summarize the knowledge you need to master in order to move on to the next chapter. We will provide as many examples and figures as necessary afin order to achieve a better understanding of the course. We'll also try to point out the pitfalls that everyone can fall into, either through inattention or poor mastery of the course.
This course brings together the notes from the analysis module dedicated to first-year students in the mathematics and computer science stream. It includes limited developments, Reiman integrals and ordinary differential equations.
The general idea behind the first chapter is to To handle Limited development, you first need to know a few basic rules. These rules all have a name afin order to identifier them right away, so you know what you're talking about. A bit like a map, this lets us know what we're seeing, where we're going and how we're getting there. Let's start with the simple, familiar rules.
In the second chapter we give the notion of a Reiman's integral in Analysis. Darboux's sums. They are introduced and studied in this chapter. Further, we give the different methods for calculating integrals
For the last chapter: we study two types of Ordinary differential equations.An ordinary differential equation of$n$ order, an ordinary differential equation of 1^{st} order and 2^{nd} order. They are introduced and studied in this chapter. Further, we give the different methods and types to solve this type of an ordinary differential equation
Each chapter is followed by a list of exercises, usually illustrating a point made earlier. It goes without saying that these exercises complement the course.

Notations

If E is a set, note by

$P(E)$	Part of E.
$A^c = C_E^A$	$A's$complement in E
$\varnothing$	Empty set
Id_x	Identity application
inf	the lower bound,
sup	the upper bound
$\lim\limits_{n\to+\infty}$	the limit when n tends towards $+\infty$
$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of natural numbers
$x \to x_0$	x tends towardsx_0
$f'(x)$	The first derivative of $f(x)$
$f^{(n)}(x)$	n^{th}The nth derivative of $f(x)$

Let I be an interval of $\mathbb{R}$, we have the following notations

C^n	"n" times derivable functions
$C^n(I)$	The set of functions of class C^n on I
$C^\infty(I)$	The set of functions of class C^∞ on I
e^x	exponential
log	Neperian logarithm

L.D	limited development
$\int_a^b$	Bounded integral
$\int$	Infinite integral
O.D.E	Ordinary differential equations
iff	if only if

Chapter 1

Limited development

To handle Limited development, you first need to know a few basic rules. These rules all have a name afin order to identifier them right away, so you know what you're talking about. A bit like a map, this lets us know what we're seeing, where we're going and how we're getting there. Let's start with the simple, familiar rules.

Theorem 1.0.1. *Let f be a function defined in the vicinity of* "0" *except perhaps at* "0" *. On dit que f admits a limited development(D.L) of order* "n" *in the vicinity of* "0" *if there exists an open interval I with center* "0" *and constants $a_0, a_1, a_2, ..., a_n$ such that $\forall 0 \neq x \in I$, we have:*

$$f(x) = a_0 + a_1x + a_2x^2 + ... + a_0x^n + x^n\varepsilon(x)$$
$$f(x) = P_n(x) + R_n(x)$$

where $\varepsilon(x)$ is a function defined on I, such that $\lim\limits_{x\to 0}\varepsilon(x) = 0$

$P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_0x^n$ s a polynomial of degree "n" *and is called the regular part $R_n(x) = x^n\varepsilon(x)$ is the rest*

Example 1. *$f(x) = \frac{3}{2} + x^2 + x^3\sin x$ does f admit a D.L? **
*D.L**of order** 4*
$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + x^4\varepsilon(x)$
$f(x) = \frac{3}{2} + 0.x + 1.x^2 + 0.x^3 + 0.x^4 + x^4\varepsilon(x)$
Then:$a_0 = \frac{3}{2}, a_1 = 0, a_2 = 1, a_3 = a_4 = 0, x^4\varepsilon(x) = x^3\sin x = x^4\frac{\sin x}{x} / \lim\limits_{x\to 0}\varepsilon(x) = \lim\limits_{x\to 0}\frac{\sin x}{x} = 1$
So f does not admit a D.L
D.L **of order 3*
$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + x^3\varepsilon(x)$
$f(x) = \frac{3}{2} + 0.x + 1.x^2 + 0.x^3 + x^3\varepsilon(x)$
So:$a_0 = \frac{3}{2}$,$a_1 = 0$,$a_2 = 1$,$a_3 = 0$ $x^3\varepsilon(x) = x^3\sin x$ / $\lim\limits_{x\to 0}\varepsilon(x) = \lim\limits_{x\to 0}\sin x = 0$
So f admets a D.L

Example 2. *$f(x) = \frac{1}{1-x}$ does f admet a D.L?*
Euclidean division therefore

$f(x) = \frac{1}{1-x} = 1 + x + x^2 + ... + x^n + x^n \frac{x}{1-x}$
$a_0 = a_1 = a_2 = ... = a_n = 1 \quad \lim\limits_{x\to 0} \varepsilon(x) = \lim\limits_{x\to 0} \frac{x}{1-x} = 0$
So f admets a D.L of order "n"

Remark 1.0.1. *1/if f admits a D.L of order "n" in the vicinity of "0" :*
$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_0 x^n + x^n \varepsilon(x)$, *So* $\lim\limits_{x\to 0} f(x) = a_0$ *2/If f admits a D.L of order "n" in the vicinity of "0", this implies that* $\lim\limits_{x\to 0} f(x) = a_0$*.This is a necessary condition for f to admit a D.L. 3/if f admits an L.D. of order "n" in the vicinity of "0", this development is unique.*

Corollary: Si f an even(respectively odd) function admits $D.L$ of order "n" in the vicinity of "0", its regular part is even(respectively odd)

Theorem 1.0.2. **If $f^{(n)}(0)$ exists then f admits a unique L.D.*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}(x)^2 + ... + \frac{f^{(n)}(0)}{n!}(x)^n + x^n \varepsilon(x)$$
$$\text{with:} \lim_{x\to 0} \varepsilon(x) = 0$$

**If f admits an L.D. of order "n" in the vicinity of "0"*

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + x^n \varepsilon(x)$$
$$\text{Hence:} a_0 = f(0),\ a_1 = \frac{f'(0)}{1!},\ a_2 = \frac{f''(0)}{2!}, ... a_n = \frac{f^{(n)}(0)}{n!}$$

Example 3. $f(x) = \frac{1}{1-x}$
$f(x) = \frac{1}{1-x} = 1 + x + x^2 + ... + x^n + x^n \frac{x}{1-x}$
we have: f is indefinitely derivable$\Rightarrow f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$
$f'(0) = 1,\ f''(0) = 2!, ..., f^{(n)}(0) = n!$
$a_0 = f(0) = 1, a_1 = \frac{f'(0)}{1!} = 1,\ a_2 = \frac{f''(0)}{2!} = 1,\ ...., a_n = \frac{f^{(n)}(0)}{n!} = 1$

Example 4. $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x^{n+1} \cdot \sin \frac{1}{x^{n+1}} & \text{if } x \neq 0 \end{cases}$
we have:$f(x) = 1 + 0.x + 0.x^2 + 0.x^3 + ...0.x^n + x^n \left(x \cdot \sin \frac{1}{x^{n+1}}\right)$
$a_0 = 0, a_1 = a_2 = ... = a_n = 0 \ \lim\limits_{x\to 0} \varepsilon(x) = \lim\limits_{x\to 0} x \cdot \sin \frac{1}{x^{n+1}} = 0 \left(\begin{array}{c} \text{because:} \sin \frac{1}{x^{n+1}} \text{is bounded} \\ -1 \leq \sin \frac{1}{x^{n+1}} \leq 1 \\ -x \leq x \cdot \sin \frac{1}{x^{n+1}} \leq x \end{array} \right)$
Therefore::f admets a D.L of order "n" in the vicinity of "0"

Usual D.L obtained by the Mac-Laurin formula

* Mac-Laurin's formula gives a L.D. of order "n" in the vicinity of "0" for all function f such that $f^{(n)}(0)$exist

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^n}{n!} f^{(n)}(0) + x^n \varepsilon(x)$$
$$\text{with:} \lim_{x\to 0} \varepsilon(x) = 0$$

Example 5.

$$\begin{aligned}
&* \sin x = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} ..... + (-1)^n \tfrac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\varepsilon(x) \\
&* \cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} ..... + (-1)^n \tfrac{x^{2n}}{(2n)!} + x^{2n}\varepsilon(x) \\
&* e^x = 1 + \tfrac{x}{1!} + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} ..... + \tfrac{x^n}{n!} + x^n\varepsilon(x) \\
&* shx = x + \tfrac{x^3}{3!} + \tfrac{x^5}{5!} + ... + \tfrac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\varepsilon(x) \\
&* chx = 1 + \tfrac{x^2}{2!} + \tfrac{x^4}{4!} + ... + \tfrac{x^{2n}}{(2n)!} + x^{2n}\varepsilon(x) \\
&(1+x)^\alpha = 1 + \tfrac{\alpha}{1!}x + \tfrac{\alpha(\alpha-1)}{2!}x^2 + ... + \tfrac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)........(\alpha-n+1)}{n!}x^n + x^n\varepsilon(x)
\end{aligned}$$

1.1 Algebraic operations on L.D.

Theorem 1.1.1. *Let f and g be two functions that admit a L.D. of order "0". Then the functions $f+g$, $f\cdot g$ and $\frac{f}{g}$ admet a L.D of order "n" in the vicinity of "0"*

Example 6.

$$\begin{aligned}
&g(x) = 2 + x + x^2\varepsilon(x), \quad f(x) = 3x - x^2 + x^2\varepsilon(x) \\
&f + g = 2 + 4x - x^2 + x^2\varepsilon(x) \\
&f \cdot g = 6x + x^2 - x^3 + x^2\varepsilon(x) \\
&\begin{cases}
\frac{1}{g} = \frac{1}{2+x+x^2\varepsilon(x)} = \frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}+x^2\varepsilon(x)} = \frac{1}{2}\cdot\frac{1}{1-y} / \quad y = -\frac{x}{2} + x^2\varepsilon(x) \\
\frac{1}{g} = \frac{1}{2}\cdot\frac{1}{1-y} = 1 + y + y^2 + y^2\varepsilon(x) = 1 + \left(-\frac{x}{2} + x^2\varepsilon(x)\right) + \left(-\frac{x}{2} + x^2\varepsilon(x)\right)^2 + x^2\varepsilon(x) \\
\frac{1}{g} = 1 - \frac{x}{2} + \frac{x^2}{4} + x^2\varepsilon(x)
\end{cases} \\
&\frac{f}{g} = f\cdot\frac{1}{g} = (3x - x^2 + x^2\varepsilon(x)) \cdot 1 - \frac{x}{2} + \frac{x^2}{4} + x^2\varepsilon(x) = \frac{3}{2}x - \frac{5}{10}x^2 + x^2\varepsilon(x)
\end{aligned}$$

1.2 D.L of a composite function

Let f and g be two functions admitting a D.L of order "n" in the vicinity of "0",such that

$$f(x) = P_n(x) + x^n\varepsilon_1(x), \quad g(x) = Q_n(x) + x^n\varepsilon_1(x)$$

Then, the function $g\circ f$ admits a D.L of order "n" in the vicinity of "0",such that

$$\begin{aligned}
g\circ f(x) &= Q_n(x) \circ P_n(x) + x^n\varepsilon(x) \\
g\circ f(x) &= Q_n(x)(P_n(x)) + x^n\varepsilon(x)
\end{aligned}$$

Example 7.

$$\begin{aligned}
&f(x) = e^{\cos x} \\
&\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} + x^4\varepsilon(x) \\
&e^y = 1 + \tfrac{y}{1!} + \tfrac{y^2}{2!} + \tfrac{y^3}{3!} + \tfrac{y^4}{4!} + y^4\varepsilon(y) \\
&e^{\cos x} = e^{\cos x - 1 + 1} = e \cdot e^{\cos x - 1} = e \cdot e^y \\
&e^{\cos x} = e \cdot \left[1 + (\cos x - 1) + \frac{(\cos x-1)^2}{2!} + \frac{(\cos x-1)^3}{3!} + \frac{(\cos x-1)^4}{4!} + x^4\varepsilon(x)\right] \\
&e^{\cos x} = e \cdot \left[1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right) + \frac{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}\right)^2}{2!} + \frac{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}\right)^3}{3!} + \frac{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}\right)^4}{4!} + x^4\varepsilon(x)\right] \\
&e^{\cos x} = e \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{24} + x^4\varepsilon(x)\right)
\end{aligned}$$

1.3 D.L obtained by restriction

Si f admits a L.D. of order "n" in the vicinity of "0", then all $p \leq n$ it admits a L.D. of order "p" in the vicinity of "0", in fact

$$
\begin{aligned}
&f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + x^n\varepsilon(x) \\
&f(x) = a_0 + a_1x + a_2x^2 + ... + a_px^p + x^p\left(x + x^2 + ... + x^{n-p}\right)\varepsilon(x) \\
&f(x) = a_0 + a_1x + a_2x^2 + ... + a_px^p + x^p\varepsilon_1(x) \\
&\varepsilon_1(x) = \left(x + x^2 + ... + x^{n-p}\right)\varepsilon(x) \\
&\lim_{x\to 0}\varepsilon(x) = 0 \Rightarrow \lim_{x\to 0}\varepsilon_1(x) = 0
\end{aligned}
$$

Example 8.

$$
\begin{aligned}
&e^y = 1 + \tfrac{y}{1!} + \tfrac{y^2}{2!} + \tfrac{y^3}{3!} + \tfrac{y^4}{4!} + y^4\varepsilon(y) \\
&e^y = 1 + \tfrac{y}{1!} + \tfrac{y^2}{2!} + y^2\left(\tfrac{y}{3!} + \tfrac{y^2}{4!} + y^2\varepsilon(y)\right) \\
&e^y = 1 + \tfrac{y}{1!} + \tfrac{y^2}{2!} + y^2\varepsilon(y)
\end{aligned}
$$

1.4 L.D obtain by derivation and integration

Theorem 1.4.1. *Let f be a function derivable on $[-a,a]$ admits a L. D of order "n" in the vicinity of "0"*

If its derivative f' also has a L.D. of order "$n-1$" in the vicinity of "0"

then the regular part of the L.D. of$f'$is the derivative of the regular part of f

Example 9.

$$
\begin{aligned}
&f(x) = \tfrac{1}{1-x} = 1 + x + x^2 + ... + x^n + x^n\varepsilon(x) \\
&f'(x) = \tfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... + nx^{n-1} + x^{n-1}\varepsilon(x) \\
&g(x) = \tfrac{1}{1+x} = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + x^n\varepsilon(x) \\
&g'(x) = \tfrac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + ... + n(-1)^n x^{n-1} + x^{n-1}\varepsilon(x) \\
&\log(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} + ... + (-1)^n \tfrac{x^{n+1}}{n+1} + x^{n+1}\varepsilon(x) \\
&\log(1-x) = \log(1+(-x)) = -x - \tfrac{x^2}{2} - \tfrac{x^3}{3} + ... - \tfrac{x^{n+1}}{n+1} + x^{n+1}\varepsilon(x) \\
&\arg thx = \tfrac{1}{2}\log\left(\tfrac{1+x}{1-x}\right) = \tfrac{1}{2}\left[\log(1+x) - \log(1-x)\right] \\
&\arg thx = x + \tfrac{x^3}{3} + \tfrac{x^5}{5} + ... + \tfrac{x^{2n+1}}{2n+1} + x^{2n+1}\varepsilon(x)
\end{aligned}
$$

1.5 L.D in the vicinity of a $x_0$point

Let f be a function defined in the vicinity of x_0 except perhaps at x_0,it admits a L.D. of order "n" in the vicinity of "x_0"if the function $x \mapsto F(X) = f(x - x_0)$ admits an L.D. of order "n" n the vicinity of "0" and,then:

$$F(X) = a_0 + a_1 X + a_2 X^2 + ... + a_n X^n + X^n \varepsilon(X), \quad \lim_{X\to 0} \varepsilon(X) = 0$$
On pose: $X = x - x_0 \Rightarrow$
$$F(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + ... + a_n (x - x_0)^n + (x - x_0)^n \varepsilon((x - x_0)),$$
$$\lim_{x\to x_0} \varepsilon(x - x_0) = 0$$

We therefore reduce to the vicinity of "x_0" at "0" by posing: $X = x - x_0$

Example 10. *The L.D de $f(x) = e^x$in the vicinity of "$x_0 = 1$"*
We pose :$X = x - 1 \Rightarrow x = X + 1$

$$e^x = e^{X+1} = e \cdot e^X = e \cdot \left[1 + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + ... + \frac{X^n}{n!} + X^n \varepsilon(X)\right]$$
$$e^x = e \cdot \left[1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + ... + \frac{(x-1)^n}{n!} + (x-1)^n \varepsilon(x-1)\right]$$

In the same way, we can consider the L.D. at infinity, reducing it to the vicinity of "0" by posing $X = \frac{1}{x}$

Example 11. $f(x) = \sqrt[3]{x^2(x^2-1)}$ *in the vicinity of infinity*

$$\sqrt[3]{x^2(x^2-1)} = \left[x^2(x^2-1)\right]^{\frac{1}{3}} = \left[x^4\left(1 - \frac{1}{x^2}\right)\right]^{\frac{1}{3}} = x^{\frac{4}{3}} \cdot \left(1 - \frac{1}{x^2}\right)^{\frac{1}{3}}$$
On pose: $x = \frac{1}{X}$
$$f(X) = \left(\frac{1}{X}\right)^{\frac{4}{3}} \cdot (1 - X^2)^{\frac{1}{3}}$$
$$(1 - X^2)^{\frac{1}{3}} = 1 - \frac{\frac{1}{3}}{1!}X^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}X^4 - .... + (-1)^n \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)...\left(\frac{1}{3}-n+1\right)}{n!}X^{2n} + X^{2n}\varepsilon(X)$$

1.6 The generalized L.D

Let$f$ be defined in the vicinity of "0". It is assumed that does not admit a L.D. in the vicinity of"0" but the function $x \mapsto x^\alpha f(x)$ admits a L.D. in the vicinity of "0", and we can write:
$$x^\alpha f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + x^n \varepsilon(x)$$
$$f(x) = \frac{1}{x^\alpha}\left(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + x^n \varepsilon(x)\right)$$
This expression is the generalized L.D. of f in "0"

Example 12. $f(x) = \cot gx = \frac{\cos x}{\sin x}$
$\lim_{x\to 0} \cot gx = \infty \Rightarrow f$*does not admit a L.D. in the vicinity vicinity of* "0"
$\lim_{x\to 0} x \cdot \cot gx = \lim_{x\to 0} x \cdot \frac{\cos x}{\sin x} = 1 \left(\textit{because } \lim_{x\to 0} \frac{\sin x}{x} = 1\right)$
Then:$x \cdot \cot gx = x \cdot \frac{\cos x}{\sin x} = x \cdot \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + x^4\varepsilon(x)}{x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\varepsilon(x)}$ $x \cdot \cot gx = 1 - \frac{x^2}{3} + \frac{x^4}{45} + x^4\varepsilon(x)$ $\cot gx = \frac{1}{x}\left(1 - \frac{x^2}{3} + \frac{x^4}{45} + x^4\varepsilon(x)\right)$

1.7 Solved exercises

Exercise 1.

Show that the following functions admit a limited development to order "n" in the vicinity of "0".

$a) f(x) = \log(2+x)$, $b) g(x) = \cos(1+x)$

Solution:

*Show that the following functions admit a limited development to order "n" in the vicinity of "0".

$a) f(x) = \log(2+x) \in C^{\infty}(]-2,+\infty[)$, in particular $C^{n+1}([0,x])$

Therefore, we can apply the Taylor-Lagrange limited development

$f(x) = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + ... + \frac{f^{(n)}(0)}{n!}(x)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$

We have:$(\log(2+x))^{(n)} = \frac{(-1)^{n+1}(n-1)!}{(2+x)^n}$

So:$f(x) = \log 2 + \frac{1}{2}x - \frac{1}{8}x^2 + ... + \frac{(-1)^{n+1}}{2^n n}x^n + \frac{(-1)^{n+2}}{(n+1)(2+c)^{n+1}}(x)^{n+1}$

$b) g(x) = \cos(1+x) \in C^{\infty}(\mathbb{R})$, in particular $C^{n+1}([0,x])$

Therefore, we can apply the Taylor-Lagrange limited development

$g(x) = g(0) + \frac{g'(0)}{1!}(x) + \frac{g''(0)}{2!}(x)^2 + ... + \frac{g^{(n)}(0)}{n!}(x)^n + \frac{g^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$

We have:$(\cos(1+x))^{(n)} = \cos\left(1+x+n\frac{\pi}{2}\right)$

So:$g(x) = 1 + \frac{\cos\left(1+\frac{\pi}{2}\right)}{1}x + \frac{\cos(1+\pi)}{2!}x^2 + ... + \frac{\cos\left(1+n\frac{\pi}{2}\right)}{n!}x^n + \frac{\cos\left(1+c+(n+1)\frac{\pi}{2}\right)}{(n+1)!}(x)^{n+1}$

Exercise 2:

Let $f$be a function that admits limited development to the order "1" in the vicinity of "0"

Show that if f is continuous at "0", f is differentiable at "0"

Solution:

Let $f$be a function that admits limited development to the order "1" in the vicinity of "0" Show that if f is continuous at "0", f is differentiable at "0" $f(x) = a_0 + a_1 x + x\epsilon(x)$ with $\lim\limits_{x\to 0}\epsilon(x) = 0$

*f is continuous at "0" $\Leftrightarrow \lim\limits_{x\to 0} f(x) = f(0) = a_0$

*$\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim\limits_{x\to 0}\frac{a_0+a_1x+x\epsilon(x)-a_0}{x} = \lim\limits_{x\to 0}(a_1+\epsilon(x)) = f'(0)$

$\Rightarrow f$ is differentiable at "0"

Exercise 3:

Let g be the function defined on $\mathbb{R}$, by

$$g(x) = \begin{cases} \frac{e^x-1}{x}, \text{if } x \neq 0 \\ 2, \end{cases}$$

a)Study the continuity of g in "0"
b) Show that $g$admits a limited development of order "n" in the vicinity of "0"
c)Form this development

Solution:

Let g be the function defined on $\mathbb{R}$, by
$g(x)=\begin{cases} \frac{e^x-1}{x}, \text{if } x\neq 0 \\ 2, \qquad \text{if } x=0 \end{cases}$
a)The continuity of g in "0"
$\lim\limits_{x\to 0} g(x)=\lim\limits_{x\to 0}\frac{e^x-1}{x}=1\neq g(0)$, so g is not continuous in "0"
b)Let's show that $g$admits a limited development of order "n" in the vicinity of"0"
We have: $x\mapsto e^x\in C^\infty(\mathbb{R})$,in particular $C^{n+1}([0,x])$, we can apply the limited development
c) The Limited development
$g(x)=g(0)+\frac{g'(0)}{1!}(x)+\frac{g"(0)}{2!}(x)^2+.....+\frac{g^{(n+1)}(0)}{(n+1)!}(x)^{n+1}+\frac{g^{(n+2)}(c)}{(n+2)!}(x)^{n+2}$
$e^x=1+x+\frac{x^2}{2!}+.....+\frac{x^{n+1}}{(n+1)!}+\frac{e^c}{(n+2)!}x^{n+2}$
$e^x-1=x+\frac{x^2}{2!}+.....+\frac{x^{n+1}}{(n+1)!}+\frac{e^c}{(n+2)!}x^{n+2}$
$\frac{e^x-1}{x}=1+\frac{x}{2!}+.....+\frac{x^n}{(n+1)!}+\frac{e^c}{(n+2)!}x^{n+1}$

Exercice 4:

Study if the function f defined by $f(x)=\sqrt[3]{tgx}$has a development limited to order "3"in the vicinity of "$\frac{\pi}{4}$"

Solution:

Let $f(x)=\sqrt[3]{tgx}\in C^\infty\left(\mathbb{R}-\left\{\frac{\pi}{2}+k\pi\right\}\right)$, in particular $C^{n+1}\left(\left[\frac{\pi}{4},x\right]\right)$,
We have: $x\to\frac{\pi}{4}, y=x-\frac{\pi}{4}\to 0$
$f(x)=\sqrt[3]{tgx}=\sqrt[3]{tg\left(y+\frac{\pi}{4}\right)}=\sqrt[3]{\frac{tgy+tg\frac{\pi}{4}}{1-tgy\cdot tg\frac{\pi}{4}}}\left(\text{car:}tg(a+b)=\frac{tga+tgb}{1-tga\cdot tgb}\right)$
$f(x)=\sqrt[3]{tgx}=\sqrt[3]{\frac{1+tgy}{1-tgy}}$
$siny=f(0)+\frac{f'(0)}{1!}(y)+\frac{f"(0)}{2!}y^2+\frac{f'''(0)}{3!}y^3+O(y^3)$
$\sin y=y-\frac{1}{3!}y^3+O(y^3)$
$\cos y=1-\frac{1}{2}y^2+O(y^3)$
$tgy=\frac{y-\frac{1}{3!}y^3+O(y^3)}{1-\frac{1}{2}y^2+O(y^3)}=y+\frac{1}{3}y^3+O(y^3)$
$\frac{1+tgy}{1-tgy}=\frac{1+y+\frac{1}{3}y^3+O(y^3)}{1-y-\frac{1}{3}y^3+O(y^3)}=1+2y+2y^2+\frac{8}{3}y^3+O(y^3)$
$(1+x)^\alpha=g(0)+\frac{g'(0)}{1!}(x)+\frac{g"(0)}{2!}(x)^2+.....+\frac{g^{(n)}(0)}{n!}(x)^n+\frac{g^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$
$(1+x)^\alpha=1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+.....+\frac{\alpha(\alpha-1).....(\alpha-(n+1))}{n!}(x)^n+\frac{g^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$

$$\sqrt[3]{tgx} = \sqrt[3]{\frac{1+tgy}{1-tgy}} = \left(\frac{1+y+\frac{1}{3}y^3+O\left(y^3\right)}{1-y-\frac{1}{3}y^3+O(y^3)}\right)^{\frac{1}{3}} = \left(1+2y+2y^2+\frac{8}{3}y^3+O\left(y^3\right)\right)^{\frac{1}{3}}$$
$$f(x) = \sqrt[3]{tgx} = 1+\frac{2}{3}y-\frac{2}{9}y^2-\frac{40}{27}y^3+O\left(y^3\right)$$
$$f(x) = \sqrt[3]{tgx} = 1+\frac{2}{3}\left(x-\frac{\pi}{4}\right)-\frac{2}{9}\left(x-\frac{\pi}{4}\right)^2-\frac{40}{27}\left(x-\frac{\pi}{4}\right)^3+O\left(\left(x-\frac{\pi}{4}\right)^3\right)$$

Exercise 5:

Give the generalized limited development in the vicinity of "0" to the order "3" of $f\left(x\right) = \frac{\cos x}{\log(1+x)}$

Solution:

We give the generalized limited development in the vicinity of "0" to order "3" of $f\left(x\right) = \frac{\cos x}{\log(1+x)}$

We have: $\underset{x\to 0}{lim} f\left(x\right) = +\infty$, therefore f does not admit a limited development

$\cos x = 1-\frac{1}{2}x^2+O\left(x^3\right)$

$\log\left(1+x\right) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+O\left(x^3\right)$

$x\cos x = x\left(1-\frac{1}{2}x^2+O\left(x^3\right)\right)$

We give the limited development in the vicinity of "0" to the order "4" of $x\ f\left(x\right) = \frac{x\cdot\cos x}{\log(1+x)}$

$x\ f\left(x\right) = \frac{x\cdot\cos x}{\log(1+x)} = \frac{x-\frac{1}{2}x^3+O\left(x^4\right)}{x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+O(x^4)} = 1+\frac{1}{2}x-\frac{7}{12}x^2-\frac{17}{24}x^3+\frac{61}{144}x^4+O\left(x^4\right)$

So: $f\left(x\right) = \frac{1}{x}+\frac{1}{2}-\frac{7}{12}x-\frac{17}{24}x^2+\frac{61}{144}x^3+O\left(x^3\right)$

Chapter 2

Reiman's integral

we give the notion of a Reiman's integral in Analysis. Darboux's sums. They are introduced and studied in this chapter. Further, we give the different methods for calculating integrals

2.1 Darboux's sums

Let f be a function defined and bounded on $[a, b]$

Let "d" be a subdivision of $[a, b]$ obtained by means of "n" points $x_0, x_1, x_2, ..., x_n$

$$d = \{a = x_0, ..., x_n = b\}$$

The positive real $\delta : \delta(d) = Max(x_{i+1} - x_i)$ is called the pitch of the subdivision.

Every subdivision "d" of $[a, b]$, is associated with two sums known as Darboux sums

$$\begin{cases} s(d) = m_0(x_1 - a) + m_1(x_2 - x_1) + ... + m_{n-1}(b - x_{n-1}) \\ s(d) = \sum\limits_{i=0}^{n} m_i(x_{i+1} - x_i) \end{cases}$$
$$\begin{cases} S(d) = M_0(x_1 - a) + M_1(x_2 - x_1) + ... + M_{n-1}(b - x_{n-1}) \\ S(d) = \sum\limits_{i=0}^{n} M_i(x_{i+1} - x_i) \end{cases}$$

*m_i, M_i are respectively the lower and upper bounds of f on $[x_i, x_{i+1}]$

*$s(d)$ and $S(d)$ are respectively the sum of the areas of the lower and upper rectangles.

2.2 Integrability criterion

The function f bounded on $[a, b]$ is said to be integrable if $S(d) - s(d) \to 0$ when $\delta(d) \to 0$ i-e: the number of points "n" tends to ∞

The sums $S(d), s(d)$ form two families are continuous their "common" limit is called the integral of f on $[a, b]$ and is denoted by

$$I\left(f\right)=\int_a^b f(x)dx$$

and is read as the sum of "a" to "b" of $f(x)dx$

$I(f)$ is the common limit of the two sums

$$I\left(f\right)=\lim_{n\to\infty}\sum_{i=0}^{n}m_i\left(x_{i+1}-x_i\right)=\lim_{n\to\infty}\sum_{i=0}^{n}M_i\left(x_{i+1}-x_i\right)$$

Example 13. $f(x)=x^2,[a,b]=[0,1]$

*"n" *points:*$x_0=0,x_1=\frac{1}{n+1},x_2=\frac{2}{n+1},.....,x_i=\frac{i}{n+1}$

*$x_{i+1}-x_i=\frac{i+1}{n+1}-\frac{i}{n+1}=\frac{1}{n+1}$

*$m_i=\left(x_i\right)^2=\left(\frac{i}{n+1}\right)^2=\frac{i^2}{(n+1)^2}$

*$M_i=\left(x_{i+1}\right)^2=\left(\frac{i+1}{n+1}\right)^2=\frac{(i+1)^2}{(n+1)^2}$

$$*\begin{cases} s=\sum\limits_{i=0}^{n}m_i\left(x_{i+1}-x_i\right)=\sum\limits_{i=0}^{n}\frac{i^2}{(n+1)^3}=\frac{1}{(n+1)^3}\sum\limits_{i=0}^{n}i^2 \\ s=\frac{1}{(n+1)^3}\left(0^2+1^2+...+n^2\right)=\frac{1}{(n+1)^3}\cdot\frac{n(n+1)(2n+1)}{6} \\ \lim\limits_{n\to\infty}s=\lim\limits_{n\to\infty}\sum\limits_{i=0}^{n}m_i\left(x_{i+1}-x_i\right)=\lim\limits_{n\to\infty}\frac{1}{(n+1)^3}\cdot\frac{n(n+1)(2n+1)}{6}=\frac{1}{3} \end{cases}$$

$$*\begin{cases} S=\sum\limits_{i=0}^{n}M_i\left(x_{i+1}-x_i\right)=\sum\limits_{i=0}^{n}\frac{(i+1)^2}{(n+1)^3}=\frac{1}{(n+1)^3}\sum\limits_{i=0}^{n}\left(i+1\right)^2 \\ S=\frac{1}{(n+1)^3}\left(1^2+2^2+...+(n+1)^2\right)=\frac{1}{(n+1)^3}\cdot\frac{(n+1)(n+2)(2n+3)}{6} \\ \lim\limits_{n\to\infty}S=\lim\limits_{n\to\infty}\sum\limits_{i=0}^{n}M_i\left(x_{i+1}-x_i\right)=\lim\limits_{n\to\infty}\frac{1}{(n+1)^3}\cdot\frac{(n+1)(n+2)(2n+3)}{6}=\frac{1}{3} \end{cases}$$

*$I\left(f\right)=\int_0^1 x^2dx=\left[\frac{1}{3}x^3\right]_0^1=\frac{1}{3}$

2.3 Properties of Reiman's integral

1/$f:[a,b]\to\mathbb{R}$ is integrable. Then f is integrable on each interval $[\alpha,\beta]\subset[a,b]$

2/Let $c\in\left]a,b\right[$ .if f is separately integrable on $[a,c]$ and $[c,b]$. Then f is integrable on $[a,b]$:

$$\int_a^c f(x)dx+\int_c^b f(x)dx=\int_a^b f(x)dx=F(x)$$

3/if f is integrable on $[a,b]$, if $c\in\left]a,b\right[$ .then

$$F(x)=\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

4/All continuous functions on $[a,b]$ are integrable

5/All monotone functions on $[a,b]$ are integrable

6/$\int_a^a f(x)dx=0=\int_a^b f(x)dx+\int_b^a f(x)dx=0\Leftrightarrow\int_a^b f(x)dx=-\int_b^a f(x)dx$

2.4 General integration processes

2.4.1 Variable change

Table of usual primitives

$1/\int x^m dx = \frac{1}{m+1}x^{m+1} + c\,(m \neq -1)$
$2/\int \frac{1}{x}dx = \ln|x| + c$
$3/\int \frac{f'}{f}dx = \ln|f| + c$
$4//\int f' \cdot f^m dx = \frac{1}{m+1}f^{m+1} + c$
$5/\int \sin x dx = -\cos x + c$
$6/\int \cos x dx = \sin x + c$
$7/\int shx dx = chx + c$
$8/\int chx dx = shx + c$
$9/\int \frac{dx}{(\cos x)^2} = tgx + c$
$10/\int \frac{dx}{1+x^2} = arctgx + c$
$11/\int e^x dx = e^x + c$
$12/\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$
$13/\int \frac{dx}{\sqrt{h+x^2}} = \log\left|x + \sqrt{h+x^2}\right| + c$
$14/\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$

Theorem 2.4.1. *Let f be a continuous function on $[a,b]$ and $Q : [\alpha,\beta] \to [a,b]$ continuously differentiable function such that $Q(\alpha) = a$, $Q(\beta) = b$.*

Then the function $f(Q(t)) \cdot Q'(t)$ and integrable on $[a,b]$ and we have:

$$\int_a^b f(x)dx = \int_\alpha^\beta f(Q(t)) \cdot Q'(t)\ dx$$

Example 14. *1/*$I = \int_0^\pi (\cos x)^2 \cdot \sin x dx$

We take: $cosx = t \to (cosx)' = -\sin x dx = 1dt \begin{cases} x = 0 \to t = 1 \\ x = \pi \to t = -1 \end{cases}$

$I = \int_0^\pi (\cos x)^2 \cdot \sin x dx = -\int_{-1}^1 t^2 dx = -\left[\frac{t^3}{3}\right]_{-1}^1 = \frac{2}{3}$

$-\int t^2 dx = -\frac{t^3}{3} + c = -\frac{(\cos x)^3}{3} + c$

2/$II = \int_0^1 \sqrt{1-x^2}dx$

We take: $x = \sin t \to dx = \cos t dt \begin{cases} x = 0 \to t = 0 \\ x = 1 \to t = \frac{\pi}{2} \end{cases}$

$II = \int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \sqrt{1-(\sin t)^2} \cdot \cos t dt = \int_0^{\frac{\pi}{2}} |\cos t| \cdot \cos t dt$

$II = \int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} (\cos t)^2 dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 2t + 1)\, dt = \frac{1}{2}\left[\left(\frac{1}{2}\sin 2t + t\right)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

2.4.2 Integration by part

Theorem 2.4.2. *Let U and V be two functions continuously differentiable on $[a,b]$.then*

$\int_a^b (U(x))' \cdot V(x)\, dx = [U(x) \cdot V(x)]_a^b - \int_a^b U(x) \cdot (V(x))'\, dx$
$(U(x) \cdot V(x))' = (U(x))' \cdot V(x) + U(x) \cdot (V(x))'$
$\int_a^b (U(x) \cdot V(x))'\, dx = \int_a^b (U(x))' \cdot V(x)\, dx + \int_a^b U(x) \cdot (V(x))'\, dx$
$[U(x) \cdot V(x)]_a^b = \int_a^b (U(x))' \cdot V(x)\, dx + \int_a^b U(x) \cdot (V(x))'\, dx$

Example 15. $I = \int_0^1 arctgx dx$

We take: $\begin{cases} U(x) = arctgx \to (U(x))' = \frac{1}{1+x^2} dx \\ (V(x))' = 1dx \to V(x) = x \end{cases}$

$I = \int_0^1 arctgx dx = [x \cdot arctgx]_0^1 - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx$

$I = \int_0^1 arctgx dx = \frac{\pi}{4} - \left[\frac{1}{2} \ln |1 + x^2|\right]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln (2)$

2.4.3 Calculates primitives

Every continuous function on $[a, b]$admits a primitive i-e:a function F:$[a, b] \to \mathbb{R}$ such that $F' = f$on$[a, b]$.

The set of primitives of f (known as the indefinite integral) is written:

$$\int f(x)dx = F(x) + c$$

Where F is a fixed primitive and "c" a constant function

$$\int_a^b f(x)dx = F(b) - F(a)$$

is called a definite integral and is unique

2.4.4 Primitive of a rational function

Every rational function is written as a finite number of rational fractions of the form

$$\frac{A}{(x-a)^k}, \frac{Mx+N}{\left((x-\alpha)^2 + \beta^2\right)^k} / a, A, \alpha, \beta, M, N \text{ des constantes}$$

$I_1 = \int \frac{dx}{(x-a)^k} = \begin{cases} k = 1, \quad I_1 = \int \frac{dx}{x-a} = \ln |x - a| + c \\ k > 1, \quad I_1 = \int \frac{dx}{(x-a)^k} = \int (x-a)^{-k} dx = \frac{(x-a)^{-k+1}}{-k+1} + c \end{cases}$

$\frac{Mx+N}{\left((x-a)^2+\beta^2\right)^k} = \frac{Mx+N}{\beta^{2k}\left[\left(\frac{x-\alpha}{\beta}\right)^2+1\right]^k}$

we take:$\frac{x-\alpha}{\beta} = X \to x = \beta X + \alpha \to dx = \beta dX$

$$I_2=\int\frac{Mx+N}{\left((x-a)^2+\beta^2\right)^k}dx=\int\frac{Mx+N}{\beta^{2k}\left[\left(\frac{x-\alpha}{\beta}\right)^2+1\right]^k}dx=\int\frac{M(\beta X+\alpha)+N}{\beta^{2k-1}[X^2+1]^k}dX$$
$$I_2=\frac{1}{\beta^{2k-1}}\int\frac{AX+B}{[X^2+1]^k}dX=\frac{1}{\beta^{2k-1}}\int\frac{AX}{[X^2+1]^k}dX+\frac{1}{\beta^{2k-1}}\int\frac{B}{[X^2+1]^k}dX$$
$$I_2=\frac{A}{\beta^{2k-1}}\int\frac{X}{[X^2+1]^k}dX+\frac{B}{\beta^{2k-1}}\int\frac{1}{[X^2+1]^k}dX=\frac{A}{\beta^{2k-1}}J_1+\frac{B}{\beta^{2k-1}}J_2$$
$$J_1=\int\frac{X}{[X^2+1]^k}dX=\frac{1}{2}\int\frac{2X}{[X^2+1]^k}dX$$
$$\begin{cases} If: k=1\rightarrow J_1=\frac{1}{2}\int\frac{2X}{X^2+1}dX=\frac{1}{2}\ln\left(X^2+1\right)+c \\ If: k>1\rightarrow J_1=\frac{1}{2}\int\frac{2X}{[X^2+1]^k}dX=\frac{1}{2}\int 2X\left[X^2+1\right]^{-k}dX=\frac{1}{2(1-k)}\left(X^2+1\right)^{1-k}+c \end{cases}$$
$$\mathrm{J}_2=\int\frac{1}{[X^2+1]^k}dX\begin{cases}\text{If}: k=1\rightarrow J_2=\int\frac{1}{X^2+1}dX=arctgX+c\\ \text{If}: k>1\rightarrow J_2=\int\frac{1}{[X^2+1]^k}dX\end{cases}$$
$$\begin{cases} U(X)=\frac{1}{[X^2+1]^k}\rightarrow (U(X))'=-2kX\left[X^2+1\right]^{-k-1}dX \\ (V(X))'=1dX\rightarrow V(x)=X \\ J_2=\int\frac{1}{[X^2+1]^k}dX=\frac{X}{[X^2+1]^k}+2k\int\frac{X^2}{[X^2+1]^{k+1}}dX \\ \int\frac{X^2}{[X^2+1]^{k+1}}dX=\int\frac{X^2+1-1}{[X^2+1]^{k+1}}dX=\int\frac{1}{[X^2+1]^k}dX-\int\frac{1}{[X^2+1]^{k+1}}dX \\ J_2=\frac{X}{[X^2+1]^k}+2k\int\frac{1}{[X^2+1]^k}dX-2k\int\frac{1}{[X^2+1]^{k+1}}dX \\ \text{on pose } J_2=I_k=\int\frac{1}{[X^2+1]^k}dX\rightarrow I_k=\frac{X}{[X^2+1]^k}+2kI_k-2kI_{k+1} \\ (1-2k)\,I_k=(1-2k)\int\frac{1}{[X^2+1]^k}dX=\frac{X}{2k[X^2+1]^k}-2kI_{k+1}\rightarrow \\ 2kI_{k+1}=\frac{X}{2k[X^2+1]^k}-(1-2k)\,I_k \\ I_1=\int\frac{1}{X^2+1}dX=arctgX+c \\ 2I_2=\frac{X}{2[X^2+1]}+I_1=\frac{X}{2[X^2+1]}+arctgX+c \end{cases}$$

2.4.5 Primitive of rational functions in $\sin x,\cos x$

Calculate $I=R(\sin x,\cos x,....)$

R is a rational function of $\sin x,\cos x$,we use the change of variable

$$t=tg\left(\frac{x}{2}\right)\rightarrow arctgt=\frac{x}{2}\rightarrow 2arctgt=x\rightarrow\begin{cases}\sin x=\frac{2t}{1+t^2}\\ \cos x=\frac{1-t^2}{1+t^2}\\ dx=\frac{2dt}{1+t^2}\end{cases}$$

Example 16. *1/* $I_1=\int\frac{dx}{\sin x}=\int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}}=\int\frac{dt}{t}=\ln|t|+c=\ln\left|tg\left(\frac{x}{2}\right)\right|+c$

2/ $I_1=\int\frac{dx}{\cos x}=\int\frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2}}=2\int\frac{dt}{1-t^2}=\left[\int\frac{dt}{1-t}+\int\frac{dt}{1+t}\right]=-\ln|1-t|+\ln|1+t|+c=$ $\ln\left|\frac{1+t}{1-t}\right|+c$

because: $\frac{1}{1-t^2}=\frac{1}{(1-t)(1+t)}=\frac{a}{1-t}+\frac{b}{1+t}=\frac{1}{2}\left[\frac{1}{1-t}+\frac{1}{1+t}\right]$

So: $I_1=\int\frac{dx}{\cos x}=\ln\left|\frac{1+t}{1-t}\right|+c=\ln\left|\frac{1+tg\left(\frac{x}{2}\right)}{1-tg\left(\frac{x}{2}\right)}\right|+c$

It also applies to all trigonometric functions.

2.4.6 Primitive of rational functions in e^x

Here we use the change of variable $t=e^x$

Example 17. *1/* $I_1 = \int \frac{dx}{1+e^x} \begin{cases} t = e^x \\ dt = e^x dx = tdx \end{cases}$

$I_1 = \int \frac{dx}{1+e^x} = \int \frac{dt}{t(1+t)} = \int \frac{dt}{t} - \int \frac{dt}{1+t} = \ln|t| - \ln|1+t| + c$
$I_1 = \int \frac{dx}{1+e^x} = \ln\left|\frac{t}{1+t}\right| + c = \ln\left|\frac{e^x}{1+e^x}\right| + c$

For hyperbolic functions, we can also use the change of variable $t = th\left(\frac{x}{2}\right)$

Example 18. *1/*$I_1 = \int \frac{dx}{x^2+2x+3} = \int \frac{dx}{2\left[\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right]}$

We take :$\frac{x+1}{\sqrt{2}} = t \to x = \sqrt{2}t - 1 \to dx = \sqrt{2}dt$
$I_1 = \int \frac{dx}{2\left[\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right]} = \int \frac{\sqrt{2}dt}{2(t^2+1)} = \frac{\sqrt{2}}{2}\int \frac{dt}{t^2+1} = \frac{\sqrt{2}}{2}arctg(t) + c$
$I_1 = \frac{\sqrt{2}}{2}arctg(\frac{x+1}{\sqrt{2}}) + c$
2/$I_2 = \int x \cdot \sqrt{x-1}dx \begin{cases} x - 1 = t^2 \\ dx = 2tdt \end{cases}$
$I_2 = \int x \cdot \sqrt{x-1}dx = \int (t^2+1)\, t \cdot 2tdt = 2\int t^2 (t^2+1)\, dt = 2\int (t^4+t^2)\, dt$
$I_2 = \int x \cdot \sqrt{x-1}dx = 2\left(\frac{t^5}{5} + \frac{t^3}{3}\right) + c = 2\frac{t^5}{5} + 2\frac{t^3}{3} + c$
$I_2 = \int x \cdot \sqrt{x-1}dx = 2\frac{(x-1)^{\frac{5}{2}}}{5} + 2\frac{(x-1)^{\frac{3}{2}}}{3} + c$

2.5 Solved exercises

Exercise 1

Calculate using the limit method
$\int_0^1 xdx, \int_0^1 x^2dx, \int_0^1 x^3dx,$
Since: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Solution:

We use the Riemann integral to calculate the following integrals $*\int_0^1 xdx = \underset{n\to+\infty}{lim} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right) /$
$x, b = 1, a = 0$
$= \underset{n\to+\infty}{lim} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} = \underset{n\to+\infty}{lim} \frac{1}{n^2} \sum_{k=1}^{n} k = \underset{n\to+\infty}{lim} \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{1}{2}$
$* \int_0^1 x^2dx = \underset{n\to+\infty}{lim} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right) / f(x) = x^2, b = 1, a = 0$

$= \underset{n\to+\infty}{lim} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 = \underset{n\to+\infty}{lim} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \underset{n\to+\infty}{lim} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}$
$*\int_0^1 x^3dx = \underset{n\to+\infty}{lim} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right) / f(x) = x^3, b = 1, a = 0$
$= \underset{n\to+\infty}{lim} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 = \underset{n\to+\infty}{lim} \frac{1}{n^4} \sum_{k=1}^{n} k^3 = \underset{n\to+\infty}{lim} \frac{1}{n^4} \frac{n^2(n+1)^2}{4} = \frac{1}{4}$

Exercise 2:

Let $x \in \mathbb{R}^+$, using the properties of geometric sequences
Show that : $\underset{x\to+\infty}{lim} \frac{1}{n} \sum_{p=0}^{n-1} e^{\frac{px}{n}} = \frac{e^x-1}{x}$
Use this result to establish, for every real number $x > 0$, the relation $\int_0^x e^t dt = e^x - 1$

Solution:

*Show that: $\underset{x\to+\infty}{lim} \frac{1}{n} \sum_{p=0}^{n-1} e^{\frac{px}{n}} = \frac{e^x-1}{x}$
We have: $\sum_{p=0}^{n-1} e^{\frac{px}{n}}$ is a sum of a geometric "n" term sequence, of reason $q = e^{\frac{x}{n}}$ and the first term is a $v_0 = 1$,Therefore
$\sum_{p=0}^{n-1} e^{\frac{px}{n}} = v_0 \frac{1-q^n}{1-q} = \frac{1-e^x}{1-e^{\frac{x}{n}}}$
So: $\underset{x\to+\infty}{lim} \frac{1}{n} \sum_{p=0}^{n-1} e^{\frac{px}{n}} = \underset{x\to+\infty}{lim} \frac{1}{n} \frac{1-e^x}{1-e^{\frac{x}{n}}}$
We have:$e^{\frac{x}{n}} - 1 \underset{0}{\sim} \frac{x}{n}$
$\underset{x\to+\infty}{lim} \frac{1}{n} \sum_{p=0}^{n-1} e^{\frac{px}{n}} = \underset{x\to+\infty}{lim} \frac{1}{n} \frac{1-e^x}{-\frac{x}{n}} = \frac{e^x-1}{x}$
We establish the relation $\int_0^x e^t dt = e^x - 1$
$\int_0^x e^t dt = \underset{n\to+\infty}{lim} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + \frac{b-a}{n}k\right) / f(x) = e^t, b = x, a = 0$
$\int_0^x e^t dt = \underset{n\to+\infty}{lim} \frac{x}{n} \sum_{k=0}^{n-1} f\left(\frac{x}{n}k\right) = \underset{n\to+\infty}{lim} \frac{x}{n} \sum_{k=0}^{n-1} e^{\frac{x}{n}k} = x \underset{n\to+\infty}{lim} \frac{1}{n} \sum_{k=0}^{n-1} e^{\frac{x}{n}k}$
$\int_0^x e^t dt = x \cdot \frac{e^x-1}{x} = e^x - 1$

Exercise 3:

Calculate the following integrals
$\int \frac{1}{x(\log x)^2}dx, \int \frac{1}{x\sqrt{1+\log x}}dx, \int \frac{x\log x}{(1+x^2)^2}dx, \int \cos x e^{\sin x}dx, \int e^{2x}\cos e^x dx, \int \frac{\sin x-\cos x}{\sin x+\cos x}dx, \int xtg^2xdx, \int \frac{1}{3+5x^2}dx,$

Solution:

Calculate the following integrals

$*\int \frac{1}{x(\log x)^2}dx = \int \frac{1}{x}\frac{1}{(\log x)^2}dx = \int \frac{1}{x}(\log x)^{-2}dx = -\frac{1}{\log x} + c$

$*\int \frac{1}{x\sqrt{1+\log x}}dx = 2\int \frac{\frac{1}{x}}{2\sqrt{1+\log x}}dx = 2\sqrt{1+\log x} + C$

$*\int \frac{x\log x}{(1+x^2)^2}dx = \begin{cases} U = \log x \to U' = \frac{1}{x}dx \\ V' = \frac{x}{(1+x^2)^2}dx \to V = \frac{-1}{2}\frac{1}{1+x^2} \end{cases}$

$\int \frac{x\log x}{(1+x^2)^2}dx = \frac{-\log x}{2(1+x^2)} + \frac{1}{2}\int \frac{1}{x(1+x^2)}dx + c$

$\int \frac{x\log x}{(1+x^2)^2}dx = \frac{-\log x}{2(1+x^2)} + \frac{1}{2}\int \left[\frac{1}{x} - \frac{x}{1+x^2}\right]dx + c$

$\int \frac{x\log x}{(1+x^2)^2}dx = \frac{-\log x}{2(1+x^2)} + \frac{1}{2}\ln|x| - \frac{1}{4}\ln(1+x^2) + c$

$*\int \cos x e^{\sin x}dx = e^{\sin x} + c$

$*\int e^{2x}\cos e^x dx = \int e^x \cdot e^x \cos e^x dx \begin{cases} U = e^x \to U' = e^x dx \\ V' = e^x \cos e^x dx \to V = \sin e^x \end{cases}$

$\int e^{2x}\cos e^x dx = e^x \sin e^x - \int e^x \sin e^x dx + c$

$\int e^{2x}\cos e^x dx = e^x \sin e^x + e^x \cos e^x + c$

$*\int \frac{\sin x-\cos x}{\sin x+\cos x}dx = -\ln|\sin x + \cos x| + c$

$*\int xtg^2xdx = \begin{cases} U = x \to U' = dx \\ V' = tg^2xdx \to V = tgx - x \end{cases}$

$\int xtg^2xdx = x(tgx - x) - \int (tgx - x)dx + c$

$\int xtg^2xdx = xtgx + \ln|\cos x| - \frac{1}{2}x^2 + c$

$*\int \frac{1}{3+5x^2}dx = \frac{1}{3}\int \frac{1}{1+\left(x\sqrt{\frac{5}{3}}\right)^2}dx = \begin{cases} t = x\sqrt{\frac{5}{3}} \\ dt = \sqrt{\frac{5}{3}}dx \end{cases}$

$\int \frac{1}{3+5x^2}dx = \frac{1}{3}\sqrt{\frac{3}{5}}\int \frac{1}{1+t^2}dt = \sqrt{\frac{1}{15}}arctgt + c = \sqrt{\frac{1}{15}}arctg\left(x\sqrt{\frac{5}{3}}\right) + c$

$*\int (1+2x)\arg thxdx = \begin{cases} U = \arg thx \to U' = \frac{1}{1-x^2}dx \\ V' = (1+2x)dx \to V = x + x^2 \end{cases}$

$\int (1+2x)\arg thxdx = (x+x^2)\arg thx - \int \frac{x+x^2}{1-x^2} + c$

$\int (1+2x)\arg thxdx = (x+x^2)\arg thx - \int \frac{x}{1-x^2}dx - \int \frac{x^2}{1-x^2}dx + c$

$\int (1+2x)\arg thxdx = (x+x^2)\arg thx - \frac{1}{2}\int \frac{2x}{1-x^2}dx + \int \frac{1-x^2-1}{1-x^2}dx + c$

$\int (1+2x)\arg thxdx = (x+x^2)\arg thx + \frac{1}{2}\int \frac{-2x}{1-x^2}dx + \int \frac{1-x^2}{1-x^2}dx + \int \frac{-1}{1-x^2}dx + c$

$\int (1+2x)\arg thxdx = (x+x^2)\arg thx + \frac{1}{2}\ln|1-x^2| + x + \frac{1}{2}\int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{1}{1+x}dx + c$

$\int (1+2x)\arg thxdx = (x+x^2)\arg thx + \frac{1}{2}\ln|1-x^2| + x - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + c$

$\int (1+2x)\arg thxdx = (x+x^2)\arg thx + \frac{1}{2}\ln|1-x^2| + x + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$

$$* \int x^2 shx dx = \begin{cases} U = x^2 \rightarrow U' = 2xdx \\ V' = shxdx \rightarrow V = chx \end{cases}$$
$$\int x^2 shx dx = x^2 chx - \int 2xchxdx \begin{cases} U = x \rightarrow U' = dx \\ V' = chxdx \rightarrow V = shx \end{cases}$$
$$\int x^2 shx dx = x^2 chx - 2\left[xshx - \int shxdx\right]$$
$$\int x^2 shx dx = x^2 chx - 2xshx - 2chx + c$$

<u>Exercise 4:</u>

Calculate primitives of rational functions
$\int \frac{x^5+x^4-8}{x^3-4x}dx, \int \frac{x+3}{(x^2+1)(x-2)^2}dx, \int \frac{10}{(x^2+4x+5)^3}dx, \int \frac{x-1}{(x^2-x+1)^2}dx, \int \frac{1}{1+\cos 2x}dx, \int \frac{\sin x}{\cos x(1+\cos x)^2}dx, \int \frac{1}{1+tgx}dx, \int tg$

<u>Solution:</u>

Calculate primitives of rational functions
$$*\int \frac{x^5+x^4-8}{x^3-4x}dx = \int \left(x^2+x+4\right)dx + \int \frac{4x^2+16x-8}{x^3-4x}dx$$
$$\int \frac{x^5+x^4-8}{x^3-4x}dx = \int \left(x^2+x+4\right)dx + \int \frac{\frac{4}{3}\left(3x^2-4\right)}{x^3-4x}dx + \int \frac{\frac{-8}{3}+16x}{x^3-4x}dx + c$$
$$\int \frac{x^5+x^4-8}{x^3-4x}dx = \int \left(x^2+x+4\right)dx + \int \frac{\frac{4}{3}\left(3x^2-4\right)}{x^3-4x}dx + \int \frac{\frac{-8}{3}+16x}{x(x-2)(x+2)}dx + c$$
$$\int \frac{x^5+x^4-8}{x^3-4x}dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x + \frac{4}{3}\ln\left|3x^2-4\right| + \int \frac{2}{x}dx + \int \frac{5}{x-2}dx + \int \frac{-3}{x+2}dx + c$$
$$\int \frac{x^5+x^4-8}{x^3-4x}dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x + \frac{4}{3}\ln\left|3x^2-4\right| + 2\ln|x| + 5\ln|x-2| - 3\ln|x+2| + c$$
$$*\int \frac{x+3}{(x^2+1)(x-2)^2}dx = \int \frac{ax+b}{x^2+1}dx + \int \frac{c}{x-2}dx + \int \frac{d}{(x-2)^2}dx$$
$$\int \frac{x+3}{(x^2+1)(x-2)^2}dx = \int \frac{\frac{3}{5}x+\frac{1}{5}}{x^2+1}dx + \int \frac{\frac{-3}{5}}{x-2}dx + \int \frac{1}{(x-2)^2}dx$$
$$\int \frac{x+3}{(x^2+1)(x-2)^2}dx = \frac{3}{10}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx - \frac{3}{5}\int \frac{1}{x-2}dx + \int \frac{1}{(x-2)^2}dx$$
$$\int \frac{x+3}{(x^2+1)(x-2)^2}dx = \frac{3}{10}\ln\left(x^2+1\right) + \frac{1}{5}arctgx - \frac{3}{5}\ln|x-2| - \frac{1}{x-2} + c$$
$$* \int \frac{10}{(x^2+4x+5)^3}dx = 10\int \frac{1}{\left((x+2)^2+1\right)^3}dx = 10\int \frac{1}{(t^2+1)^3}dt$$
We take: $I_n = \int \frac{1}{(t^2+1)^n}dt$
$$I_n = \int \frac{t^2+1}{(t^2+1)^{n+1}}dt = \int \frac{1}{2}t \times \frac{2t}{(t^2+1)^{n+1}}dt + \int \frac{1}{(t^2+1)^{n+1}}dt$$
$$I_n = \int \frac{t^2+1}{(t^2+1)^{n+1}}dt = \int \frac{1}{2}t \times \frac{2t}{(t^2+1)^{n+1}}dt + I_{n+1} \Leftrightarrow \begin{cases} U = \frac{1}{2}t \rightarrow U' = \frac{1}{2}dt \\ V' = \frac{2t}{(t^2+1)^{n+1}}dt \rightarrow V = \frac{-1}{n(t^2+1)^n} \end{cases}$$
$$I_n = \int \frac{t^2+1}{(t^2+1)^{n+1}}dt = \frac{-t}{2n(t^2+1)^n} + \int \frac{-1}{2n(t^2+1)^n}dt + I_{n+1}$$
$$I_n = \int \frac{t^2+1}{(t^2+1)^{n+1}}dt = \frac{-t}{2n(t^2+1)^n} + \frac{1}{2n}I_n + I_{n+1}$$
$$I_{n+1} = \frac{t}{2n(t^2+1)^n} + \left(1-\frac{1}{2n}\right)I_n$$
We have: $I_1 = \int \frac{1}{t^2+1}dt = arctgt + c$
$$I_2 = \frac{t}{2(t^2+1)} + \left(1-\frac{1}{2}\right)I_1 \rightarrow I_2 = \frac{t}{2(t^2+1)} + \frac{1}{2}arctgt + c$$
$$I_3 = \frac{t}{4(t^2+1)^2} + \left(1-\frac{1}{4}\right)I_2 \rightarrow I_3 = \frac{t}{4(t^2+1)^2} + \frac{3}{4}I_2$$
$$I_3 = \frac{t}{4(t^2+1)^2} + \frac{3t}{8(t^2+1)} + \frac{3}{8}arctgt + c$$
$$\int \frac{10}{(x^2+4x+5)^3}dx = 10\int \frac{1}{\left((x+2)^2+1\right)^3}dx = 10\int \frac{1}{(t^2+1)^3}dt = \frac{10t}{4(t^2+1)^2} + \frac{30t}{8(t^2+1)} + \frac{30}{8}arctgt + c$$

$\int \frac{10}{(x^2+4x+5)^3}dx = 10\int \frac{1}{\left((x+2)^2+1\right)^3}dx = \frac{10(x+2)}{4(x^2+4x+5)^2} + \frac{30(x+2)}{8(x^2+4x+5)} + \frac{30}{8}arctg\,(x+2) + c$

$*\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{1}{2}\int \frac{2x-1}{(x^2-x+1)^2}dx + \frac{1}{2}\int \frac{-1}{(x^2-x+1)^2}dx$

$\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)}dx - \frac{1}{2}\int \frac{1}{\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}dx$

$\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)} - \frac{1}{2}\int \frac{1}{\left(t^2+\frac{3}{4}\right)^2}dt \left(\text{ on pose: } t = x - \frac{1}{2}\right)$

$\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)} - \frac{1}{2}\int \frac{1}{\frac{9}{4}\left(\frac{t^2}{\frac{3}{4}}+1\right)^2}dt$

$\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)} - \frac{1}{2}\int \frac{1}{\frac{9}{4}\left(\left(\frac{2t}{\sqrt{3}}\right)^2+1\right)^2}dt$

$\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)} - \frac{1}{2}\int \frac{1}{\frac{9}{4}(y^2+1)^2}\frac{\sqrt{3}}{2}dy \left(\text{ We take: } y = \frac{2t}{\sqrt{3}}\right)$

$\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)} - \frac{1}{3\sqrt{3}}\int \frac{1}{(y^2+1)^2}dy$

We take: $I_n = \int \frac{1}{(y^2+1)^n}dt$

We have: $I_{n+1} = \frac{t}{2n(y^2+1)^n} + \left(1 - \frac{1}{2n}\right)I_n$

We have: $I_1 = \int \frac{1}{y^2+1}dt = arctgy + c$

$I_2 = \frac{y}{2(y^2+1)} + \left(1 - \frac{1}{2}\right)I_1 \to I_2 = \frac{y}{2(y^2+1)} + \frac{1}{2}arctgy + c$

$\Rightarrow \int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)} - \frac{y}{6\sqrt{3}(y^2+1)} - \frac{1}{6\sqrt{3}}arctgy + c$

$\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)} - \frac{\frac{2t}{\sqrt{3}}}{6\sqrt{3}\left(\left(\frac{2t}{\sqrt{3}}\right)^2+1\right)} - \frac{1}{6\sqrt{3}}arctg\left(\frac{2t}{\sqrt{3}}\right) + c$

$\left(x - \frac{1}{2}\right)\int \frac{x-1}{(x^2-x+1)^2}dx = \frac{-1}{2(x^2-x+1)} - \frac{\left(x-\frac{1}{2}\right)}{9\left(\left(\frac{2x-1}{\sqrt{3}}\right)^2+1\right)} - \frac{1}{6\sqrt{3}}arctg\left(\frac{2x-1}{\sqrt{3}}\right) + c$

$*\int \frac{1}{1+\cos 2x}dx = \frac{1}{2}\int \frac{dx}{\cos^2 x} = \frac{1}{2}tgx + c\left(\cos 2x = \cos^2 x - \sin^2 x \to 1 + \cos 2x = 2\cos^2 x\right)$

$*\int \frac{\sin x}{\cos x(1+\cos x)^2}dx = \Big\{ t = tg\frac{x}{2} \to \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2} \to dx = \frac{2dt}{1+t^2}$

$\int \frac{\sin x}{\cos x(1+\cos x)^2}dx = \int \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)^2}\frac{2dt}{1+t^2} = \int \frac{t\left(1+t^2\right)}{1-t^2}dt$

$\int \frac{\sin x}{\cos x(1+\cos x)^2}dx = \int \frac{t+t^3}{1-t^2}dt = \int -tdt + \int \frac{2t}{1-t^2}dt$

$\int \frac{\sin x}{\cos x(1+\cos x)^2}dx = \frac{-1}{2}t^2 - \ln|1-t^2| + c$

$\int \frac{\sin x}{\cos x(1+\cos x)^2}dx = \frac{-1}{2}\left(tg\frac{x}{2}\right)^2 - \ln\left|1 - \left(tg\frac{x}{2}\right)^2\right| + c$

$*\int \frac{1}{1+tgx}dx = \int \frac{\cos x}{\cos x+\sin x}dx \left(tgx = \frac{\sin x}{\cos x}\right)$

$\begin{cases} \int \frac{\cos x+\sin x}{\cos x+\sin x}dx = x + c_1 \ldots\ldots\ldots (1) \\ \int \frac{\cos x-\sin x}{\cos x+\sin x}dx = \ln|\cos x + \sin x| + c_2 \ldots\ldots (2) \end{cases}$

$(1) + (2) = \int \frac{1}{1+tgx}dx = \int \frac{\cos x}{\cos x+\sin x}dx = \frac{1}{2}x + \frac{1}{2}\ln|\cos x + \sin x| + c$

$*\int tg^3xdx = \int \sin^2 x \cdot \frac{\sin x}{\cos^3 x}dx = \begin{cases} U = \sin^2 x \to U' = 2\sin x\cos xdx \\ V' = \frac{\sin x}{\cos^3 x}dx \to V = \frac{1}{2\cos^2 x} \end{cases}$

$\int tg^3xdx = \frac{1}{2}\frac{\sin^2 x}{\cos^2 x} - \int \frac{\sin x}{\cos x}dx$

$\int tg^3xdx = \frac{1}{2}tg^2x + \ln|\cos x| + c$

Exercise 5:

Let α be a strictly positive real

1) For what value of α is the integral $I=\int_0^{2\pi}\log\left(1-2\alpha\cos x+\alpha^2\right)dx$ is defined?
2) Calculate$I$ when it is defined

Solution:

Let α be a strictly positive real
1) The integral I is defined when: $1-2\alpha\cos x+\alpha^2>0$ on $[0,2\pi]$
But $1-2\alpha\cos x+\alpha^2=0$ if cos $x=\frac{1+\alpha^2}{2\alpha}\,(\alpha>0)$
Since $\frac{1+\alpha^2}{2\alpha}\geq 1$ and $\frac{1+\alpha^2}{2\alpha}=1\Leftrightarrow\alpha=1$
Therefore, the integral $I=\int_0^{2\pi}\log\left(1-2\alpha\cos x+\alpha^2\right)dx$ will be defined,, $\forall\alpha>0$ and $\alpha\neq 1$
2) We calculate I
assume that $\alpha\neq 1$ on $[0,2\pi]$

$$I=\underset{n\to+\infty}{lim}\frac{b-a}{n}\sum_{k=1}^{n}f\left(a+\frac{b-a}{n}k\right)/f(x)=1-2\alpha\cos x+\alpha^2,b=2\pi,a=0$$

$$I=\lim_{n\to+\infty}\sum_{p=1}^{n}\frac{2\pi}{n}\log\left(1-2\alpha\cos\frac{2\pi p}{n}+\alpha^2\right)$$

$$\Leftrightarrow I=\lim_{n\to+\infty}\frac{2\pi}{n}\log\left(\prod_{p=1}^{n}\left(1-2\alpha\cos\frac{2\pi p}{n}+\alpha^2\right)\right)$$

Or:$1-2\alpha\cos\frac{2\pi p}{n}+\alpha^2$ has two roots complex roots $e^{\frac{2i\pi p}{n}}$ and $e^{\frac{-2i\pi p}{n}}$
As a result: $\prod_{p=1}^{n}\left(1-2\alpha\cos\frac{2\pi p}{n}+\alpha^2\right)=\prod_{p=1}^{n}\left(\alpha-e^{\frac{2i\pi p}{n}}\right)\left(\alpha-e^{\frac{-2i\pi p}{n}}\right)$

$$\Leftrightarrow\prod_{p=1}^{n}\left(1-2\alpha\cos\frac{2\pi p}{n}+\alpha^2\right)=\prod_{p=1}^{n}\left(\alpha-e^{\frac{2i\pi p}{n}}\right)\prod_{p=1}^{n}\left(\alpha-e^{\frac{-2i\pi p}{n}}\right)$$

$$\Leftrightarrow\prod_{p=1}^{n}\left(1-2\alpha\cos\frac{2\pi p}{n}+\alpha^2\right)=(\alpha^n-1)(\alpha^n-1)=(\alpha^n-1)^2$$

The result is:$I=\lim_{n\to+\infty}\frac{2\pi}{n}\log\left((\alpha^n-1)^2\right)=\lim_{n\to+\infty}\frac{4\pi}{n}\log|\alpha^n-1|$
*If $0<\alpha<1:I=0$, ($car:\alpha^n$ is a geometric sequence)
*If:$\alpha>1$,we have:

$$\log|\alpha^n-1|=\log(\alpha^n-1)=\log\alpha^n\left(1-\frac{1}{\alpha^n}\right)$$

$$\Leftrightarrow\log|\alpha^n-1|=\log\alpha^n+\log\left(1-\frac{1}{\alpha^n}\right)=n\log\alpha+\log\left(1-\frac{1}{\alpha^n}\right)$$

$$\Leftrightarrow I=\lim_{n\to+\infty}\frac{4\pi}{n}\left(n\log\alpha+\log\left(1-\frac{1}{\alpha^n}\right)\right)$$

$$I=4\pi\log\alpha$$

Chapter 3

Ordinary differential equations(O.D.E)

we study two types of Ordinary differential equations.An ordinary differential equation of$n$ order, an ordinary differential equation of 1^{st} order and 2^{nd} order. They are introduced and studied in this chapter. Further, we give the different methods and types to solve this type of an ordinary differential equation

Definition 3.0.1. *An ordinary differential equation of order "n" is an equation of the form*

$$F\left(x, y(x), y'(x), \ldots\ldots, y^{(n)}(x)\right) = 0 \ldots\ldots\ldots (E)$$

Where F is a function of $(n+2)$ variable

A solution of such an equation on an interval $I \subset \mathbb{R}$:is a function $y : I \to \mathbb{R}$ which is "n" times differentiable and verifies the equation (E)

Remark 3.0.1. *1/Find a primitive is solving the O.D.E: $y = f(x, y)$*

2/Integrating the O.D.E means finding the solution

Example 19. $y' = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow \int dy = \int \frac{1}{x} dx$

$y' = \frac{1}{x} \Rightarrow y = \ln|x| + c$

3.1 An ordinary differential equation of 1^{st} order

Ordinary differential equations of 1^{st} order

$$F\left(x, y(x), y'(x)\right) = 0$$

If equation (E) remains at $n = 1$ and can be written as form

$$y' = f(x, y)$$

$f$is a function defined by: $\begin{array}{l} I \subset \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x, y) \mapsto f(x, y) \end{array}$

and $y : x \mapsto y(x)$ is the unknown function(solution)

Problème de Cauchy

every O.D.E with an initial condition (x_0, y_0) is a Cauchy problem.

Example 20. $\begin{cases} y' = \frac{1}{x} \\ y(1) = 2 \rightarrow \begin{cases} x_0 = 1 \\ y_0 = 2 \end{cases} \end{cases}$

$y' = \frac{1}{x} \Rightarrow y = \ln|x| + c \rightarrow 2 = \ln|1| + c \rightarrow 2 = c$

$\Rightarrow y = \ln|x| + 2$ *eis the particular solution of the Cauchy problem*

3.1.1 Ordinary differential equations with separate variables

An ODE with separate variables is a differential equation with the formula

$$y' = \frac{f(x)}{g(y)} \Leftrightarrow y'g(y) = f(x)$$

if: F is a primitive of f on I

if: G is a primitive of g on I

So:

$$\begin{aligned} & y' = \tfrac{f(x)}{g(y)} \Leftrightarrow y'g(y) = f(x) \\ & \tfrac{dy}{dx} g(y) = f(x) \Leftrightarrow g(y)dy = f(x)dx \\ & \textstyle\int g(y)dy = \int f(x)dx \Leftrightarrow G(y) = F(x) + c \end{aligned}$$

Example 21. $x^2y' - e^{-y} = 0$

$x^2y' = e^{-y} \Leftrightarrow x^2\frac{dy}{dx} = e^{-y} \Leftrightarrow \frac{dy}{e^{-y}} = \frac{dx}{x^2} \Leftrightarrow e^y dy = \frac{dx}{x^2}$

$x^2y' = e^{-y} \Leftrightarrow \int e^y dy = \int \frac{dx}{x^2} \Leftrightarrow e^y = -\frac{1}{x} + c$

$x^2y' = e^{-y} \Leftrightarrow y = \ln\left|-\frac{1}{x} + c\right|$

3.1.2 Homogeneous ordinary differential equations

An homogeneous ODE is a differential equation of the type

$$y' = f\left(\frac{y}{x}\right)$$

To solve it, we pose: $U = \frac{y}{x}$

$U = \frac{y}{x} \Rightarrow y = U \cdot x \Rightarrow y' = U' \cdot x + U = f\left(\frac{y}{x}\right) = f(U)$

$\Rightarrow U' \cdot x + U = f(U) \Rightarrow U' \cdot x = f(U) - U \Rightarrow \frac{dU}{dx} \cdot x = f(U) - U$

$\Rightarrow\Rightarrow \int \frac{dU}{f(U)-U} = \int \frac{dx}{x} \; (f(U) - U \neq 0)$

Example 22. $y'(x + y) - y = 0$

$y'(x + y) - y = 0 \Rightarrow y'\left(1 + \frac{y}{x}\right) - \frac{y}{x} = 0$

We take: $U = \frac{y}{x} \Rightarrow y' = U' \cdot x + U$

$y'\left(1 + \frac{y}{x}\right) = \frac{y}{x} \Rightarrow (U' \cdot x + U)(1 + U) = U$

$\Rightarrow U' \cdot x + U + U' \cdot x \cdot U + U^2 = U$

$\Rightarrow U' \cdot x(1 + U) = -U^2 \Rightarrow \frac{dU}{dx} \cdot x(1 + U) = -U^2$

$\Rightarrow -\frac{(1+U)}{U^2}dU = \frac{dx}{x} \Rightarrow \int -\frac{(1+U)}{U^2}dU = \int \frac{dx}{x}$
$\Rightarrow \int -\frac{1}{U^2}dU - \int \frac{1}{U}dU = \int \frac{dx}{x} \Rightarrow -\frac{1}{U} - \ln|u| = \ln|x| + c$

3.1.3 Linear ordinary differential equations

A linear ODE is a differential equation of the type

$$a(x)\,y' + b(x)\,y = c(x)$$

$a(x), b(x)$ and $c(x)$ are continuous continuous functions and $a(x) \neq 0$
This equation can be written:

$$y' = \alpha(x)\,y + \beta(x)$$

$\alpha(x)$ and $\beta(x)$ are continuous functions

$$a(x)\,y' + b(x)\,y = 0$$

is a homogeneous linear equation with separate variables

3.1.4 Solving linear ordinary differential equations

1/Solving the homogeneous equation

$a(x)\,y' + b(x)\,y = 0$:is a homogeneous equation with separate variables

$$\begin{aligned}
&a(x)\,y' = -b(x)\,y \Rightarrow a(x)\,\tfrac{dy}{dx} = -b(x)\,y\\
&\Rightarrow \tfrac{dy}{y} = -\tfrac{b(x)}{a(x)}dx \Rightarrow \int \tfrac{dy}{y} = -\int \tfrac{b(x)}{a(x)}dx\\
&\Rightarrow \ln|y| = -\int \tfrac{b(x)}{a(x)}dx + c\\
&\Rightarrow y = \pm\exp\left(-\int \tfrac{b(x)}{a(x)}dx + c\right)\\
&y_0 = Ke^{F(x)} \quad / \quad F(x) = -\int \tfrac{b(x)}{a(x)}dx \;\; / \; k = \pm e^c
\end{aligned}$$

So: $y_0 = Ke^{F(x)}$: is solution of the homogeneous equation
2/The particular solution is assumed to be of the form

$$y_p = K(x)\,e^{F(x)}/K(x)\,\text{is a function of"}x\text{"}$$

Using the constant variation method, we assume that:

$$y_p = K(x)\,e^{F(x)} \Rightarrow y_p' = K'(x)\,e^{F(x)} + F'(x)K(x)\,e^{F(x)}$$

we transfer to the equation:

$$a(x)\,y' + b(x)\,y = c(x)$$

we find:

$$a\left(x\right)\left[K'\left(x\right)e^{F(x)}+F'(x)K\left(x\right)e^{F(x)}\right]+b\left(x\right)\left[K\left(x\right)e^{F(x)}\right]=c(x)$$
$$a\left(x\right)K'\left(x\right)e^{F(x)}=c(x)\Rightarrow K'\left(x\right)=\frac{c(x)}{a(x)}e^{-F(x)}$$
$$\Rightarrow\frac{dk}{dx}=\frac{c(x)}{a(x)}e^{-F(x)}\Rightarrow k(x)=\int\frac{c(x)}{a(x)}e^{-F(x)}dx$$
$$y_p=K\left(x\right)e^{F(x)}\Rightarrow y_p=\int\frac{c(x)}{a(x)}e^{-F(x)}dx\cdot e^{F(x)}$$

So:

$$y=y_0+y_p$$
$$y=Ke^{F(x)}\quad+\int\frac{c(x)}{a(x)}e^{-F(x)}dx\cdot e^{F(x)}$$

Example 23.

$\sin xy'-\cos xy=x.....\left(E\right)$
$*)\sin xy'-y\cos x=0.....\left(E_0\right)$
$\Rightarrow\sin x\frac{dy}{dx}=y\cos x\Rightarrow\frac{dy}{y}=\frac{\cos x}{\sin x}dx$
$\Rightarrow\int\frac{dy}{y}=\int\frac{\cos x}{\sin x}dx\Rightarrow\ln|y|=\ln|\sin x|+c$
$\Rightarrow y=\pm e^c\cdot\sin x\Rightarrow y_0=k\cdot\sin x\quad,k\in\mathbb{R}$
y_0 is the solution of $\left(E_0\right)$, to find $y_0$we assume:
$*)y_p=k\left(x\right)\cdot\sin x\Rightarrow y_p'=k'\left(x\right)\cdot\sin x+k\left(x\right)\cdot\cos x$
So:$\sin xy'-\cos xy=x\Rightarrow\sin x\left[k'\left(x\right)\cdot\sin x+k\left(x\right)\cdot\cos x\right]-\cos x\left[k\left(x\right)\cdot\sin x\right]=x$
$\Rightarrow\sin xk'\left(x\right)\cdot\sin x=x\Rightarrow k'\left(x\right)=\frac{x}{\sin^2 x}\Rightarrow\frac{dk}{dx}=\frac{x}{\sin^2 x}$
$\Rightarrow\int dk=\int\frac{x}{\sin^2 x}dx\begin{cases}u=x\rightarrow u'=dx\\ v'=\frac{1}{\sin^2 x}dx\rightarrow v=\frac{-1}{tgx}\end{cases}$
$\Rightarrow\int dk=\int\frac{x}{\sin^2 x}dx=\frac{-x}{tgx}+\int\frac{1}{tgx}dx$
$\Rightarrow k\left(x\right)=\int\frac{x}{\sin^2 x}dx=\frac{-x}{tgx}+\int\frac{\sin x}{\cos x}dx\Rightarrow k\left(x\right)=\frac{-x}{tgx}-\ln|\cos x|$
$y_p=k\left(x\right)\cdot\sin x\Rightarrow y_p=\left[\frac{-x}{tgx}-\ln|\cos x|\right]\cdot\sin x$
The solution to the linear equation $\left(E\right)$ is:$y=y_0+y_p\Rightarrow y=k\cdot\sin x+\left[\frac{-x}{tgx}-\ln|\cos x|\right]\cdot\sin x$

Example 24.

$y'+2xy=2xe^{-x^2}.........\left(E\right)$
$*)y'+2xy=0.......\left(E_0\right)$
$\Rightarrow\frac{dy}{dx}=-2xy\Rightarrow\frac{dy}{y}=-2xdx\Rightarrow\int\frac{dy}{y}=\int-2xdx$
$\ln|y|=-x^2+c\Rightarrow y=\pm e^c\cdot e^{-x^2}\Rightarrow y_0=k\cdot e^{-x^2}\quad,k\in\mathbb{R}$
y_0 is the solution of $\left(E_0\right)$, to find y_0 we assume:
$*)y_p=k\left(x\right)\cdot e^{-x^2}\Rightarrow y_p'=k'\left(x\right)\cdot e^{-x^2}-2xk\left(x\right)\cdot e^{-x^2}$
So: $y'+2xy=2xe^{-x^2}\Rightarrow k'\left(x\right)\cdot e^{-x^2}-2xk\left(x\right)\cdot e^{-x^2}+2xk\left(x\right)\cdot e^{-x^2}=2xe^{-x^2}$
$k'\left(x\right)\cdot e^{-x^2}=2xe^{-x^2}\Rightarrow\frac{dk}{dx}=2x\Rightarrow\int dk=\int 2xdx\Rightarrow k(x)=x^2$
$y_p=k\left(x\right)\cdot e^{-x^2}\Rightarrow y_p=x^2\cdot e^{-x^2}$
The solution to the linear equation $\left(E\right)$ is:$y=y_0+y_p\Rightarrow y=k\cdot e^{-x^2}+x^2\cdot e^{-x^2}$

3.1.5 Bernouli's ordinary differential equations

Bernouli's equation in the form

$$a(x)\,y' + b(x)\,y = c(x)\,y^k, \qquad k \in \mathbb{R}/\{0,1\}$$

To solve it, we return to an ordinary linear differential equation
*)Multiply the equation by y^{-k}

$$\begin{aligned}&a(x)\,y' + b(x)\,y = c(x)\,y^k \Rightarrow a(x)\,y' \cdot y^{-k} + b(x)\,y \cdot y^{-k} = c(x)\\&\Rightarrow a(x)\,y' \cdot y^{-k} + b(x)\,y^{1-k} = c(x)\end{aligned}$$

*)We take:$z = y^{1-k} \Rightarrow z' = (1-k)\,y^{-k}y'$

$$\begin{aligned}&a(x)\,y' \cdot y^{-k} + b(x)\,y^{1-k} = c(x) \Rightarrow \tfrac{a(x)}{1-k}z' + b(x)\,z = c(x)\\&z = z_0 + z_p \Rightarrow y = z^{\frac{1}{1-k}}\end{aligned}$$

Example 25. $y' - xy = y^{\frac{1}{2}}$
**) Multiply the equation by* $y^{-k} = y^{-\frac{1}{2}}$

$$\begin{aligned}&y' - xy = y^{\frac{1}{2}} \Rightarrow y^{-\frac{1}{2}} \cdot y' - xy \cdot y^{-\frac{1}{2}} = 1\\&\Rightarrow y^{-\frac{1}{2}} \cdot y' - x \cdot y^{\frac{1}{2}} = 1\end{aligned}$$

**)We take:* $z = y^{1-k} \Rightarrow z = y^{1-\frac{1}{2}} = y^{\frac{1}{2}} \Rightarrow z' = \frac{1}{2}y^{\frac{-1}{2}} \cdot y'$

$$y^{-\frac{1}{2}} \cdot y' - x \cdot y^{\frac{1}{2}} = 1 \Rightarrow 2z' - x \cdot z = 1 ......(E)$$

**) Solving the homogeneous equation*

$$\begin{aligned}&2z' - x \cdot z = 0.......(E_0)\\&\Rightarrow 2\tfrac{dz}{dx} = xz \Rightarrow 2\tfrac{dz}{z} = xdx\\&\Rightarrow \int 2\tfrac{dz}{z} = \int xdx \Rightarrow 2\ln|z| = \tfrac{1}{2}x^2 + c\\&\Rightarrow z = \pm e^c \cdot e^{\frac{1}{4}x^2} \Rightarrow z_0 = k \cdot e^{\frac{1}{4}x^2}, \quad k \in \mathbb{R}\end{aligned}$$

**)z_0 is the solution of (E_0) ,to find z_0 we assume:*

$$\begin{aligned}&z_p = k(x) \cdot e^{\frac{1}{4}x^2} \Rightarrow z_p' = k'(x) \cdot e^{\frac{1}{4}x^2} + \tfrac{1}{2}xk(x) \cdot e^{\frac{1}{4}x^2}\\&2z' - x \cdot z = 1 \Rightarrow 2\left(k'(x) \cdot e^{\frac{1}{4}x^2} + \tfrac{1}{2}xk(x) \cdot e^{\frac{1}{4}x^2}\right) - x\left(k(x) \cdot e^{\frac{1}{4}x^2}\right) = 1\\&2k'(x) \cdot e^{\frac{1}{4}x^2} = 1 \Rightarrow \tfrac{dk}{dx} = \tfrac{1}{2}e^{-\frac{1}{4}x^2} \Rightarrow k(x) = \tfrac{1}{2}\int e^{-\frac{1}{4}x^2}dx\\&z_p = k(x) \cdot e^{\frac{1}{4}x^2} \Rightarrow z_p = \tfrac{1}{2}e^{\frac{1}{4}x^2} \cdot \int e^{-\frac{1}{4}x^2}dx\\&z = z_0 + z_p \Rightarrow z = k \cdot e^{\frac{1}{4}x^2} + \tfrac{1}{2}e^{\frac{1}{4}x^2} \cdot \int e^{-\frac{1}{4}x^2}dx\\&z = y^{\frac{1}{2}} \Rightarrow y = z^2 \Rightarrow y = \left[k \cdot e^{\frac{1}{4}x^2} + \tfrac{1}{2}e^{\frac{1}{4}x^2} \cdot \int e^{-\frac{1}{4}x^2}dx\right]^2\end{aligned}$$

3.2 Ordinary differential equations of 2^{nd} order

Definition 3.2.1. *A second-order linear differential equation is an equation of the form*

$$a(x)y" + b(x)y' + c(x)y = f(x)$$

where :$a(x), b(x), c(x)$ et $f(x)$ are continuous functions
The homogeneous equation (the equation without a second member)

$$a(x)y" + b(x)y' + c(x)y = 0$$

The2^{nd} order linear differential equation with constant coefficients is an equation of the form

$$ay" + by' + cy = f(x)\ldots..(E)$$

where : a, b, c are constants and $f(x)$ are continuous functions
The homogeneous equation (the equation without a second member)is

$$ay" + by' + cy = 0\ldots..(E_0)$$

To solve this equation, we find if there are solutions of the type

$$\begin{gathered} y = e^r, \quad r \in \mathbb{R} \\ y' = re^r \Rightarrow y" = r^2 e^r \end{gathered}$$

We take it in the equation

$$ay" + by' + cy = 0\ldots..(E_0)$$

we find:

$$\begin{aligned} & ay" + by' + cy = 0 \Rightarrow ar^2 e^r + bre^r + ce^r = 0 \\ & e^r\left(ar^2 + br + c\right) = 0 \Rightarrow ar^2 + br + c = 0 \end{aligned}$$

is a characteristic equation of the homogeneous equation without second member
$*)$ 1st case: $\triangle = 0$, we will have a double root: $r = r_1 = r_2 = \frac{-b}{2a}$
The functions $y_1 = e^{rx}$ et $y_2 = re^{rx}$ are two solutions of the homogeneous equation without a second member
So the maximum general solution of the homogeneous equation without a second member is given by:

$$\begin{aligned} & y = \lambda_1 e^{rx} + \lambda_2 x e^{rx} \\ & y = e^{rx}\left(\lambda_1 + \lambda_2 x\right), \lambda_1, \lambda_2 \in \mathbb{R} \end{aligned}$$

$*)$ 2nd case: $\triangle > 0$, the characteristic equation admits two real rootsr_1 et r_2,So The functions $y_1 = e^{r_1 x}$ et $y_2 = e^{r_2 x}$ are two particularly solution of the homogeneous equation without a second member
So the maximum general solution of the homogeneous equation without second member is given by:

$$y = \lambda_1 e^{r_1 x} + \lambda_2 e^{r_2 x}, \lambda_1, \lambda_2 \in \mathbb{R}$$

$*)$ 3rd case: $\triangle < 0$, the characteristic equation has two complex roots

$$\begin{cases} r_1 = \alpha + i\beta \\ r_2 = \alpha - i\beta \end{cases}$$

Then the functions $y_1 = \cos(\beta x) e^{\alpha x}$ *and* $y_2 = \sin(\beta x) e^{\alpha x}$ *are two particular solutions of the homogeneous equation without a second member*

So the maximum general solution of the homogeneous equation without a second member is given by:

$$y = \lambda_1 \cos(\beta x) e^{\alpha x} + \lambda_2 \sin(\beta x) e^{\alpha x}, \lambda_1, \lambda_2 \in \mathbb{R}$$
$$y = e^{\alpha x} (\lambda_1 \cos(\beta x) + \lambda_2 \sin(\beta x))$$

Example 26. $:y" - 5y' + 6y = 0 \Rightarrow r^2 - 5r + 6 = 0$

$\triangle = 25 - 24 = 1 \begin{cases} r_1 = \frac{5-1}{2} = 2 \to y_1 = e^{r_1 x} = e^{2x} \\ r_2 = \frac{5+1}{2} = 3 \to y_2 = e^{r_2 x} = e^{3x} \end{cases} y = \lambda_1 e^{2x} + \lambda_2 e^{3x}, \lambda_1, \lambda_2 \in \mathbb{R}$

Example 27. $y" + 4y' + 4y = 0 \Rightarrow r^2 + 4r + 4 = 0$

$\triangle = 16 - 16 = 0 \Rightarrow r = r_1 = r_2 = \frac{-4}{2} = -2 \Rightarrow y = e^{-2x} (\lambda_1 + \lambda_2 x), \lambda_1, \lambda_2 \in \mathbb{R}$

Example 28. $:y" + y' + y = 0 \Rightarrow r^2 + r + 1 = 0$

$\triangle = 1 - 4 = -3 = 3i^2 \begin{cases} r_1 = \frac{-1-i\sqrt{3}}{2} = 2 \to y_1 = \cos(\beta x) e^{\alpha x} = \cos\left(\frac{\sqrt{3}}{2}x\right) e^{\frac{-1}{2}x} \\ r_2 = \frac{-1+i\sqrt{3}}{2} = 3 \to y_2 = \sin(\beta x) e^{\alpha x} = \sin\left(\frac{\sqrt{3}}{2}x\right) e^{\frac{-1}{2}x} \end{cases} y = e^{\frac{-1}{2}x} \left(\lambda_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + \lambda_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right),$

Solving the non-homogeneous equation

$$ay" + by' + cy = f(x)..... (E)$$

Theorem 3.2.1. *The general solution* y *of the non-homogeneous equation is the sum of the maximum solution of the homogeneous equation without second member* y_0 *and the particular solution* y_1 *such that:*

$$y = y_0 + y_1$$

Particular solution

if: $f(x) = Ae^{\alpha x}/A$: constant

1)α is not the root of the characteristic equation$(ar^2 + br + c = 0)$,then

$$y_1 = \frac{Ae^{\alpha x}}{\alpha^2 + b\alpha + c}$$

2)α is the root of the characteristic equation$(ar^2 + br + c = 0)$,then

$$y_1 = \frac{Axe^{\alpha x}}{2\alpha + b}$$

Example 29. $y" - 3y' + 2y = e^x$

**The homogeneous equation::*$y" - 3y' + 2y = 0 \Rightarrow r^2 - 3r + 2 = 0$

$\triangle = 9 - 8 = 1 \begin{cases} r_1 = \frac{3-1}{2} = 1 \to y_1 = e^{r_1 x} = e^x \\ r_2 = \frac{3+1}{2} = 2 \to y_2 = e^{r_2 x} = e^{2x} \end{cases} y_0 = \lambda_1 e^x + \lambda_2 e^{2x}, \lambda_1, \lambda_2 \in \mathbb{R}$

$*\begin{cases} A=1 \\ \alpha=1 \end{cases}$ *$\alpha=1$ is a simple root, so*

$$y_1=\frac{xe^x}{2\,(1)-3}=-xe^x$$

The general solution y is:

$$\begin{array}{l} y=y_0+y_1 \\ y=\lambda_1 e^x+\lambda_2 e^{2x}-xe^x,\lambda_1,\lambda_2\in\mathbb{R} \end{array}$$

if: $f(x)=A\cos(\omega x)+B\sin(\omega x)\,/A,B$:constants
1/f is not a root of the characteristic equation, then:

$$y_1=k_1\cos(\omega x)+k_2\sin(\omega x)$$

2/f is a root of the characteristic equation, then:

$$y_1=x\,(k_1\cos(\omega x)+k_2\sin(\omega x))$$

Example 30. :$y"+y=\cos x$
**The homogeneous equation:*$y"+y=0\Rightarrow r^2+1=0$
$r^2=-1=i^2\begin{cases} r_1=i=1\rightarrow y_1=\cos(\beta x)\,e^{\alpha x}=\cos(x) \\ r_2=-i=2\rightarrow y_2=\cos(\beta x)\,e^{\alpha x}=\sin(x) \end{cases}\quad y_0=\lambda_1\cos x+\lambda_2\sin x$
**/$f(x)=\cos x$ is a root of the characteristic equation then:*

$$y_1=x\,(k_1\cos x+k_2\sin x)$$

We have:

$$\begin{array}{l} y_1=x\,(k_1\cos x+k_2\sin x) \\ y_1'=k_1\cos x+k_2\sin x+x\,(-k_1\sin x+k_2\cos x) \\ y_1"=-k_1\sin x+k_2\cos x-k_1\sin x+k_2\cos x-x\,(k_1\cos x+k_2\sin x) \end{array}$$

So:

$$\begin{array}{l} y"+y=\cos x\Rightarrow \\ -k_1\sin x+k_2\cos x-k_1\sin x+k_2\cos x-x\,(k_1\cos x+k_2\sin x) \\ +x\,(k_1\cos x+k_2\sin x)=\cos x \\ \Rightarrow -2k_1\sin x+2k_2\cos x=\cos x\begin{cases} -2k_1=0\rightarrow k_1=0 \\ 2k_2=1\rightarrow k_2=\frac{1}{2} \end{cases} \end{array}$$

Then:

$$y_1=\frac{1}{2}x\sin x$$

The general solution to the equation $y"+y=\cos x$ is given by:

$$\begin{array}{l} y=y_0+y_1 \\ y=\lambda_1\cos x+\lambda_2\sin x+\frac{1}{2}x\sin x,\lambda_1,\lambda_2\in\mathbb{R} \end{array}$$

2^{nd} method for finding the solution particular

if y_p is a particular solution, it must verify the following system

$$y_p = \lambda_1 y_1 + \lambda_2 y_2 \left\{ \begin{array}{l} \lambda_1' y_1 + \lambda_2' y_2 = 0 \\ \lambda_1' y_1' + \lambda_2' y_2' = f(x) \end{array} \right.$$

Example 31. $y" - 3y' + 2y = e^{5x}$

**The homogeneous equation*: $y" - 3y' + 2y = 0 \Rightarrow r^2 - 3r + 2 = 0$

$\triangle = 9 - 8 = 1 \left\{ \begin{array}{l} r_1 = \frac{3-1}{2} = 1 \rightarrow y_1 = e^{r_1 x} = e^x \\ r_2 = \frac{3+1}{2} = 2 \rightarrow y_2 = e^{r_2 x} = e^{2x} \end{array} \right. \; y_0 = \lambda_1 e^x + \lambda_2 e^{2x}, \lambda_1, \lambda_2 \in \mathbb{R}$

**We find the particular solution by the method of variation of constants*

$$\begin{array}{l}
y_p = \lambda_1 e^x + \lambda_2 e^{2x} \left\{ \begin{array}{l} \lambda_1' y_1 + \lambda_2' y_2 = 0 \\ \lambda_1' y_1' + \lambda_2' y_2' = f(x) \end{array} \right. \\
y_p = \lambda_1 e^x + \lambda_2 e^{2x} \left\{ \begin{array}{l} \lambda_1' e^x + \lambda_2' e^{2x} = 0 \ldots\ldots (1) \\ \lambda_1' e^x + 2\lambda_2' e^{2x} = e^{5x} \ldots.. (2) \end{array} \right. \\
(1) - (2) \Leftrightarrow -\lambda_2' e^{2x} = -e^{5x} \Rightarrow \lambda_2' = e^{3x} \Rightarrow \lambda_2 = \frac{1}{3} e^{3x} + c_2 \\
\textit{replace } \lambda_2 \textit{ in } (1), \textit{we find :} \\
\lambda_1' e^x + e^{3x} e^{2x} = 0 \Rightarrow \lambda_1' = -e^{4x} \Rightarrow \lambda_1 = -\frac{1}{4} e^{4x} + c_1
\end{array}$$

So:

$$\begin{array}{l}
y = \lambda_1 e^x + \lambda_2 e^{2x} \\
y = \left(-\frac{1}{4} e^{4x} + c_1\right) e^x + \left(\frac{1}{3} e^{3x} + c_2\right) e^{2x} \\
y = -\frac{1}{4} e^{5x} + c_1 e^x + \frac{1}{3} e^{5x} + c_2 e^{2x} \\
y = \frac{1}{12} e^{5x} + c_1 e^x + c_2 e^{2x}, c_1, c_2 \in \mathbb{R}
\end{array}$$

3.3 Solved exercises

Exercise 1.

Solve the following 1^{st}order differential equations following
1) $y' - \frac{1}{x}y = \ln x$
2)$y' + y\sin x = \sin x$

Solution:

1) $y' - \frac{1}{x}y = \ln x$
*First we solve the homogeneous equation
$y' - \frac{1}{x}y = 0 \Leftrightarrow y' = \frac{1}{x}y$
$\Leftrightarrow \frac{dy}{dx} = \frac{1}{x}y \Leftrightarrow \frac{dy}{y} = \frac{1}{x}dx$
$\Leftrightarrow \int \frac{dy}{y} = \int \frac{1}{x}dx \Leftrightarrow \ln|y| = \ln|x| + c$
$\Leftrightarrow y = \pm e^c \cdot x \Leftrightarrow y = k \cdot x$ with $k = \pm e^c$
*Constant variation method
The constant k is varied as a function of x hence
$y = k(x) \cdot x \Rightarrow y' = k' \cdot x + k$
then replace in $y' - \frac{1}{x}y = \ln x$, we find
$y' - \frac{1}{x}y = \ln x \Leftrightarrow k' \cdot x + k - \frac{1}{x}(k \cdot x) = \ln x$
$k' \cdot x = \ln x \Leftrightarrow \frac{dk}{dx}x = \ln x \Leftrightarrow \int dk = \int \frac{\ln x}{x}dx$
$k(x) = \frac{1}{2}(\ln x)^2 + c$
Therefore $y_g = x\left(\frac{1}{2}(\ln x)^2 + c\right) / c \in \mathbb{R}$
2)$y' + y\sin x = \sin x$
*First we solve the homogeneous equation
$y' + y\sin x = 0 \Leftrightarrow y' = -y\sin x$
$\Leftrightarrow \frac{dy}{dx} = y\sin x \Leftrightarrow \frac{dy}{y} = -\sin x dx$
$\Leftrightarrow \int \frac{dy}{y} = \int -\sin x dx \Leftrightarrow \ln|y| = \cos x + c$
$\Leftrightarrow y = \pm e^c \cdot e^{\cos x} \Leftrightarrow y = k \cdot e^{\cos x}$ with $k = \pm e^c$
*Constant variation method
The constant k is varied as a function of x hence
$y = k(x) \cdot e^{\cos x} \Rightarrow y' = k' \cdot e^{\cos x} - k\sin x \cdot e^{\cos x}$
then replace in $y' + y\sin x = \sin x$, we find
$y' + y\sin x = \sin x \Leftrightarrow k' \cdot e^{\cos x} - k\sin x \cdot e^{\cos x} + k \cdot e^{\cos x}(\sin x) = \sin x$
$k' \cdot e^{\cos x} = \sin x \Leftrightarrow \frac{dk}{dx}e^{\cos x} = \sin x \Leftrightarrow \int dk = \int \sin x \cdot e^{-\cos x}dx$
$k(x) = e^{-\cos x} + c$
Therefore $y_g = e^{\cos x}(e^{-\cos x} + c) = 1 + c \cdot e^{\cos x} / c \in \mathbb{R}$

Exercise 2:

Solve the following 1^{st} order differential equations following 1)$y' + xy = xy^2$
2)$y' + \frac{y}{x} - y^2 = \frac{-1}{x^2}$ such that $s(x) = \frac{1}{x}$ is a particular solution

Solution:

1)$y' + xy = xy^2$ this is a Bernoulli equation with $\alpha = 2$, then divide the equation on y^2,we find:

$\frac{y'}{y^2} + \frac{x}{y} = x$

Then we make the following change of variable

$z = y^{-1} = \frac{1}{y} \Rightarrow z' = \frac{-y'}{y^2}$

So we get: $\frac{y'}{y^2} + \frac{x}{y} = x \Rightarrow -z' + xz = x$

which is a linear equation

*First we solve the homogeneous equation

$-z' + xz = 0 \Leftrightarrow z' = xz$

$\Leftrightarrow \frac{dz}{dx} = xz \Leftrightarrow \frac{dz}{z} = xdx$

$\Leftrightarrow \int \frac{dz}{z} = \int xdx \Leftrightarrow \ln|z| = \frac{1}{2}x^2 + c$

$\Leftrightarrow z = \pm e^c \cdot e^{\frac{1}{2}x^2} \Leftrightarrow y = k \cdot e^{\frac{1}{2}x^2}$ with $k = \pm e^c$

*Constant variation method

The constant k is varied as a function of x hence

$z = k(x) \cdot e^{\frac{1}{2}x^2} \Rightarrow z' = k' \cdot e^{\frac{1}{2}x^2} + k \cdot x \cdot e^{\frac{1}{2}x^2}$

then replace in $-z' + xz = x$, we find

$-z' + xz = x \Leftrightarrow -k' \cdot e^{\frac{1}{2}x^2} - k \cdot x \cdot e^{\frac{1}{2}x^2} + x \cdot k \cdot e^{\frac{1}{2}x^2} = x$

$k' \cdot e^{\frac{1}{2}x^2} = -x \Leftrightarrow \frac{dk}{dx} e^{\frac{1}{2}x^2} = -x \Leftrightarrow \int dk = \int -x \cdot e^{-\frac{1}{2}x^2} dx$

$k(x) = e^{-\frac{1}{2}x^2} + c$

Therefore $z_g = e^{\frac{1}{2}x^2} \left(e^{-\frac{1}{2}x^2} + c \right) = 1 + c \cdot e^{\frac{1}{2}x^2} / c \in \mathbb{R}$

So: $y_g = \frac{1}{1 + c \cdot e^{\frac{1}{2}x^2}} / c \in \mathbb{R}$

2)$y' + \frac{y}{x} - y^2 = \frac{-1}{x^2}$ such that $s(x) = \frac{1}{x}$ is a particular solution

This is a Riccati equation, as $s(x) = \frac{1}{x}$ is a particular solution of the equation, then we make the change of variable $y = z + \frac{1}{x}$,d'où $y' = z' - \frac{1}{x^2}$, hence

$y' + \frac{y}{x} - y^2 = \frac{-1}{x^2} \Leftrightarrow z' - \frac{z}{x} - z^2 = 0$

which is a Bernoulli equation with $\alpha = 2$, then divide the equation on z^2, we find:

$\frac{z'}{z^2} - \frac{1}{xz} = 1$

Then we make the following change of variable

$t = z^{-1} = \frac{1}{z} \Rightarrow t' = \frac{-z'}{z^2}$

So we get: $\frac{z'}{z^2} - \frac{1}{xz} = 1 \Rightarrow -t' - \frac{t}{x} = 1$

which is a linear equation

*First we solve the homogeneous equation

$-t' - \frac{t}{x} = 0 \Leftrightarrow t' = -\frac{t}{x}$

$\Leftrightarrow \frac{dt}{dx} = -\frac{t}{x} \Leftrightarrow \frac{dt}{t} = -\frac{1}{x} dx$

$\Leftrightarrow \int \frac{dt}{t} = \int \frac{1}{x} dx \Leftrightarrow \ln|t| = -\ln|x| + c$

$\Leftrightarrow t = \pm e^c \cdot \frac{1}{x} \Leftrightarrow t = \frac{k}{x}$ with $k = \pm e^c$

*Constant variation method

The constant k is varied as a function of x hence

$t = \frac{k(x)}{x} \Rightarrow t' = \frac{k' \cdot x - k}{x^2}$

then replace in$-t' - \frac{t}{x} = 1$, we find

$-t' - \frac{t}{x} = 1 \Leftrightarrow -\frac{k' \cdot x - k}{x^2} - \frac{k}{x^2} = 1$
$-\frac{k'}{x} = 1 \Leftrightarrow \frac{dk}{dx} = x \Leftrightarrow \int dk = -\int x dx$
$k(x) = -\frac{1}{2}x^2 + c$
Therefore $t_g = -\frac{1}{2}x + \frac{c}{x} / c \in \mathbb{R}$
So: $z_g = \frac{2x}{2c - x^2} / c \in \mathbb{R}$
$y_g = z_g + \frac{1}{x} \Leftrightarrow y_g = \frac{2x}{2c - x^2} + \frac{1}{x} / c \in \mathbb{R}$

Exercise 3:

Solve the following differential equations
1)$y" - 2y' + y = x$ such that: $y(0) = y'(0) = 0$
2)$y" + 4y = tgx$

Solution:

Solve the following differential equations
1)$y" - 2y' + y = x$ such that: $y(0) = y'(0) = 0$
We start by solving the homogeneous equation $y" - 2y' + y = 0$. Its characteristic equation is $r^2 - 2r + 1 = 0$, of which "1" is a double root. The general solutions of the homogeneous equation are therefore the functions

$$\lambda e^x + \mu x e^x$$

We will search for a particular solution in the form of a polynomial of degree"1".
But $y(x) = ax + b$ is a solution to the differential equation if and only if only if
$-2a + ax + b = x \Leftrightarrow (a - 1)x + (b - 2a) = 0$:
Since a real polynomial is identically zero if and only if all its coefficients are zero, we deduce that $a = 1$ and $b = 2$. The solutions of the quation are functions of the form

$$\lambda e^x + \mu x e^x + (x + 2)$$

If we add the conditions $y(0) = y'(0) = 0$,we obtain the equations
$\lambda + 2 = 0$ and $\lambda + \mu + 1 = 0$;
Let $\lambda = -2$ and $\mu = 1$.The only solution to the equation is therefore the function

$$(x - 2)e^x + (x + 2)$$

2) $y" + 4y = tgx$
First, we solve the homogeneous equation without a second member $y" + 4y = 0$. We can introduce the characteristic equation $r^2 + 4 = 0$, or directly note that the functions
$y_1 = sin(2x)$ and $y_2 = cos(2x)$
are solutions. To determine a solution of the equation with second member, we apply the method of variation of the constant by searching for a solution written as :

$$y(x) = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x)$$

the derived functions λ_1' and λ_2' must verify the system

$$\begin{cases} \lambda_1'(x)y_1(x) + \lambda_2'(x)y_2(x) = 0 \\ \lambda_1'(x)y_1'(x) + \lambda_2'(x)y_2'(x) = tgx \end{cases}$$

Replace y_1 and y_2 by their respective values, and find the find the system

$$\begin{cases} \lambda_1'(x)sin(2x) + \lambda_2'(x)\cos(2x) = 0 \\ \lambda_1'(x)\cos(2x) - \lambda_2'(x)sin(2x) = \frac{1}{2}tgx \end{cases}$$

It comes

$\lambda_1'(x) = \frac{1}{2}\cos(2x)\frac{\sin x}{\cos x} = \frac{1}{2}\sin(2x)\frac{-\sin x}{\cos x}$

using $cos(2x) = 2cos^2(x) - 1$. The last part of the right member is of the form $\frac{u}{u'}$. We can integrate and find, $\lambda_1(x) = \frac{-1}{4}\cos(2x) + \frac{1}{2}\ln(\cos x)$

In the same way, we find

$\lambda_2'(x) = -sin^2(x) = \frac{-1}{2}(1 - cos(2x))$

Let

$\lambda_2(x) = -sin^2(x) = \frac{-1}{2}(x - \frac{1}{2}\sin(2x))$

Exercise 4:

Integrate the following differential equations

1)$y" + 2y' + 5y = e^{-x}(2\cos 2x - 3\sin 2x)$

2)$y" + 4y' + 4y = \frac{e^{-2x}}{1+x^2}$

Solution:

1)$y" + 2y' + 5y = e^{-x}(2\cos 2x - 3\sin 2x)$ .... (E)

-The homogeneous equation $y" + 2y' + 5y = 0$. Its equation characteristic equation $r^2 + 2r + 5 = 0$, whose roots are $-1 - 2i$ and $-1 + 2i$. The solutions of the homogeneous equation are functions of the form

$y = e^{-x}(c_1cos(2x) + c_2sin(2x))/\ c_1, c_2 \in \mathbb{R}$

We have

$f(x) = e^{-x}(2cos(2x) - 3\sin(2x))$

We take

$y = e^{mx}z = e^{-x}z$,

$y' = -e^{-x}z + z'e^{-x} = (z' - z)e^{-x}$,

$y" = (z" - z')e^{-x} - (z' - z)e^{-x} = (z" - 2z' + z)e^{-x}$

We replace $y; y'; y"$ in (E):

$y" + 2y' + 5y = (z" + 4z)\,e^{-x} = e^{-x}(2\cos 2x - 3\sin 2x)$

$$\begin{cases} z" + 4z = 2\cos 2x - 3\sin 2x ....(E_0) \\ z" + 4z = 0 ....(E_0') \end{cases}$$

$(E_0') \Leftrightarrow r^2 + 4 = 0 \Leftrightarrow r_1 = 2i$ and $r_2 = -2i$

We have:$\pm ip = \pm 2i$ are solutions of (E_0'), then

$z_p = x(Acos2x + Bsin2x)\ /A, B \in \mathbb{R}$

$z_p' = x(-2Asin2x + 2Bcos2x) + Acos2x + Bsin2x$

$z"_p = x(-4Acos2x - 4Bsin2x) + (-4Asin2x + 4Bcos2x)$

we replace z_p, $z_p', z"_p$ in (E'):

$-4Asin2x + 4Bcos2x = 2\cos 2x - 3\sin 2x$

$\begin{cases} B = \frac{1}{2} \\ A = \frac{3}{4} \end{cases}$

Hence

$z_p = x(\frac{3}{4}cos2x + \frac{1}{2}sin2x)$

So

$yp = e^{-x}z_p = x(\frac{3}{4}cos2x + \frac{1}{2}sin2x)e^{-x}$

Then $y = y_0 + y_p$ is the solution to the given equation

2) $y" + 4y' + 4y = \frac{e^{-2x}}{1+x^2}$

-The homogeneous equation $y" + 4y' + 4y = 0$.Its associated characteristic equation

$r^2 + 4r + 4 = 0$,of which -2 is a double root. The real solutions of the homogeneous equation are the functions of the form

$y = c_1e^{-2x} + c_2xe^{-2x}/c_1, c_2 \in \mathbb{R}$

We apply the constant variation method

$\begin{cases} c_1'e^{-2x} + c_2'xe^{-2x} = 0 .... (1) \\ -2c_1'e^{-2x} + c_2'\left(-2xe^{-2x} + e^{-2x}\right) = \frac{e^{-2x}}{1+x^2} .... (1) \end{cases}$

From $2 \times (1) + (2)$:

$c_2'e^{-2x} = \frac{e^{-2x}}{1+x^2}$

$c_2' = \frac{1}{1+x^2} \Rightarrow c_2(x) = arctgx + k_2/k_2 \in \mathbb{R}$

From(1) :

$c_1' + c_2'x = 0 \Rightarrow c_1' = -\frac{x}{1+x^2}$

$c_1(x) = \frac{-1}{2}\ln\left(1 + x^2\right) + k_1/k_1 \in \mathbb{R}$

Hence

$y = c_1e^{-2x} + c_2xe^{-2x}$

$y = \left(\frac{-1}{2}\ln\left(1 + x^2\right)\right)e^{-2x} + \left(arctgx + k_2\right)xe^{-2x}/k_1, k_2 \in \mathbb{R}$

$y = \left(k_2xe^{-2x} + k_1e^{-2x}\right) + \left(\frac{-1}{2}\ln\left(1 + x^2\right) + xarctgx\right)e^{-2x}/k_1, k_2 \in \mathbb{R}$

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