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FACULTY of SCIENCES

DEPARTMENT OF MATHEMATICS

Variational Inequalities Problem

Course support

Mathematics - Master 2

Option: Optimization and Control

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2025/206

Contents

Introduction

This course support is dedicated to the Variational Inequality Problem (VIP), designed for Master's 2 students in Optimization and Optimal Control option.

Variational inequalities provide a powerful, unified mathematical framework that generalizes and encompasses a wide range of fundamental problems in applied mathematics, economics, and engineering. This includes optimization problems (both constrained and unconstrained), complementarity problems, fixed-point problems, and Nash equilibrium problems in game theory. Therefore, knowing how to formulate and solve (VIP) is a crucial skill for handling complex real-world systems that involve optimality and equilibrium.

This document is structured to guide the student from the foundational principles to the forefront of numerical techniques.

Chapter One establishes the core concepts, introducing the formal definition of the variational inequality problem, its geometric interpretation, and its profound connections with optimization and other mathematical problems.

Chapter Two delves into the theory behind (VIP). We will classify different types of (VIP) based on the properties of the underlying mapping (e.g., monotonicity, strong monotonicity) and rigorously examine the crucial questions of existence and uniqueness of solutions, which form the base of any numerical approach.

Chapter Three marks the transition into computational methods. We will explore the family of projection methods, the foundational algorithms for solving (VIP).

The student will learn about their derivation and their convergence properties.

Chapter Four builds upon this foundation by focusing on the advanced and highly efficient class of projection and contraction methods. These algorithms, which enhance basic projection steps with specially designed contraction steps, offer robust convergence without restrictive assumptions.

By the end of this course, the student will has gained a comprehensive understanding of both the theory of variational inequalities and the numerical tools essential for solving them, providing a strong foundation for his research and professional work in optimization and optimal control.

Chapter 1

Variational Inequalities Problem

Let's take the space $\mathbb{R}^n$ where n is a finite non negative integer (the dimension of the space is finite). The norm and inner scalar considered here are as follows

- $\forall x, y \in \mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$.
- $\forall x \in \mathbb{R}^n$, $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$.

This chapter focuses on the variational inequalities problem. The first section introduces its definition and geometrical interpretation. As (VIP) generalizes several fundamental mathematical problems, the second section explores the connections between (VIP) and these related problems.

Definition 1.0.1 *Let C be a non-empty closed convex of $\mathbb{R}^n$ and F a mapping of $\mathbb{R}^n$ in it self $(F : \mathbb{R}^n \longrightarrow \mathbb{R}^n)$. Variational Inequalities Problem denoted $VIP(F, C)$ or simply (VIP) is to*

$$VIP(F,C) \quad \begin{cases} \text{Find } \overline{x} \in C \text{ such that} \\ \langle F(\overline{x}), x - \overline{x}\rangle \geq 0, \ \forall x \in C \end{cases} \tag{1.1}$$

1.1 Overview history

Historically, the variational inequalities problem (VIP) originated in the 1960s. Specifically, Hartman and Stampacchia (1966) first introduced (VIP) in the context of partial differential equations $(PDEs)$ for infinite-dimensional mechanics applications [16, 26]. Later, Dafermos developed the (VIP) theory in finite dimensions when he noticed that the optimality conditions of a transport problem naturally aligned with (VIP)'s structure. During the 1970s and 1980s, significant progress was made in establishing (VIP)'s fundamental theory, including the existence, uniqueness, and stability of solutions. Today, research remains active, with ongoing efforts to develop efficient numerical resolution methods for this problem. For more details, see [5, 6, 7, 11, 21].

We denote by S the set of solutions associated with (VIP) supposed non-empty.

1.1.1 Geometric interpretation

1. $\overline{x} \in S \Longleftrightarrow F(\overline{x})$ and any vector of the form $x - \overline{x}$ $(\forall x \in C)$ must form an angle $\leq 90^0$.

2. $$\overline{x} \in S \Longleftrightarrow -F(\overline{x}) \in \mathcal{N}_C(\overline{x}) \tag{1.2}$$

Recall the definition of $\mathcal{N}_C(\overline{x})$ the normal cone of C at the point x

$$\mathcal{N}_C(\overline{x}) = \{y \in \mathbb{R}^n / \langle y, x - \overline{x}\rangle \leq 0,\ \forall x \in C\}.$$

Indeed,

$$\begin{aligned} -F(\overline{x}) \in \mathcal{N}_C(\overline{x}) &\Longleftrightarrow \langle -F(\overline{x}), x - \overline{x} \rangle \leq 0, \ \forall x \in C \\ &\Longleftrightarrow \langle F(\overline{x}), x - \overline{x} \rangle \geq 0, \ \forall x \in C \\ &\Longleftrightarrow \overline{x} \in S. \end{aligned}$$

Or, we can also write

$$\overline{x} \in S \Longleftrightarrow 0 \in F(\overline{x}) + \mathcal{N}_C(\overline{x}) \tag{1.3}$$

Equation (1.3) is known as the generalized equation.

We have, $F(\overline{x}) + \mathcal{N}_C(\overline{x}) = \{F(\overline{x}) + y / y \in \mathcal{N}_C(\overline{x})\}$, then

$0 \in F(\overline{x}) + \mathcal{N}_C(\overline{x}) \Longleftrightarrow \exists y \in \mathcal{N}_C(\overline{x})$ such that $0 = F(\overline{x}) + y$.

Thus, $y = -F(\overline{x})$ and since $y \in \mathcal{N}_C(\overline{x})$, then $-F(\overline{x}) \in \mathcal{N}_C(\overline{x})$.

According to (1.2) ,we conclude that $\overline{x} \in S$.

1.2 Motivations

Our study of the variational inequalities problem (VIP) is motivated by two principal reasons:

Broad Applicability: (VIP) arises naturally in diverse domains, serving as a foundational tool in economic equilibrium theory, traffic network analysis, and mechanical contact problems, among others. Its ability to model complex phenomena explains its widespread using across applied mathematics.

Unifying Framework: More fundamentally, (VIP) provides a unifying framework that generalizes and connects fundamental mathematical problems, including:

Nonlinear equation systems

Complementarity problems

First-order optimality conditions in convex optimization

Nash equilibrium formulations in game theory

Variational formulations of optimal control problems

This generality makes (VIP) a central tool for both theoretical analysis and algorithmic development.

In what follows, we will establish the relationship between (VIP) and these problems

1.2.1 Systems of equations

Proposition 1.2.1 *Let $C = \mathbb{R}^n$, $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\overline{x}$ a solution of $VIP\left(F, \mathbb{R}^n\right)$. Then,*

$$\overline{x} \in S \iff F\left(\overline{x}\right) = 0.$$

Proof. $\Longleftarrow)$

$$\begin{aligned} F\left(\overline{x}\right) = 0 \quad & \Longrightarrow \left\langle F\left(\overline{x}\right), x - \overline{x} \right\rangle = 0, \ \forall x \in \mathbb{R}^n \\ & \text{(Variationel equality)} \\ & \Longrightarrow \overline{x} \in S. \end{aligned}$$

$\Longrightarrow)$

$\overline{x} \in S \Longrightarrow \left\langle F\left(\overline{x}\right), x - \overline{x} \right\rangle \geq 0, \forall x \in \mathbb{R}^n.$

We take $x = \overline{x} - F(\overline{x}) \in \mathbb{R}^n$, then we get

$$\begin{aligned} \langle F(\overline{x}), \overline{x} - F(\overline{x}) - \overline{x} \rangle \geq 0 & \Longrightarrow \langle F(\overline{x}), -F(\overline{x}) \rangle \geq 0 \\ & \Longrightarrow -\|F(\overline{x})\|^2 \geq 0 \\ & \Longrightarrow \|F(\overline{x})\|^2 \leq 0 \\ & \Longrightarrow \|F(\overline{x})\| = 0 \\ & \Longrightarrow F(\overline{x}) = 0. \end{aligned}$$

Therefore, $\overline{x}$is a solution of the system $F(x) = 0$.

Second method

We know that if C is an open set, then its normal cone coincides with the singleton $\{0\}$,($\mathcal{N}_C(x) = \{0\}, \forall x \in C$).

We have $\mathbb{R}^n$is an open set, so $\mathcal{N}_{\mathbb{R}^n}(\overline{x}) = \{0\}$.

Moreover from (1.2), we have $\overline{x} \in S \Longleftrightarrow -F(\overline{x}) \in \mathcal{N}_C(\overline{x})$.

Since $C = \mathbb{R}^n$, then

$$\begin{aligned} \overline{x} \in S & \Longleftrightarrow -F(\overline{x}) \in \mathcal{N}_{\mathbb{R}^n}(\overline{x}) \\ & \Longleftrightarrow -F(\overline{x}) \in \{0\} \\ & \Longleftrightarrow F(\overline{x}) = 0. \end{aligned}$$

■

1.2.2 Complementarity problem

If C is a cone ($\forall x \in C, \forall \lambda \geq 0, \lambda x \in C$), then (VIP)admits an equivalent form known as the complementarity problem.

Definition 1.2.1 *Let C a cone of $\mathbb{R}^n$ and F a mapping ($F : C \longrightarrow \mathbb{R}^n$), the general complementarity problem (GCP) is defined as follows*(GCP) $\begin{cases} \text{Find } \overline{x} \in C \text{ such that} \\ F(\overline{x}) \in C^*, \langle \overline{x}, F(\overline{x}) \rangle = 0 \end{cases}$, *where* C^* *denotes the dual cone of* C *defined by*$C^* = \{y \in \mathbb{R}^n / \langle x, y \rangle \geq 0,\ \forall x \in C\}$.

Proposition 1.2.2 *Let* C *a cone of* $\mathbb{R}^n$ *and* F *a mapping* ($F : C \longrightarrow \mathbb{R}^n$)*, then we have*

$$\overline{x} \in S \iff \overline{x} \text{ is solution of } (GCP).$$

Proof. $\Longrightarrow$)

$$\overline{x} \in S \Longrightarrow \langle F(\overline{x}), x - \overline{x} \rangle \geq 0, \forall x \in C \qquad (a)$$

Since C is a cone, then $0 \in C$. Taking $x = 0$,

$$\begin{aligned} (a) \quad &\Longrightarrow \langle F(\overline{x}), -\overline{x} \rangle \geq 0 \\ &\Longrightarrow \langle F(\overline{x}), \overline{x} \rangle \leq 0 \qquad (*) \end{aligned}$$

For $x = 2\overline{x}$,

$$\begin{aligned} (a) \quad &\Longrightarrow \langle F(\overline{x}), 2\overline{x} - \overline{x} \rangle \geq 0 \\ &\Longrightarrow \langle F(\overline{x}), \overline{x} \rangle \geq 0 \qquad (**) \end{aligned}$$

From $(*), (**)$, we obtain $\langle F(\overline{x}), \overline{x} \rangle = 0 \qquad (i)$.

Also according to the last equation and the fact that $\overline{x}$ is a solution for (VIP), we have

$$\begin{aligned} \langle F(\overline{x}), x - \overline{x} \rangle \geq 0,\ \forall x \in C \quad &\iff \langle F(\overline{x}), x \rangle - \langle F(\overline{x}), \overline{x} \rangle \geq 0,\ \forall x \in C \\ &\iff \langle F(\overline{x}), x \rangle \geq 0,\ \forall x \in C \\ &\iff F(\overline{x}) \in C^* \qquad (ii) \end{aligned}$$

From (i) and (ii), we deduce that $\overline{x}$ is a solution for (GCP).

$\Longleftarrow)$

$$\begin{aligned}
\overline{x} \text{ is a solution for } (GCP) &\Longrightarrow F(\overline{x}) \in C^* \text{ and } \langle F(\overline{x}), \overline{x}\rangle = 0 \\
&\Longrightarrow \langle F(\overline{x}), x\rangle \geq 0, \forall x \in C \text{ and } \langle F(\overline{x}), \overline{x}\rangle = 0 \\
&\Longrightarrow \langle F(\overline{x}), x\rangle - \langle F(\overline{x}), \overline{x}\rangle \geq 0, \forall x \in C \\
&\Longrightarrow \langle F(\overline{x}), x - \overline{x}\rangle \geq 0, \forall x \in C \\
&\Longrightarrow \overline{x} \in S.
\end{aligned}$$

■

There are special cases of the general complementarity problem which are very important for modelling many practical problems. We introduce here, the case where $C = \mathbb{R}^n_+$ known by the complementarity problem.

Since the dual cone of the positive orthant $\mathbb{R}^n_+$ is itself, then (CP) is defined as follows

$$(CP) \quad \begin{cases} \text{Find } \overline{x} \geq 0 \text{ such that} \\ F(\overline{x}) \geq 0, \langle \overline{x}, F(\overline{x})\rangle = 0 \end{cases}.$$

Orthogonality conditions

$\langle \overline{x}, F(\overline{x})\rangle = 0$ in component expression is given by $\overline{x}_i F_i(\overline{x}) = 0, \forall i = 1, \ldots, n$. This condition explains the term of complementarity which means that if one of the components is strictly positive, then the other is certainly zero.

The complementarity problem (CP) traces its origins to Richard W. Cottle's doctoral dissertation in 1964, marking a pivotal development in mathematical programming. The problem class admits two fundamental formulations based on the nature of the mapping F.

Linear complementarity problem (LCP): When F is affine $F(x) = Mx + q$ $(M \in \mathbb{R}^{n\times n}, q \in \mathbb{R}^n)$, (CP) reduces to an (LCP). This special case became the subject of intensive study during 1965-1980, with C. E. Lemke's pivotal algorithm (1965) and subsequent developments by Eaves (1971) and Murty (1972) establishing its theoretical and computational foundations.

Nonlinear complementarity problem $(NLCP)$: The general case where F is nonlinear includes important subclasses such as:

$(P_0/NLCP)$ class (generalizing of P_0-Matrix theory).

Monotone $(NLCP)$ class (arising in convex optimization).

The fundamental connection between $VIP(F, \mathbb{R}^n_+)$ and (CP) is characterized by the following proposition, which establishes their equivalence under standard constraint qualifications when $C = \mathbb{R}^n_+$. This relationship provides crucial insights into the variational structure of complementarity conditions.

Proposition 1.2.3 *Let $C = \mathbb{R}^n_+$ and F a mapping $(F : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n)$, then we have*

$$\overline{x} \in S \iff \overline{x} \textit{ is a solution of } (CP)$$

First method: We can use the same way of demonstrating given in the case of (GCP).
Second method: Replace $x = \overline{x} + e_i \in \mathbb{R}^n_+$ in $VIP\left(F, \mathbb{R}^n_+\right)$ witn
$e_i = \left(0, \dots, 0, \underset{i}{1}, 0, \dots, 0\right)$.

$$\begin{aligned} \overline{x} \in S \text{ of } VIP\left(F, \mathbb{R}^n_+\right) \quad &\Longrightarrow \langle F(\overline{x}), \overline{x} + e_i - \overline{x}\rangle \geq 0 \\ &\Longrightarrow \langle F(\overline{x}), e_i\rangle \geq 0 \\ &\Longrightarrow F_i(\overline{x}) \geq 0. \end{aligned}$$

We vary the index $i = 1, \dots, n$, then we obtain $F_1(\overline{x}) \geq 0, F_2(\overline{x}) \geq 0, \dots, F_n(\overline{x}) \geq 0$, consequenstly $F(\overline{x}) \geq 0$. Now to have $\langle \overline{x}, F(\overline{x})\rangle = 0$ and the inverse implication, we use the same way of demonstration given in the case of (GCP).

Remark 1.2.1 *1. Of course, the problem (LCP) generalizes linear programs (LP) and quadratic programs (QP). Since (VIP) generalizes (LCP), then it is the same for (LP) and (QP).*

2. (CP) (and similarly for (GCP)) is a special case of (VIP), but the opposite is not true.

Affine problems

In what follows, we introduce several other special cases of the $VIP(F,C)$ where either F or C has some other interesting structures. To begin, let F be the affine mapping given by: $F(x) = Mx + q$, $x, q \in \mathbb{R}^n$, and $M \in \mathbb{R}^{n \times n}$; in this case, we write $VIP\,(M, q, C)$ to mean $VIP(F,C)$. If in addition C is a polyhedral set, we attach the adjective "affine" and use the notation $AVIP\,(M, q, C)$ to describe the all

affine (VIP). Finally, if C is a polyhedral set but F is not necessarily affine, we use the adjective "linearly constrained" to describe this $VIP(F,C)$ and we denoted by $LCVIP(F,C)$. Unlike the $AVIP\,(M, q, C)$ where both the defining mapping and set are affinely structured, the

$VIP\,(M, q, C)$ and the $LCVIP(F,C)$ have only one affine member in the pair (F,C). An important linearly constrained (VIP) is the box constrained $VIP(F,C)$ denoted by $(BVIP)$, where the set C is a closed rectangle given by

$$C = \{x \in \mathbb{R}^n / a_i \leq x_i \leq b_i, i = 1, \cdots, n\} = \prod_{i=1}^{n} [a_i, b_i] = [a, b]\,, where\ a, b \in \mathbb{R}^n.$$

Here a_i and a_i are possibly infinite scalars satisfying $-\infty \leq a_i < b_i \leq +\infty$,

$\forall i = 1, \cdots, n$.

Then we can write,

$$
\begin{aligned}
&C = \{x \in \mathbb{R}^n / -x \leq -a, x \leq b\} \\
&C = \left\{ x \in \mathbb{R}^n / \begin{bmatrix} -I \\ I \end{bmatrix} x \leq \begin{pmatrix} -a \\ b \end{pmatrix} \right\} \\
&C = \{x \in \mathbb{R}^n / Ax \leq B\} \ \ where\ A = \begin{bmatrix} -I \\ I \end{bmatrix} \ and\ B = \begin{pmatrix} -a \\ b \end{pmatrix}.
\end{aligned}
$$

This extended framework includes as a particular case the (CP), which corresponds to a being the zero vector and b being the vector all of whose components are equal to $+\infty$.

We summarize the various classes of problems and their connections in the diagram below. The notation $P \Longrightarrow Q$ means that we can derive problem Q by specializing problem P.

$$
\begin{array}{ccccccc}
 & & VIP\,(M,q,C) & & & & VIP\,(M,q,C) \\
 & & \Uparrow & & & & \Downarrow \\
 & & VIP(F,C) & \Longrightarrow & LCVIP(F,C) & \Longrightarrow & AVIP\,(M,q,C) \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
(NCP) & \Longleftarrow & (CP) & \Longleftarrow & (BVIP) & \Longrightarrow & (LCP) \\
 & & \Downarrow & & & & \\
 & & (LCP) & & & &
\end{array}
$$

1.2.3 Constrained optimization problem

Variational inequalities problems arise from a variety of interesting sources. Foremost among these sources are differentiable constrained optimization problems. Nevertheless, not all (VIP) are naturally derived from optimization problems. Part of the objective of this subsection is to show that a (VIP) is a nontrivial extension of a standard optimization problem.

An optimization problem is characterized by an objective function which must be maximized or minimized on $\mathbb{R}^n$ or on a subset of $\mathbb{R}^n$. In both cases the optimization problem with or without constraints can be formulated as (VIP). The following two propositions define the relationship between an optimization problem and (VIP).

Proposition 1.2.4 *Consider the constrained optimization problem*

$$(OP) \qquad \begin{cases} \min f(x) \\ x \in C \end{cases},$$

where the objective function f is defined and continuously differentiable on the closed convex set C $(\varnothing \neq C \subseteq \mathbb{R}^n)$. Let $\overline{x}$ a solution of (OP), then $\overline{x}$ is a solution of the following (VIP)

$$\langle \nabla f(\overline{x}), x - \overline{x} \rangle \geq 0, \forall x \in C.$$

Proof. Let x be any point in C and $\overline{x}$ the local minimum of (OP). Taking the convex combination of the two points x and $\overline{x}$, then we have

$$tx + (1-t)\overline{x} \in C \quad \text{and } t \in [0,1].$$

Using the limited Taylor expansion of f in the neighborhood of $\overline{x}$, then

$$f(y) = f(\overline{x}) + \langle \nabla f(\overline{x}), y - \overline{x} \rangle, \forall y \in V(\overline{x}).$$

For $y = tx + (1-t)\overline{x}$, and $t > 0$ small enough,

$$\begin{aligned} f(tx + (1-t)\overline{x}) &= f(\overline{x}) + \langle \nabla f(\overline{x}), t(x - \overline{x}) \rangle \\ (f(tx + (1-t)\overline{x}) - f(\overline{x}))/t &= \langle \nabla f(\overline{x}), (x - \overline{x}) \rangle . \end{aligned}$$

Since $\overline{x}$ is a local minimum of f on C, then $(f(tx + (1-t)\overline{x}) - f(\overline{x})) \geq 0$, in addition we have $t > 0$. Therefore, we deduce that $\langle \nabla f(\overline{x}), x - \overline{x} \rangle \geq 0, \forall x \in C$ and consequently $\overline{x}$ is a solution of $VIP(\nabla f, C)$.

The last inequality is called the stationary point condition for (OP).

Proposition 1.2.5 *If f is a convex function and $\overline{x}$ a solution of $VIP(\nabla f, C)$, then $\overline{x}$ is also a solution of (OP).*

Proof. Here we use the definition of the convexity of f in terms of the its gradient

$$f \text{ is convex on } C \Longleftrightarrow f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle, \forall x, y \in C.$$

Let $\overline{x}$ a solution of $VIP(\nabla f, C)$, then we have

$$\langle \nabla f(\overline{x}), x - \overline{x} \rangle \geq 0, \forall x \in C$$

If we take $y = \overline{x}$ in the previous definition of the convexity of the f function, we find

$$\begin{aligned} f(x) - f(\overline{x}) &\geq \langle \nabla f(\overline{x}), (x - \overline{x}) \rangle \geq 0, \forall x \in C \\ f(x) - f(\overline{x}) &\geq 0, \forall x \in C. \end{aligned}$$

Therefore, $\overline{x}$ is a global minimum of f on C ($\overline{x}$ a solution of (OP)).

Consequently, for a convex problem , i.e., with f a convex function and C a convex set, the VIP $(\nabla f, C)$ is equivalent to the optimization problem (OP).

In the case where the condition of symmetry of the jacobian of mapping F is satisfied,

$VIP(F,C)$ can be transformed into an optimization problem. This transformation is given by the following theorem.

Theorem 1.2.1 *Assume that the mapping $F=(F_1,F_2,\ldots,F_n)$ is continuously differentiable on D (an open convex) and that its Jacobian matrix is given by*

$$J_F(x)=\begin{pmatrix} \frac{\partial F_1}{\partial x_1}(x) & \frac{\partial F_1}{\partial x_2}(x) & \cdots & \frac{\partial F_1}{\partial x_n}(x) \\ \frac{\partial F_2}{\partial x_1}(x) & \frac{\partial F_2}{\partial x_2}(x) & \cdots & \frac{\partial F_2}{\partial x_n}(x) \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}(x) & \frac{\partial F_n}{\partial x_2}(x) & \cdots & \frac{\partial F_n}{\partial x_n}(x) \end{pmatrix}.$$

1- If $J_F\ (x)$ is symmetric, then there exists a real-valued function $f:D\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}$ such that $\nabla f(x)=F(x)$ and $f(x)=\int F(x)^T dx$, $\forall x\in D$.

2- In addition if $J_F(x)$ is positive semidefinite on D and $\overline{x}$ a solution of $VIP(\nabla f,C)$, then $\overline{x}$ is also a solution of (OP) on C, where C is a closed convex set included in D $(C\subset D)$.

Proof.

1. This is a result of differential calculus.

2. Since $\nabla f(x)=F(x)$, then $H_f(x)=J_F(x)$ on D, where $H_f(x)$ is the Hessian matrix of f.

3. If $J_F(x)$ is positive semi-definite on D, then H_f is also positive semi-definite on D. Therefore, f is convex on D, then f is also convex on D (where $D\subset C$ is a closed convex) . By Proposition 1.2.5, x is a solution of $VIP(\nabla f,D)$, then $\overline{x}$ is a solution of (OP) on D.

There is a very interesting case where we can transform (VIP) to an equivalent optimization problem without requiring the jacobian matrix of F to be symmetric. The following lemma describes this equivalence.

Lemma 1.2.1

$$\overline{x} \in S \Longleftrightarrow \overline{x} \textit{ is a solution for } (P1),$$

where

$$(P1) \quad \begin{cases} \min \; F(\overline{x})^T x \\ x \in C \end{cases}$$

Proof. Indeed,

$$\begin{aligned} \overline{x} \in S \quad &\Longleftrightarrow \quad \langle F(\overline{x}), x - \overline{x} \rangle \geq 0, \; \forall x \in C \\ &\Longleftrightarrow \quad \langle F(\overline{x}), x \rangle \geq \langle F(\overline{x}), \overline{x} \rangle, \; \forall x \in C \\ &\Longleftrightarrow \quad \langle F(\overline{x}), \overline{x} \rangle = \min \langle F(\overline{x}), x \rangle, \forall x \in C \\ &\Longleftrightarrow \quad \overline{x} = \arg\min_{x \in C} (P1). \end{aligned}$$

■

In the case where we take the set C is a polyhedron, $(P1)$ became a linear program. Therefore, several algorithms known for their efficiency can be used for its resolution. Unfortunately, this is not the case, because the value of $F(\overline{x})$ in the objective function of $(P1)$ is unknown.

But this relationship established between (VIP) and $(P1)$ is of great interest for writing the Karush-Kuhn-Tucker (KKT) system associated with (VIP), as we will see later.

1.2.4 Variational inequalities problem and the KKT system

It is well known that the (KKT) conditions have played a key role in all aspects of constrainted optimization problems. It turns out that these conditions can be easily extended to (VIP). We start this extension for $(LCVIP)$.

Proposition 1.2.6 *Let be the following problem* $(LCVIP)$ *with* C *be given by*

$$C = \{x \in \mathbb{R}^n / Ax \leq B,\ Dx = E\},\ A \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{l\times n} \text{ and } B \in \mathbb{R}^m, E \in \mathbb{R}^l. \tag{1.4}$$

Then,

$$\overline{x} \in S \Longleftrightarrow \exists\ \lambda \in \mathbb{R}_+^m \text{ and } \mu \in \mathbb{R}^l \text{ such that}$$

$$\begin{cases} F(\overline{x}) + A^T\lambda + D^T\mu = 0 \\ D\overline{x} - E = 0 \\ \lambda_i (A_i\overline{x} - B_i) = 0, \quad i = 1, \cdots, m \end{cases} \tag{1.5}$$

with A_i *is the ith line of the matrix* A *and* B_i *is the ith component of the vector* B.

Proof. According to the Lemma 1.2.1, we can associate the (VIP) with an equivalent optimization problem of the form $(P1)$ that we denote $(LP1)$ where the constraint set C is defined by (1.4).

$$\overline{x} \in S \Longleftrightarrow \overline{x} \text{ is the following linear program } (LP_1)$$

$$(LP1) \quad \begin{cases} \min(f(x) = \langle F(\overline{x}), x\rangle) \\ \qquad Ax \leq B \\ \qquad Dx = E \end{cases}$$

Since the constraints are affine (linear) holds at every point in C. Given that $\overline{x}$ is a solution to $(LP1)$, the (KKT) conditions are necessary sufficient for optimality of $\overline{x}$. Therefore, we get directly the system (1.5).

In the other hand, since $\overline{x}$ is a solution to $(P1)$ and thus to (VIP), then the proof is achevied. ■

The next part, we will extend the above result for a more general set C by giving the following proposition.

Proposition 1.2.7 *Let the following problem (VIP) with C be given by*

$$C = \{x \in \mathbb{R}^n / h_i(x) \leq 0, i = 1, \cdots, m \text{ and } g_j(x) = 0, j = 1, \cdots, l\},$$

where the functions h_i and g_j are continuously differentiable on C.

The following two statements are valid.

- Let $\overline{x} \in S$. If the constraints qualification holds at $\overline{x}$, then there exist vectors $\lambda \in \mathbb{R}_+^m$ and $\mu \in \mathbb{R}^l$ such that

$$\overline{x} \in S \Longrightarrow \exists\ \lambda \in \mathbb{R}_+^m \text{ and } \mu \in \mathbb{R}^l \text{ such that}$$

$$\begin{cases} F(\overline{x}) + \sum_{i=1}^{m} \lambda_i h_i(\overline{x}) + \sum_{j=1}^{l} \mu_j g_j(\overline{x}) = 0 \\ g(\overline{x}) = 0 \\ \lambda_i h_i(\overline{x}) = 0, \quad i = 1, \cdots, m \end{cases} \tag{1.6}$$

- Conversely, if each function g_j is affine and each function h_i is convex on C, and if $(\overline{x}, \lambda, \mu)$ satisfies (1.6), then $\overline{x}$ solves the $VIP(F, C)$.

Proof. By following the same steps of the demonstration Proposition 1.2.6, we obtain the desired result. ■

1.2.5 Fixed point problem

The link between (VIP) and the fixed point problem is established mainly by using the projection application. The definition of this application is given below.

Definition 1.2.2 *Let C be a non-empty closed convex of $\mathbb{R}^n$. Then, for any point $x \in \mathbb{R}^n$, there exists a unique point $\overline{x} \in C$ such that*

$$\|\overline{x} - x\| \leq \|y - x\|, \forall y \in C,$$

or

$$\overline{x} = P_C(x) = \underset{y \in C}{\arg\min} \|y - x\| = \underset{y \in C}{\arg\min} \left(\frac{1}{2} \|y - x\|^2 \right).$$

P_C denotes the orthogonal projection application on C and $\overline{x}$ the orthogonal projection (or orthogonal project) of x onto C.

Theorem 1.2.2 *Let C be a non-empty closed convex of $\mathbb{R}^n$. Then, for $x \in \mathbb{R}^n$, we have*

$$\overline{x} = P_C(x) \Longleftrightarrow \langle \overline{x} - x, y - \overline{x} \rangle \geq 0, \forall y \in C.$$

This inequality is known as the classical projection inequality.

Proof. We have $\overline{x} = P_C(x)$, thus $\overline{x}$ is a solution of the following optimization problem

$$\begin{cases} \min \left(\frac{1}{2} \|y - x\|^2 = f(y) \right) \\ \qquad\qquad y \in C \end{cases}.$$

The latter problem has a unique solution $\overline{x}$ because f is continuously differentiable and strictly convex on C.

The solution $\overline{x}$ is characterised by the following inequality

$$\langle \nabla f(\overline{x}), y - \overline{x} \rangle \geq 0, \forall y \in C.$$

Since $\nabla f(y) = y - x$, the previous inequality becomes

$$\langle \overline{x} - x, y - \overline{x} \rangle \geq 0, \forall y \in C,$$

Hence the result. ■

Theorem 1.2.3 *Let C be a non-empty closed convex of $\mathbb{R}^n$ and $F : C \longrightarrow \mathbb{R}^n$. Then,*

$$\overline{x} \in S \Longleftrightarrow \overline{x} \textit{ is the fixed point of the mapping } P_C \circ (I_{\mathbb{R}^n} - \alpha F), \forall \alpha > 0$$

In other words

$$\overline{x} = P_C(\overline{x} - \alpha F(\overline{x})), \forall \alpha > 0.$$

Proof. Suppose that $\overline{x} \in S$,

$$\begin{aligned}
\langle F(\overline{x}), x - \overline{x} \rangle \geq 0, \forall x \in C \quad & \Longleftrightarrow -\alpha \langle F(\overline{x}), x - \overline{x} \rangle \leq 0, \forall x \in C, \forall \alpha > 0 \\
& \Longleftrightarrow \langle (\overline{x} - \alpha F(\overline{x})) - \overline{x}, x - \overline{x} \rangle \leq 0, \forall x \in C, \forall \alpha > 0 \\
& \Longleftrightarrow \langle \overline{x} - (\overline{x} - \alpha F(\overline{x})), x - \overline{x} \rangle \geq 0, \forall x \in C, \forall \alpha > 0.
\end{aligned}$$

Using the result of Theorem 3, it implies that $\overline{x} = P_C(\overline{x} - \alpha F(\overline{x}))$. ■

1.2.6 Equilibrium problem

The equilibrium problem denoted by $EP(f, C)$, in its most general form, is defined as follows

$$\begin{cases} \textit{Find } x \in C \textit{ such that} \\ f(x, y) \geq 0, \textit{for all } y \in C, \end{cases}$$

Where C is a closed, convex subset of $\mathbb{R}^n$ and $f : C \times C \to \mathbb{R}$ is a bifunction (a function of two arguments).

It was Oettli who in 1994 the first coined the term equilibrium problem during the annual conference of the Indian Mathematical Society and his paper was published in the journal (*MSIMS*) [2]. It is one of the most cited papers in optimization theory.

The power of this formulation is that it allow to include, in a common framework, a large set of problems.

For example consider

$$f(x, y) = \varphi(y) - \varphi(x) \ \ and\ C \subseteq \mathbb{R}^n, \tag{1.7}$$

with $\varphi : C \longrightarrow \mathbb{R}^n$.

Then the solution set of the problem $EP(f, C)$ coincides with the set of global minimizers of the function φ on C.

Now if we consider

$$f(x, y) = \varphi(x) - \varphi(y) \ \ and\ C \subseteq \mathbb{R}^n. \tag{1.8}$$

Then, symmetrically, the solutions of the equilibrium problem $EP(f, C)$ are the global maximizers of the function φ on C.

Thus, the concept of an equilibrium problem seems to unify both minimization and maximization problems.

Additionally, in the case where the objective function φ is supposed to be differentiable and convex on a closed convex set C. Then, it is a well-known that

$\overline{x}$ is a global minimizer of φ on C , if and only if

$$\overline{x} \text{ is a global minimizer of } \varphi \text{ on } C \Longleftrightarrow \langle \nabla\varphi\left(\overline{x}\right), y-\overline{x}\rangle \geq 0, \forall y \in C. \tag{1.9}$$

The above inequality expresses the necessary and sufficient condition for the global optimality of the solution $\overline{x}$. Thus, in context of convex optimization, a first relationship between equilibrium problem and variational inequality occurs since

$$EP(f_{\varphi}, C) = \arg\min_{C} \varphi = VIP\left(\nabla\varphi, C\right) \text{ with } f_{\varphi}\left(x, y\right) = \varphi\left(y\right) - \varphi\left(x\right). \tag{1.10}$$

Now, we will explore the relation that can be stated between equilibrium problem and variational inequalities problem whenever the function f is not coming from an optimization problem.

Given a subset C of $\mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and the associated variational inequalities problem $VIP(F, C)$, an immediate link with an equilibrium problem can be stated by simply considering the bifunction f_F defined on C by

$$f_F(x; y) = \langle F(x), y - x\rangle,$$

then we get

$$EP(f_F, C) = VIP(F, C).$$

This equality being valid without any hypothesis. Thus, we can consider that variational inequality problem is particular class of equilibrium problems.

Let us exanime the reverse question that is under which conditions an equilibrium problem $EP(f, C)$ can be seen as a variational inequality problem.

To establish this result, one of the most common needed assumptions in the literature is the following

1. $f(x,x) = 0$ for all $x \in C$.

2. For any $x \in \mathbb{R}^n$, the function $y \in C \longrightarrow f(x,y)$ is a convex function.

The first condition shows that if $\overline{x}$ is a solution of the equilibrium problem, then $\overline{x}$ a minimzer of the function $f(\overline{x}, y)$ on C.

Now assume that f is a differentiable convex function respecting to y and C a closed convex set. Then, we can write the necessary and sufficient optimality condition as

$$\langle \nabla f_y \left(\overline{x}, \overline{x}\right), y - \overline{x} \rangle \geq 0, \forall y \in C.$$

This shows that $\overline{x}$ solves the problem $VIP(F_f, C)$ where, for each $x \in \mathbb{R}^n$,

$F_f(x) = \nabla f_y(x,x)$.

Further if $\overline{x}$ solves $VIP(F_f, C)$, then by (1.9) and the convexity of f respecting to the second variable y, it is clear that $\overline{x}$ minimizes $f\left(\overline{x}, .\right)$ on C. Additionaly, since $f(\overline{x}, \overline{x}) = 0$, we conclude that $\overline{x}$ solves $EP(f, C)$. Thus, the solution set of $EP(f, C)$ coincides with the solution set of $VIP(F_f, C)$ once we assume that f is differentiable and convex respecting to the second variable, that is

$$EP(f, C) = VIP(F_f(x), C) where F_f(x) = \nabla f_y(x, x).$$

Exercises

Exercise 1.2.1 *Let F be the mapping defined by:*

- $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

  $$(x, y, z) \longmapsto (yz^2, xz^2 + z, 2xyz + 2z + y)$$

- $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

  $$(x, y, z) \longmapsto (2x - y + 3, 2y - x - z - 1, 2z - y + 2)$$

  In both cases of F:

1. *Find the primitive of F on $\mathbb{R}^3$ if it exists.*
2. *If so, determine the optimization problem equivalent to $VIP\,(F, C)$ where $C = \mathbb{R}^3$.*

Solution 1.2.1 *a)* $F\,(x, y, z) = \begin{pmatrix} F_1\,(x, y, z) \\ F_2\,(x, y, z) \\ F_3\,(x, y, z) \end{pmatrix} = \begin{pmatrix} yz^2 \\ xz^2 + z \\ 2xyz + 2z + y \end{pmatrix}$

We determine the jacobian matrix of F :

$$J_F\,(x, y, z) = \begin{pmatrix} 0 & z^2 & 2zy \\ z^2 & 0 & 2xz + 1 \\ 2zy & 2xz + 1 & 2xy + 2 \end{pmatrix}.$$

$J_F\,(x, y, z)^T = J_F\,(x, y, z)$, *then $J_F\,(x, y, z)$ is a symmetric matrix.*

Determonation of f the primitive of F :

We have that:

$$F = \nabla f \Leftrightarrow \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}.$$

Thus,

$$\begin{aligned} f(x,y,z) &= \int \tfrac{\partial f(x,y,z)}{\partial x} dx \\ &= \int F_1(x,y,z)\, dx \\ &= \int yz^2 dx \\ &= xyz^2 + h(y,z). \end{aligned}$$

Then,

$$\begin{aligned} \tfrac{\partial f(x,y,z)}{\partial y} = F_2(x,y,z) &\iff xz^2 + \tfrac{\partial h(y,z)}{\partial y} = xz^2 + z \\ &\iff \tfrac{\partial h(y,z)}{\partial y} = z \\ &\iff h(y,z) = yz + g(z). \end{aligned}$$

$$f(x,y,z) = xyz^2 + yz + g(z).$$

Finally,

$$\begin{aligned} \tfrac{\partial f(x,y,z)}{\partial z} = F_3(x,y,z) &\iff 2xz + y + g^{'}(z) = 2xyz + 2z + y \\ &\iff g^{'}(z) = 2z \\ &\iff g(z) = z^2 + k, (k \in \mathbb{R}). \end{aligned}$$

So,

$$f(x,y,z) = xyz^2 + yz + z^2 + k, (k \in \mathbb{R}).$$

f *is a convex function if and only if its hessian matrix is positive semidefinite.*

We have that $\nabla^2 f(x,y,z) = J_F(x,y,z)$. *We calculate the principale minors:*

$\triangle_1 = 0.$

$\triangle_2 = \begin{vmatrix} 0 & z^2 \\ z^2 & 0 \end{vmatrix} = -z^4 < 0,\ z \neq 0.$

Therefore, this $VIP(F,C)$ *does not admit an equivalent optimization problem.*

$$b)\quad F(x,y,z) = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} = \begin{pmatrix} 2x-y+3 \\ 2y-x-z-1 \\ 2z-y+2 \end{pmatrix}$$

We determine the jacobian matrix of F :

$$J_F(x,y,z) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

$J_F(x,y,z)^T = J_F(x,y,z)$, *then* $J_F(x,y,z)$ *is a symmetric matrix.*

Determonation of f *the primitive of* F :

We have that:

$$\begin{aligned} f(x,y,z) &= \int \tfrac{\partial f(x,y,z)}{\partial x} dx \\ &= \int F_1(x,y,z)\,dx \\ &= \int (2x-y+3)\,dx \\ &= x^2 - yx + 3x + h(y,z). \end{aligned}$$

Then,

$$\begin{aligned} \tfrac{\partial f(x,y,z)}{\partial y} = F_2(x,y,z) &\iff -x + \tfrac{\partial h(y,z)}{\partial y} = 2y - x - z - 1 \\ &\iff \tfrac{\partial h(y,z)}{\partial y} = 2y - z - 1 \\ &\iff h(y,z) = y^2 - zy - y + g(z). \end{aligned}$$

$$f(x,y,z) = x^2 - yx + 3x + y^2 - zy - y + g(z).$$

Finally,

$$\begin{aligned} \tfrac{\partial f(x,y,z)}{\partial z} = F_3(x,y,z) &\iff -y + g'(z) = 2z - y + 2 \\ &\iff g'(z) = 2z + 2 \\ &\iff g(z) = z^2 + 2z + k, (k \in \mathbb{R}). \end{aligned}$$

So,

$$f(x,y,z) = x^2 - yx + 3x + y^2 - zy - y + z^2 + 2z + k, (k \in \mathbb{R}).$$

We calculate the principale minors of $J_F(x,y,z)$:

$\triangle_1 = 0.$

$\triangle_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$\triangle_2 = \det(J_F(x,y,z)) = 4.$

Then, $J_F(x,y,z)$ positive semidefinite. Hence, this $VIP(F,C)$ admit an equivalent optimization problem defined by

$$\begin{cases} \min\limits_{(x,y,z)\in\mathbb{R}^3} f(x,y,z) = x^2 - yx + 3x + y^2 - zy - y + z^2 + 2z + k \end{cases}$$

Exercise 1.2.2 *Consider the following $VIP(F,C)$ where*

$$F(x) = Ax \quad with\ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad and\ C = \mathbb{R}^2.$$

1. *Does it exist any equivalent optimization problem to this $VIP(F,C)$.*

2. *Determine the solutions of this $VIP(F,C)$ by different methods.*

Exercise 1.2.3 *Let consider the following $VIP(F,C)$ where*

$$F(x) = \begin{pmatrix} x_1 + 1 \\ 2x_2 + 1 \end{pmatrix} \quad and\ C = \mathbb{R}^2_+.$$

1. *Determine the dual cone of C and the normal cone of C at the origin.*

2. *Knowing that the orthogonal projection of any point x of $\mathbb{R}^2$ onto C is given by*

$$\left[P_{\mathbb{R}^2_+}(x)\right]_i = \max(0, x_i), \quad i = 1, 2.$$

   determine the solutions $\overline{x}$ of this $VIP(F, C)$ by using the fixed point formula.

3. *Draw the geomitricly $\overline{x}$ and the cones $C^*, \mathcal{N}_C(\overline{x})$.*

Exercise 1.2.4 *Let $VIP_a(F, C)$ be a variational inequalities problem where the mapping F and C are given by*

$$F(x) = \left(\frac{2}{1 - \|x\|^2}\right) x + a, a \in \mathbb{R}^3 and C = \left\{x \in \mathbb{R}^3 / \|x\|^2 \leq 1\right\}.$$

Exercise 1.2.5 1. *Show that $VIP(F_a, C)$ admits an equivalent optimization problem (P_a) that must be determined.*

2. *Consider $C_a = \left\{x \in \mathbb{R}^3 / h_1(x) = \|x\|^2 \leq \frac{1}{4}, h_2(x) = \langle a, x\rangle \leq 0\right\}$. Using the conditions of Karush Kuhn Tuker (KKT):*

   (a) *Solve $VIP(F_a, C_a)$ for the case $a = 0$.*

   (b) *Suppose that $a \neq 0$ and we denote by $\overline{x}$ the solution of$VIP(F_a, C_a)$ where $h_2(\overline{x}) < 0$. Determine the multipliers of (KKT) λ_1, λ_2 associated (Discuss only the case $\lambda_1 > 0$) and the solution $\overline{x}$.*

Solution 1.2.2 *Let be $VIP(F, C)$ where F and are given by*

$$F_a(x) = \left(\frac{2}{1 - \|x\|^2}\right) x + a, \ a \in \mathbb{R}^3,$$

C is a closed convex set included in $D = \left\{x \in \mathbb{R}^3 / \|x\|^2 < 1\right\}$.

1. We determine the jacobian matrix of F

$$F_a(x) = \left(\frac{2}{1-\left(x_1^2+x_2^2+x_3^2\right)}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} F_1(x_1,x_2,x_3) \\ F_2(x_1,x_2,x_3) \\ F_3(x_1,x_2,x_3) \end{pmatrix}.$$

$$\begin{aligned}
J_F(x) &= \begin{pmatrix} \frac{2\left(1-\|x\|^2\right)+4x_1^2}{\left(1-\|x\|^2\right)^2} & \frac{4x_1x_2}{\left(1-\|x\|^2\right)^2} & \frac{4x_1x_3}{\left(1-\|x\|^2\right)^2} \\ \frac{4x_1x_2}{\left(1-\|x\|^2\right)^2} & \frac{2\left(1-\|x\|^2\right)+4x_2^2}{\left(1-\|x\|^2\right)^2} & \frac{4x_2x_3}{\left(1-\|x\|^2\right)^2} \\ \frac{4x_1x_3}{\left(1-\|x\|^2\right)^2} & \frac{4x_2x_3}{\left(1-\|x\|^2\right)^2} & \frac{2\left(1-\|x\|^2\right)+4x_3^2}{\left(1-\|x\|^2\right)^2} \end{pmatrix} \\
&= \frac{2}{\left(1-\|x\|^2\right)} I_3 + \frac{4}{\left(1-\|x\|^2\right)^2}\begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{pmatrix} \\
&= \frac{2}{\left(1-\|x\|^2\right)} I_3 + \frac{4}{\left(1-\|x\|^2\right)^2} xx^T.
\end{aligned}$$

Since $J_F(x)$ is the sum of symmetric matrices, then $J_F(x)$ is also a symmetric matrix. Therefore, it exists $f_a : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\nabla f(x) = F(x)$.

$J_F(x)$ is positive semidefinite on C if only if $\langle J_F(x)\,d, d\rangle \geq 0, \forall\ d \in \mathbb{R}^3$, $\forall$ $x \in C$.

$$\begin{aligned}
\langle J_F(x)\,d, d\rangle &= \left\langle \left(\frac{2}{\left(1-\|x\|^2\right)} I_3 + \frac{4}{\left(1-\|x\|^2\right)^2} xx^T\right) d, d \right\rangle \\
&= \frac{2}{\left(1-\|x\|^2\right)} \|d\|^2 + \frac{4}{\left(1-\|x\|^2\right)^2} \left\langle x^T d, x^T d\right\rangle \\
&= \frac{2}{\left(1-\|x\|^2\right)} \|d\|^2 + \frac{4}{\left(1-\|x\|^2\right)^2} \left(x^T d\right)^2 \\
&\geq 0, \forall\ d \in \mathbb{R}^3 \text{ and } \forall\ x \in C.
\end{aligned}$$

Then, $VIP(F_a, C)$ admits an equivalent optimization problem (OP).

$$\begin{cases}\min\limits_{x\in C} f_a(x) \Longleftrightarrow VIP(F_a, C) \quad où \ \nabla f_a = F_a \quad \text{on } C\end{cases}$$

Determination of f_a:

$$\begin{aligned}
f(x_1,x_2,x_3) &= \int \tfrac{\partial f(x_1,x_2,x_3)}{\partial x_1} dx_1 = \int F_1(x_1,x_2,x_3)\, dx_1 \\
&= \int \left(\tfrac{2x_1}{\left(1-\|x\|^2\right)} + a_1 \right) dx_1 \\
&= -\ln\left(1-\|x\|^2\right) + a_1 x_1 + g(x_2,x_3).
\end{aligned}$$

$$\begin{aligned}
\tfrac{\partial f(x_1,x_2,x_3)}{\partial x_2} = F_2(x_1,x_2,x_3) &\Longleftrightarrow \tfrac{2x_1}{\left(1-\|x\|^2\right)} + \tfrac{\partial g(x_2,x_3)}{\partial x_2} = \left(\tfrac{2x_2}{\left(1-\|x\|^2\right)} + a_2 \right) \\
&\Longleftrightarrow \tfrac{\partial g(x_2,x_3)}{\partial x_2} = a_2 \\
&\Longleftrightarrow g(x_2,x_3) = a_2 x_2 + g(x_3).
\end{aligned}$$

$$\begin{aligned}
\tfrac{\partial f(x_1,x_2,x_3)}{\partial x_3} = F_3(x_1,x_2,x_3) &\Longleftrightarrow \tfrac{2x_3}{\left(1-\|x\|^2\right)} + g'(x_3) = \left(\tfrac{2x_3}{\left(1-\|x\|^2\right)} + a_3 \right) \\
&\Longleftrightarrow g'(x_3) = a_3 \\
&\Longleftrightarrow g(x_3) = a_3 x_3 + K, \quad K \in \mathbb{R}.
\end{aligned}$$

Donc, $f(x_1,x_2,x_3) = -\ln\left(1-\|x\|^2\right) + a^T x + K, \quad K \in \mathbb{R}.$

2. We consider $C_a = \left\{ x \in \mathbb{R}^3 / h_1(x) = \|x\|^2 \leq \frac{1}{4}, h_2(x) = \langle a, x \rangle \leq 0 \right\}$ and using conditions of (KKT):

   (a) Resolution of $VIP(F_a, C_a)$ for $a = 0$:

   $\overline{x}$ is a solution of $VIP(F_0, C_0)$ iff $\exists \lambda \geq 0$ such that

   $$\begin{cases} F(\overline{x}) + \lambda \nabla h_1(\overline{x}) = 0 \\ \lambda h_1(\overline{x}) = 0 \end{cases} \Longleftrightarrow \begin{cases} \left(\frac{2}{1-\|\overline{x}\|^2}\right)\overline{x} + 2\lambda\overline{x} = 0 \quad \cdots \quad (*) \\ \lambda\left(\|\overline{x}\|^2 - \frac{1}{4}\right) = 0 \quad \cdots \quad (**) \end{cases}$$

   If $\lambda > 0 \Longrightarrow \|\overline{x}\|^2 = \frac{1}{4}$, we replace in $(*)$, we obtain $2\left(\frac{4}{3} + \lambda\right)\overline{x} = 0.$

   that is impossible, because $\|\overline{x}\|^2 = \frac{1}{4} \Longrightarrow \|\overline{x}\| = \frac{1}{2} \Longrightarrow \overline{x} \neq 0$ and $\lambda > 0.$

   If $\lambda = 0$, then from $(*)$, we have $\left(\frac{2}{1-\|\overline{x}\|^2}\right)\overline{x} = 0 \Longrightarrow \overline{x} = 0.$

   (b) Suppose that $a \neq 0$ and $h_2(\overline{x}) < 0$ where $\overline{x}$ is a solution of $VIP(F_a, C_a)$.

$\overline{x}$ is a solution of $VIP(F_a, C_a)$ iff $\exists \lambda_1 \geq 0, \lambda_2 \geq 0$ such that:

$$\begin{cases} F(\overline{x}) + \lambda_1 \nabla h_1(\overline{x}) + \lambda_2 \nabla h_2(\overline{x}) = 0 \quad \cdots \quad (***) \\ \lambda_1 h_1(\overline{x}) = 0 \\ \lambda_2 h_2(\overline{x}) = 0. \end{cases}$$

Since $h_2(\overline{x}) < 0 \Longrightarrow \lambda_2 = 0$.

For λ_1, we discuss only the case $\lambda_1 > 0$

If $\lambda_1 > 0 \Longrightarrow h_1(\overline{x}) = 0 \Longrightarrow \|\overline{x}\|^2 = \frac{1}{4}$.

$(***)$ becomes $\left(\frac{2}{1-\|\overline{x}\|^2}\right)\overline{x} + a + 2\lambda_1\overline{x} = 0 \Longrightarrow \overline{x} = -\frac{1}{2\left(\frac{4}{3}+\lambda_1\right)}a$.

We calculate $\|\overline{x}\|^2$ using the fact that $\|\overline{x}\|^2 = \frac{1}{4}$, we get

$$\begin{aligned} \|\overline{x}\|^2 = \frac{1}{4} \quad & \Longleftrightarrow \left\langle -\frac{a}{2\left(\frac{4}{3}+\lambda_1\right)}, -\frac{a}{2\left(\frac{4}{3}+\lambda_1\right)} \right\rangle = \frac{1}{4} \\ & \Longleftrightarrow \frac{1}{4\left(\frac{4}{3}+\lambda_1\right)^2}\|a\|^2 = \frac{1}{4} \\ & \Longleftrightarrow 9\lambda_1^2 + 24\lambda_1 + 16 - 9\|a\|^2 = 0. \end{aligned}$$

We solve this second order equation of unknown λ_1, we obtain

$\lambda_1' = \|a\| - \frac{4}{3} > 0$, accepted if $\|a\| > \frac{4}{3}$.

$\lambda_1'' = -\|a\| - \frac{4}{3} < 0$, rejected.

Thus, the solution $\overline{x} = -\frac{1}{2\|a\|}a$.

Consequently, $\left(\lambda_1, \lambda_2, \overline{x}^T\right) = \left(\|a\| - \frac{4}{3}, 0, -\frac{1}{2\|a\|}a^T\right)$.

Exercise 1.2.6 *Let $VIP_a(F,C)$ be a variational inequalities problem where the mapping F and C are given by*

$$F(x) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + x_2 - 1 \end{pmatrix} \text{ and } C = \left\{x \in \mathbb{R}^2 / -x_1 - x_2 \geq 2\right\}.$$

1. *Does it exist any equivalent optimization problem (OP) to this $VIP(F,C)$. If so, determine it by taking $f(0,0) = 0$.*

2. *Solve (OP) by using KKT conditions.*

3. *Determine the expression of the orthogonal projection onto C and verify that the solution obtained for (OP) is also the solution $VIP\,(F, C)$ by using the fixed point formula.*

Exercise 1.2.7

Let $g, h : C \longrightarrow \mathbb{R}$ be functions (continuously differentiable on C) and C a non-empty closed convex of $\mathbb{R}^n$.

Suppose that $\overline{x} = \arg\max\limits_{x \in C} h\,(x)$ and we put $f = g + h$ on C.

1. Show that if f, g and h are convex, then

   $\overline{x} = \arg\min\limits_{x \in C} f\ (x) \Longleftrightarrow \overline{x}$ is a solution of $VIP(\nabla g, C)$.

   We recall that $\lim\limits_{\alpha \longrightarrow 0} \frac{g(x_0 + \alpha d) - g(x_0)}{\alpha} = \langle \nabla g\,(x_0)\,, d \rangle$.

2. Show that if g and h are convex, then

   $\langle \nabla g\,(\overline{x})\,, x - \overline{x} \rangle + h\,(x) - h\,(\overline{x}) \geq 0 \Longleftrightarrow \langle \nabla g\,(x)\,, x - \overline{x} \rangle + h\,(x) - h\,(\overline{x}) \geq 0$, $\forall x \in C$.

Chapter 2

Classification, existence and uniqueness solutions results of (VIP)

The classification of variational inequalities problems depends essentially on the monotonicity properties of the mapping F. The strongly monotone case is the most significant, as it provides the primary framework for theoretical and algorithmic advances. The structure of the constraint set C can also serve as a criterion for categorization.

2.1 Classes of monotonicity

In this section, we introduce several monotonicity properties of vector mappings that are naturally satisfied by the gradient maps of convex functions of the respective kinds. Indeed, the class of monotone mappings play a similarly important role in the (VIP) as the class of convex functions in optimization.

Definition 2.1.1 *Let C be a non-empty of $\mathbb{R}^n$ and $F : C \longrightarrow \mathbb{R}$.*

- *F is said to be monotone (M) on C if*

$$\langle F(x) - F(y), x - y\rangle \geq 0, \forall x, y \in C.$$

- *F is said to be strictly monotone $(StriM)$ on C if*

$$\langle F(x) - F(y), x - y\rangle > 0, \forall x, y \in C \text{ and } x \neq y$$

- *F is said to be strongly monotone $(StroM)$ on C if there is $\mu > 0$ such that*

$$\langle F(x) - F(y), x - y\rangle \geq \mu \|x - y\|^2, \forall x, y \in C.$$

- *F is said to be paramonotone (PaM) on C if F is monotone on C and*

$$\langle F(x) - F(y), x - y\rangle = 0 \Longrightarrow F(x) = F(y), \forall x, y \in C.$$

- *F is said to be pseudomonotone (PsM) on C if one of the two conditions is satisfied*

$$\langle F(x), y - x\rangle \geq 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0, \forall x, y \in C.$$

$$\langle F(x), y - x\rangle > 0 \Longrightarrow \langle F(y), y - x\rangle > 0, \forall x, y \in C.$$

- *F is said to be quasimonotone (QM) on C if*

$$\langle F(x), y - x\rangle > 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0, \forall x, y \in C.$$

The last two definitions describe types of generalized monotonicity. The relationships between these and other monotonicity classes are summarized in the following diagram.

$$(StroM) \Longrightarrow (StriM) \Longrightarrow (PaM) \Longrightarrow (M) \Longrightarrow (PsM) \Longrightarrow (QM)$$

The concepts of convexity and monotonicity are closely related. This is most evident in the fact that the convexity of a continuously differentiable function f ($f : \mathbb{R}^n \longrightarrow \mathbb{R}$) can be determined by the monotonicity type of its gradient. The following theorem provides this crucial characterization.

Theorem 2.1.1 *Let C be a non-empty open convex of $\mathbb{R}^n$ and $f : C \longrightarrow \mathbb{R}$ a differentiable function where ∇f is its gradient. Then, we have the following equivalences*

1. *f is convex on $C \Longleftrightarrow \nabla f$ is monotone on C.*
2. *f is strictly convex on $C \Longleftrightarrow \nabla f$ is strictly monotone on C.*
3. *f is strongly convex on $C \Longleftrightarrow \nabla f$ is strongly monotone on C.*

Proof. $\Longrightarrow$)

Assume f is convex on C. By the definition of convexity, for any $x, y \in C$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda y + (1-\lambda)x) \leq \lambda f(y) + (1-\lambda)f(x).$$

This can be rearranged as

$$f(x + \lambda(y - x)) - f(x) \leq \lambda(f(y) - f(x)).$$

Dividing both sides by $\lambda > 0$,

$$\frac{f(x+\lambda(y-x))-f(x)}{\lambda} \leq f(y)-f(x).$$

Taking the limit as $\lambda \to 0^{+}$, the left-hand side becomes the directional derivative of f at x in the direction $(y-x)$.

Since f is differentiable, this is equal to $\langle \nabla f(x), y-x \rangle$.

Therefore,

$$\langle \nabla f(x), y-x \rangle \leq f(y)-f(x).$$

Similarly, we do the step another time by changing x by y and y by x, we obtain

$$\langle \nabla f(y), x-y \rangle \leq f(x)-f(y).$$

Now, adding the two inequalities,

$$\langle -\nabla f(x)+\nabla f(y), x-y \rangle \leq 0.$$

Multiplying both sides by -1, we get

$$\langle \nabla f(x)-\nabla f(y), x-y \rangle \geq 0.$$

This is exactly the condition for ∇f to be monotone on C.

$\Longleftarrow)$

Define a function $g : [0,1] \to \mathbb{R}^{n}$ by

$$g(t) = f(x+t(y-x)), \forall x, y \in C.$$

Note that g is differentiable (since f is differentiable) and we have

$$g'(t) = \langle \nabla f(x + t(y - x)), y - x \rangle .$$

Now, take any $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Let $u = x + t_1(y - x)$ and $v = x + t_2(y - x)$. These points are in C because C is convex.

Consider the difference

$$\begin{aligned} g'(t_2) - g'(t_1) &= \langle \nabla f(v), y - x \rangle - \langle \nabla f(u), y - x \rangle \\ &= \langle \nabla f(v) - \nabla f(u), y - x \rangle . \end{aligned}$$

Note that $v - u = (t_2 - t_1)(y - x)$, so $y - x = \frac{1}{(t_2 - t_1)}(v - u)$.

Substituting this in

$$g'(t_2) - g'(t_1) = \left\langle \nabla f(v) - \nabla f(u), \frac{1}{(t_2 - t_1)}(v - u) \right\rangle = \frac{1}{(t_2 - t_1)} \langle \nabla f(v) - \nabla f(u), (v - u) \rangle .$$

By the monotonicity assumption, $\langle \nabla f(v) - \nabla f(u), (v - u) \rangle \geq 0$ and since $t_1 < t_2$, , we have

$$g'(t_2) - g'(t_1) \geq 0.$$

This shows that $g\prime$ is a non-decreasing function on $[0, 1]$. A standard result in single-variable calculus states that a function with a non-decreasing derivative is convex. Therefore, g is convex on $[0, 1]$.

Since x and y were arbitrary points in C, this implies that f is convex on the convex set C.

Indeed, we put $x = x + 0 \times (y - x)$ and $y = x + 1 \times (y - x)$, therefore we obtain

$$\begin{aligned} f(\lambda x + (1-\lambda) y) &= f(\lambda (x + 0 \times (y - x)) + (1-\lambda)(x + 1 \times (y - x))) \\ &= f(x + (\lambda \times 0 + (1-\lambda) \times 1)(y - x)) \\ &= g(\lambda \times 0 + (1-\lambda) \times 1) \\ &\leq \lambda g(0) + (1-\lambda) g(1) \ (\textit{since } g \textit{ is convex on } [0,1]) \\ &= \lambda f(x + 0 \times (y - x)) + (1-\lambda) f(x + 1 \times (y - x)) \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

■

Similarly, for a continuously differentiable mapping F its monotonicity properties are directly related to the positive definiteness of its jacobian matrix J_F, as we will show in the theorem below.

Theorem 2.1.2 *Let C be a non-empty open convex of $\mathbb{R}^n$ and $F : C \longrightarrow \mathbb{R}^n$ a differentiable function where J_F is its jacobian matrix. Then, we have the following equivalences*

1. *F is monotone on $C \iff J_F(x)$ is positive semi-definite on C.*
2. *F is strictly monotone on $C \iff J_F(x)$ is positive definite on C.*
3. *F is strongly monotone on $C \iff J_F(x)$ is positive strongly definite on C $\left(\exists \mu > 0, \langle J_F(x) y, y \rangle \geq \mu \|y\|^2, \forall x, y \in C\right)$.*

Remark 2.1.1

- *In the case of a symmetric matrix, to demonstrate the type of its positive definiteness, it is sufficient to use the eigenvalue criterion or the principal minors criterion.*
- *In the case of the absence of symmetry, to verify the property in question, we can introduce the tool of the symmetric part associated with this matrix $\left(sym(A) = \frac{1}{2}\left(A + A^T\right)\right)$.*

We can also verify the property of strong monotonicity by another way as it is described in the proposition below.

Proposition 2.1.1 *F is $(StroM)$ of constant μ $(\mu > 0)$ on $C \Longleftrightarrow (J_F\ (x) - \mu I)$ is positive semi-definite matrix for all $x \in C$.*

Proof. $(\Longrightarrow)$ Assume F is strongly monotone with constant $\mu > 0$. Take any $x \in C$ and any direction $d \in \mathbb{R}^n$. Since C is open (or we consider the interior), for small $t > 0$, $x + td \in C$.

By strong monotonicity

$$\langle F(x + td) - F(x), (x + td) - x\rangle \geq \mu \left\|x + td - x\right\|^2.$$

Simplify

$$t \left\langle F(x + td) - F(x), d\right\rangle \geq \mu t^2 \left\|d\right\|^2.$$

Divide both sides by $t^2 > 0$,

$$\left\langle \frac{F(x + td) - F(x)}{t}, d\right\rangle \geq \mu \left\|d\right\|^2.$$

Take the limit as $t \longrightarrow 0^+$

$$\langle J_F\,(x)\, d, d\rangle \geq \mu \left\|d\right\|^2.$$

This holds for all $d \in \mathbb{R}^n$.

Therefore,

$$\langle (J_F\,(x) - \mu I)\, d, d\rangle \geq 0.$$

So, $J_F\ (x) - \mu I$ is positive semi-definite for every $x \in C$.

$\Longleftarrow)$

We need to show that for all $x, y \in C$, that exists $\mu > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \mu \left\| x - y \right\|^2 .$$

Since F is continuously differentiable, by the mean value theorem (in several variables), for any $x, y \in C$, there exists

$z = y + \theta\,(x - y)$ for some $\theta \in (0, 1)$ such that

$$F(x) - F(y) = J_F\,(z)\,(x - y)\,.$$

Then,

$$\langle F(x) - F(y), x - y\rangle = \langle J_F\,(z)\,(x - y)\,, x - y\rangle$$

By the positive semi-definiteness of $J_F\ (z) - \mu I$ on C, we get

$$\langle (J_F\,(z) - \mu I)\,(x - y)\,, x - y\rangle \geq 0.$$

So,

$$\langle J_F\,(z)\,(x - y)\,, x - y\rangle \geq \mu \left\| x - y \right\|^2$$

Therefore,

$$\langle F(x) - F(y), x - y\rangle \geq \mu \left\| x - y \right\|^2$$

which implies the strong monotonicity of F on C. ■

2.2 Basic existence and uniqueness results of *(VIP)'s* solutions

We briefly present the main existence and uniqueness results for the solutions of (VIP), which will be useful later. We point out that the majority of the results are consequences of the generalization of the results obtained from optimization and complementarity fields.

Theorem 2.2.1 *Let C be a non-empty compact convex of $\mathbb{R}^n$ and F a continuous mapping on C, then $VIP\left(F, C\right)$ admits at least a solution.*

Proof. The proof uses a fixed point argument based on the projection operator that we recall here.

$$\overline{x} \in S \Longleftrightarrow \overline{x} = \ P_C\left(\overline{x} - \alpha F\left(\overline{x}\right)\right), \forall \alpha > 0.$$

Then, we apply Brouwer's Fixed Point Theorem that recall:

Any continuous application Φ from a non-empty compact convex C of an Euclidean space with values in C admits a fixed point.

Define a mapping $\Phi : C \longrightarrow C$ by

$$\Phi\left(x\right) = P_C\left(x - \alpha F\left(x\right)\right), \forall \alpha > 0,$$

We must check that

- C is non-empty, compact, and convex (given).
- Φ s continuous.

Since F is continuous by assumption.

The mapping $x \longmapsto x - \alpha F\left(x\right)$ is continuous (as a sum of continuous functions).

The projection operator P_C is Lipschitz continuous.

Since Φ is a composition of continuous functions (P_C and $x-\alpha F(x)$), it is continuous.

Therefore, by Brouwer's fixed point theorem, the continuous mapping Φ has at least one fixed point $\overline{x} \in C$ and consequently $\overline{x} \in S$. ■

Theorem 2.2.2 *Let C be a non-empty closed convex of $\mathbb{R}^n$ and F a continuous and monotone mapping on C. Then,*

$$\overline{x} \in S \iff \langle F(x), x-\overline{x}\rangle \geq 0, \forall x \in C. \tag{2.1}$$

Proof. Let $\overline{x} \in S \iff \langle F(\overline{x}), x-\overline{x}\rangle \geq 0, \forall x \in C.$

In other hand, we have F is monotone on C, that means

$\langle F(x) - F(y), x-y\rangle \geq 0, \forall x, y \in C.$

We put $y = \overline{x}$, thus

$$\begin{aligned}\langle F(x) - F(\overline{x}), x-\overline{x}\rangle \geq 0, \forall x \in C \quad &\Longrightarrow \langle F(x), x-\overline{x}\rangle \geq \langle F(\overline{x}), x-\overline{x}\rangle \geq 0, \forall x \in C \\ &\Longrightarrow \langle F(x), x-\overline{x}\rangle \geq 0, \forall x \in C.\end{aligned}$$

Conversly, let $\overline{x} \in C$ satisfaying

$$\langle F(x), x-\overline{x}\rangle \geq 0, \forall x \in C \tag{2.2}$$

and we take x any point of C, then we get $\overline{x} + \lambda(x-\overline{x}) \in C$ for $\lambda \in [0,1]$.

We put in (2.2) $x = \overline{x} + \lambda(x-\overline{x})$, so

$$\begin{aligned}\langle F(\overline{x}+\lambda(x-\overline{x})), \overline{x}+\lambda(x-\overline{x})-\overline{x}\rangle \geq 0 & \Longrightarrow \lambda\langle F(\overline{x}+\lambda(x-\overline{x})),(x-\overline{x})\rangle \geq 0 \\ & \Longrightarrow \langle F(\overline{x}+\lambda(x-\overline{x})),(x-\overline{x})\rangle \geq 0, \\ & \textit{since } \lambda \in [0,1] \\ & \Longrightarrow \lim_{\lambda \longrightarrow 0}\langle F(\overline{x}+\lambda(x-\overline{x})),(x-\overline{x})\rangle \geq 0 \\ & \Longrightarrow \left\langle F\left(\overline{x}+\lim_{\lambda \longrightarrow 0}\lambda(x-\overline{x})\right),(x-\overline{x})\right\rangle \geq 0, \\ & \textit{since } F \textit{ is continuous on } C \\ & \Longrightarrow \langle F(\overline{x}), x-\overline{x}\rangle \geq 0, \forall x \in C. \\ & \Longrightarrow \overline{x} \in S.\end{aligned}$$

■

Theorem 2.2.3 *Let C be a non-empty closed convex of $\mathbb{R}^n$ and F a continuous monotone mapping on C, then S (the solutions set of $VIP(F,C)$) is a closed convex set.*

Proof. We start by proving that S is closed.

Let $\{\overline{x}_n\}$ be a sequence included in S and converging to $\overline{x}$.

Therefore,

$$\begin{aligned}\overline{x}_n \in S & \Longrightarrow \langle F(\overline{x}_n), x-\overline{x}_n\rangle \geq 0, \forall x \in C \\ & \Longrightarrow \lim_{n \longrightarrow +\infty}\langle F(\overline{x}_n), x-\overline{x}_n\rangle \geq 0, \forall x \in C \\ & \Longrightarrow \left\langle F\left(\lim_{n \longrightarrow +\infty}\overline{x}_n\right), x-\lim_{n \longrightarrow +\infty}\overline{x}_n\right\rangle \geq 0, \forall x \in C, \\ & \textit{since } F \textit{ is continuous on } C \\ & \Longrightarrow \langle F(\overline{x}), x-\overline{x}\rangle \geq 0, \forall x \in C \\ & \Longrightarrow \overline{x} \in S\end{aligned}$$

We conclude that S is a closed set.

Now, we move to demostrate that S is convex.

Suppose that $\overline{x}_1$, $\overline{x}_2$ two points in S and we search to show that

$$(1-\lambda)\,\overline{x}_1+\lambda\overline{x}_2\in S\ with\ \lambda\in[0,1].$$

We know that $\overline{x}_1\in S\Longrightarrow\overline{x}_1\in C$ $(S\subset C)$ and the same thing for $\overline{x}_2$.

So $(1-\lambda)\,\overline{x}_1+\lambda\overline{x}_2\in C,\ \lambda\in[0,1]$, since C is concex.

According to the previous theorem (Theorem 2.2.2), we have

$$F\ \text{ is monotone on } C \text{ and } \overline{x}_1,\overline{x}\in S\Longleftrightarrow\begin{cases}\langle F(x),x-\overline{x}_1\rangle\geq 0\\ \langle F(x),x-\overline{x}_2\rangle\geq 0\end{cases},\forall x\in C.$$

Then, we get

$$\begin{aligned}&(1-\lambda)\,\langle F(x),x-\overline{x}_1\rangle+\lambda\,\langle F(x),x-\overline{x}_2\rangle\geq 0,\forall x\in C\\ &\Longleftrightarrow\langle F(x),(1-\lambda)\,(x-\overline{x}_1)\rangle+\langle F(x),\lambda\,(x-\overline{x}_2)\rangle\geq 0,\forall x\in C\\ &\Longleftrightarrow\langle F(x),(1-\lambda)\,(x-\overline{x}_1)+\lambda\,(x-\overline{x}_2)\rangle\geq 0,\forall x\in C\\ &\Longleftrightarrow\langle F(x),x-(1-\lambda)\,\overline{x}_1+\lambda\overline{x}_2\rangle\geq 0,\forall x\in C\\ &\Longleftrightarrow(1-\lambda)\,\overline{x}_1+\lambda\overline{x}_2\in S\ (Using\ Theorem\ 2.2.2).\end{aligned}$$

Hence, S is a convex set. ■

Remark 2.2.1 *The two last results remain true in the case where F is only pseudomonotone on C.*

Proposition 2.2.1 *Let C be a non-empty closed convex of $\mathbb{R}^n$ and F a continuous pseudomonotone mapping on C, then*

$$S = \{\overline{x} \in C / \langle F(x), x - \overline{x} \rangle \geq 0, \forall x \in C\} = A$$
$$= \underset{x \in C}{\cap} \{\overline{x} \in C / \langle F(x), x - \overline{x} \rangle \geq 0\}.$$

Proof. Suppose that $\overline{x} \in S$ and we recall F is pseudomonotone on C, so we have

$$\langle F(y), x - y \rangle \geq 0 \Longrightarrow \langle F(x), x - y \rangle \geq 0, \forall x, y \in C$$

We put $y = \overline{x} \in S \subset C$, thus $\langle F(x), x - \overline{x} \rangle \geq 0, \forall x \in C$.

Consequently, $\overline{x} \in A$,that implies $S \subseteq A$.

Reciprocally, we consider that $\overline{x} \in A$ and x is any point of C.

We put $y = \overline{x} + \lambda (x - \overline{x}) \in C$ (C is convex set). Since $\overline{x} \in A$ and $y \in C$, we obtain

$$\langle F(\overline{x} + \lambda (x - \overline{x})), \overline{x} + \lambda (x - \overline{x}) - \overline{x} \rangle \geq 0$$

In a similar way, we demonstrate that $A \subseteq S$ by following the steps presented in the proof of the Theorem 2.2.2.

Finally, we conclude that $S = A$.

Here, we give another method to prove that S is a closed convex set.

We fix $x \in C$, so $\{\overline{x} \in C / \langle F(x), x - \overline{x} \rangle \geq 0\}$ represents a half space which is closed convex set. Since S is an intersection of a set of a half spaces, therefore S is also a closed convex set. ■

Theorem 2.2.4 *Let C be a non-empty closed convex of $\mathbb{R}^n$ and F a continuous coercive mapping on C, then S is non-empty compact set.*

We recall that F is said to be coercive on $C \iff$ there is $x_0 \in C$ and $\alpha \geq 0$ such that

$$\lim_{\|x\| \longrightarrow +\infty} \frac{\langle F(x), x - x_0 \rangle}{\|x\|^{\alpha}} > 0.$$

Theorem 2.2.5 *Let C be a non-empty closed convex of $\mathbb{R}^n$ and F a strictly monotone mapping on C. If $VIP(F, C)$ admits solutions, then the solution is unique.*

Proof. Suppose that $\overline{x}_1$, $\overline{x}_2$ are two different solutions for $VIP(F, C)$.

We have,

$$\begin{aligned} \overline{x}_1 \in S \quad &\iff \langle F(\overline{x}_1), x - \overline{x}_1 \rangle \geq 0, \forall x \in C \\ \overline{x}_2 \in S \quad &\iff \langle F(\overline{x}_2), x - \overline{x}_2 \rangle \geq 0, \forall x \in C \end{aligned}$$

We take $x = \overline{x}_2$ in the first inequality and $x = \overline{x}_1$ in the second inequality, that implies

$$\begin{cases} \langle F(\overline{x}_1), \overline{x}_2 - \overline{x}_1 \rangle \geq 0 \\ \langle F(\overline{x}_2), \overline{x}_1 - \overline{x}_2 \rangle \geq 0 \end{cases}$$

Consider the sum of the two inequalities, we obtain

$$\langle -F(\overline{x}_1) + F(\overline{x}_2), \overline{x}_1 - \overline{x}_2 \rangle \geq 0 \Longrightarrow \langle F(\overline{x}_1) - F(\overline{x}_2), \overline{x}_1 - \overline{x}_2 \rangle \leq 0.$$

But F is stricly monotone mapping on C i.e

$$\langle F(x) - F(y), x - y \rangle > 0, \forall x, y \in C \text{ and } x \neq y$$

Thus, we get a contradiction, so the two solutions $\overline{x}_1$, $\overline{x}_2$ must be identical $(\overline{x}_1 = \overline{x}_2)$.

■

Theorem 2.2.6 *Let C be a non-empty closed convex of $\mathbb{R}^n$ and F a continuous strongly monotone mapping on C. Then, the solution of $VIP\,(F, C)$ exists and it is unique.*

Proof. Since F is strongly monotone mapping on C implies that F is strictly monotone and coercive on C, we deduce directly by using Theorems 2.2.5 and 2.2.6 the result of this theorem. ■

Corollary 2.2.1 *Suppose that $\overline{x}$ is a solution of $VIP\,(F, C)$, we have*

If $\overline{x} \in int\,(C)$, then $F\,(\overline{x}) = 0$.

Proof. Since $\overline{x} \in int\,(C)$, there exists an open ball $B\,(\overline{x}, \varepsilon) \subset C$ for some $\varepsilon > 0$. For any arbitrary vector $d \in \mathbb{R}^n$, we can choose $t > 0$ sufficiently small such that $x = \overline{x} - td \in B\,(\overline{x}, \varepsilon) \subset C$. (This is because $\overline{x}$ is an interior point, so we can move a small amount in any direction and remain inside C).

Now, since $\overline{x}$ is a solution to $VIP(F, C)$, we have

$$\langle F\,(\overline{x})\,, x - \overline{x}\rangle \geq 0$$

Substituting $x = \overline{x} - td$, we get

$$\langle F\,(\overline{x})\,, \overline{x} - td - \overline{x}\rangle = -t\,\langle F\,(\overline{x})\,, d\rangle \geq 0$$

Dividing both sides by $-t$, we obtain

$$\langle F\,(\overline{x})\,, d\rangle \leq 0$$

But d was an arbitrary vector in $\mathbb{R}^n$, so we can take $d = F\,(\overline{x})$.

Thus, the last inequality becomes

$$\langle F(\overline{x}), F(\overline{x})\rangle = \|F(\overline{x})\|^2 \leq 0.$$

But we know that $\|F(\overline{x})\|^2 \geq 0$.

Therefore, $F(\overline{x})$ must equal 0. ■

Exercises

Exercise 2.2.1 1. *Show that P_C, the orthogonal projection operator onto C, where C is a non-empty closed convex set of $\mathbb{R}^n$, is a monotone mapping from $\mathbb{R}^n$ to C.*

Solution 2.2.1 *P_C is monotone on $\mathbb{R}^n$ if* $\langle P_C(x) - P_C(y), x - y\rangle \geq 0, \forall x, y \in \mathbb{R}^n$.

We recall the classical projection inequality:

$$\langle P_C(u) - u, v - P_C(u)\rangle \geq 0, \forall u \in \mathbb{R}^n, \forall v \in C.$$

We put $u = x, v = P_C(y)$, *then*

$$\langle P_C(x) - x, P_C(y) - P_C(x)\rangle \geq 0. \tag{(1)}$$

And we put $u = y, v = P_C(x)$, *then*

$$\langle P_C(y) - y, P_C(x) - P_C(y)\rangle \geq 0. \tag{(2)}$$

By addition (1) *and* (2), *we obtain*

$\langle -P_C(x) + x, P_C(x) - P_C(y)\rangle + \langle P_C(y) - y, P_C(x) - P_C(y)\rangle \geq 0 \Longleftrightarrow$

$\langle x - y - (P_C(x) - P_C(y)), P_C(x) - P_C(y)\rangle \geq 0 \Longleftrightarrow$

$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|(P_C(x) - P_C(y))\|^2 \geq 0.$

Exercise 2.2.2 *Let F be a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that*

$$F(x) = Ax = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 2 \\ 1 & 1 & \theta \end{pmatrix}$$

For what values of θ this mapping is strictly monotone on $\mathbb{R}^n$.

Exercise 2.2.3 *Consider the the mapping $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by*

$$F(x, y) = \left(x^3 - x, y\right).$$

1. *Show that F isn't monotone on $\mathbb{R}^2$.*
2. *Let $C = \{x \in \mathbb{R}^2 / x \geq 1\}$, show that F is monotone on C.*

Exercise 2.2.4 *We consider a $VIP(F, C)$ where the mapping F is pseudomonotonic on C and S is the set of its solutions assumed to be non-empty.*

$$\begin{aligned} S &= \{\overline{x} \in C / \langle F(x), x - \overline{x}\rangle \geq 0, \forall x \in C\} = A \\ &= \bigcap_{x \in C} \{\overline{x} \in C / \langle F(x), x - \overline{x}\rangle \geq 0\}. \end{aligned}$$

1. *Show that $S = A$.*
2. *Deduce that S is a closed convex set.*

Exercise 2.2.5 *Let F be a mapping ($F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$), A a matrix of $\mathbb{R}^{m\times n}$ and $a \in \mathbb{R}^m$.*

We define the mapping Q ($Q : \mathbb{R}^{n\times m} \longrightarrow \mathbb{R}^{n\times m}$) as follows

$$Q(u) = Q(x.y) = \begin{pmatrix} F(x) - A^T y \\ Ax - a \end{pmatrix} \text{ with } u = (x; y) \in \mathbb{R}^{n\times m}.$$

1. *Show that F is monotone on $\mathbb{R}^n \Longleftrightarrow Q$ is monotone on $\mathbb{R}^{n\times m}$.*
2. *Does the equivalence remain true in the case of strong monotonicity?*

Exercise 2.2.6 *Let C be a non-empty closed convex set of $\mathbb{R}^n$, we have*

$$\overline{x} = P_C(x) \Longleftrightarrow \overline{x} = \underset{y\in C}{\arg\min} \left(\frac{1}{2} \|y - x\|^2 = f(y) \right).$$

Determine the explicit formulas of the projection in the following cases

1. $C = \left\{ y \in \mathbb{R}^n / \alpha^T y = \beta \right\}, \alpha \neq 0.$
2. $C = \left\{ y \in \mathbb{R}^n / \alpha^T y \leq \beta \right\}, \alpha \neq 0.$
3. $C = \mathbb{R}^2_+.$
4. $C = \prod_{i=1}^{n} [a_i, b_i].$
5. $C = \left\{ y \in \mathbb{R}^n / \|y\|^2 \leq r^2 \right\}, r > 0.$

Chapter 3

Projection methods for solving (VIP)

The iterative algorithms developed to solve the variational inequalities problem (VIP) are systematically derived from its underlying mathematical characterizations, which are expressed through various equivalent formulations. These latter naturally give rise to several fundamental computational strategies. Notable examples include

- Projection methods, [14, 15, 17, 18, 20, 24, 25, 27, 28]
- Projection and contraction methods [3, 9, 12, 13]
- Optimization methods, [8]
- Penalty methods [1, 19],
- Interior point methods, [4]
- Decomposition methods, [10]
- Solving (VIP) as a system of equations [21], ...

3.1 Projection methods

Projection methods are conceptually simple numerical techniques for solving monotone variational inequalities problems. All methods in this class share two key characteristics. First, their implementation requires the ability to efficiently compute projections onto the closed convex set C. This requirement limits their applicability, particularly when projections are computationally expensive or complex. Second, these methods are derivative-free and involve no complex computations beyond the projection itself. Consequently, for sets where projection is efficient, these methods are extremely simple and cheap, making them suitable for very large-scale problems. We now present a historical and technical overview of the first projection methods introduced for the (VIP).

We provide a detailed convergence analysis for the first two methods. For the others, we outline the fundamental principles, present the associated algorithms, and state the main convergence results without proof, as the full details are lengthy and complex. The aim of this second part is to present a brief introduction of the subsequent developments proposed after the basic projection method and the method of Korpolevich.

3.1.1 Basic projection method

The basic projection method is an elementary procedure is predicated on the fixed-point theorem. Although this simple method is of limited practical use, it is recognized as the developmental origin of all subsequent algorithms in the class of projection methods.

Recall that

$$\overline{x} \in S \Longleftrightarrow \overline{x} = P_C\left(\overline{x} - \alpha F\left(\overline{x}\right)\right), \qquad \alpha > 0. \tag{3.1}$$

From this result, we obtain the iterative scheme of the basic projection method

$$x^{k+1} = P_C\left(x^k - \alpha F\left(x^k\right)\right), \qquad \alpha > 0. \tag{3.2}$$

Similarly, we can see that this scheme is identical to that of the projected gradient method known in otpimization with constraints.

Consider the following optimization problem

$$\begin{cases} \min f(x) \\ x \in C. \end{cases}$$

Assume that C is a non-empty subset of $\mathbb{R}^n$ and f is a differentiable function on C.

If $C = \mathbb{R}^n$, the gradient method is

$$x^{k+1} = x^k - \alpha_k \nabla f\left(x^k\right), \tag{3.3}$$

where α_k is the displacement step and $\nabla f\left(x^k\right)$ the descent direction.

If $C \subsetneq \mathbb{R}^n$ is a closed convex, then the gradient method becomes the projected gradient method given by

$$x^{k+1} = P_C\left(x^k - \alpha_k \nabla f\left(x^k\right)\right). \tag{3.4}$$

The purpose of introducing the projection into this method is to ensure that the iterates generated x^k always remain in the set C. In addition, we have assumed that C is a closed convex to ensure that the projection onto this set exists and is unique.

If we take in (4) (any application) instead of ∇f in the scheme of the projected gradient method, we find the scheme of the basic projection procedure.

Algorithm 1

Initialization: $k = 0$, $x^0 \in C$ and $\varepsilon > 0$ a given precision.

Stopping criterion: If $\left\| x^k - P_C\left(x^k - \alpha F\left(x^k\right)\right)\right\| \leq \varepsilon$, stop $x^k \in S$.

Iteration: Otherwise, we put $x^{k+1} = P_C\left(x^k - \alpha F\left(x^k\right)\right)$,

$k = k + 1$ and repeat.

End of algorithm.

The algorithm depends largely on the choice of step size α. We give the following theorem concerning the values of α for which Algorithm 1 converges.

Theorem 3.1.1 *Let $\varnothing \neq C \subseteq \mathbb{R}^n$ be a closed convex and F an application $(F : \mathbb{R}^n \to \mathbb{R}^n)$ satisfying the strong monotonicity and Lipschitz continuity conditions, then the sequence $\left\{x^k\right\}$ generated by Algorithm 1 converges to the unique solution of (VIP) which is the fixed point of the application $P_C(I - \alpha F)$ for $\alpha \in \left]0, \frac{2\mu}{L^2}\right[$.*

We recall that

- F is said to be Lipschitz continuous on $C \iff$ There is $L > 0$,

  $\|F(x) - F(y)\| \leq L\|x - y\|, \forall x, y \in C.$

- F is said to be strongly monotone on $C \iff$ There is $\mu > 0$,

  $\langle F(x) - F(y), x - y\rangle \geq \mu\|x - y\|^2, \forall x, y \in C.$

Before giving the proof of this theorem, we need the following results.

Proposition 3.1.1 *Let $\varnothing \neq C \subseteq \mathbb{R}^n$ be a closed convex. Then, P_C (the orthogonal projection on C) is a non-expansive application, in other words*

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \forall x, y \in \mathbb{R}^n.$$

Definition 3.1.1 *Let K be a real number such that $0 < K < 1$ and A be an application of D in itself ($\varnothing \neq D \subseteq \mathbb{R}^n$). We say that A is a contraction application on D of constant K if for all two distinct points of D, we have*

$$\|A(x) - A(y)\| \leq K \|x - y\|, \forall x, y \in D.$$

Proposition 3.1.2 *If A is a contraction application, then A has a fixed point and for any $x^0 \in D$, the sequence generated by the scheme $\left\{x^{k+1} = A\left(x^k\right)\right\}$ converges to this fixed point.*

So, if we show that $P_C(I - \alpha F)$ is contraction, then this application has a fixed point $\overline{x}$ which is in this case the unique solution of (VIP). Moreover, the sequence $\left\{x^{k+1} = P_C\left(x^k - \alpha F\left(x^k\right)\right)\right\}$ (the iterative scheme of Algorithm 1) converges to $\overline{x}$.

Proof. Let $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}
\|P_C(x - \alpha F(x)) - P_C(y - \alpha F(y))\|^2 &\leq \|x - y - \alpha(F(x) - F(y))\|^2 \\
&= \|x - y\|^2 + \alpha^2 \|(F(x) - F(y))\|^2 \\
&-2\alpha \langle x - y, F(x) - F(y)\rangle \\
&\textit{Unsing the assumptions of strong} \\
&\textit{monotonicity and Lipschitz continuity} \\
&\leq \|x - y\|^2 + \alpha^2 L^2 \|x - y\|^2 - 2\alpha\mu \|x - y\|^2 \\
&= (1 + \alpha^2 L^2 - 2\alpha\mu) \|x - y\|^2.
\end{aligned}$$

So,

$$\|P_C(x - \alpha F(x)) - P_C(y - \alpha F(y))\| \leq \sqrt{(1 + \alpha^2 L^2 - 2\alpha\mu)} \|x - y\|$$

Therefore, for $P_C(I - \alpha F)$ to be contraction, it suffices that

$\sqrt{(1 + \alpha^2 L^2 - 2\alpha\mu)} < 1$.

$$(1 + \alpha^2 L^2 - 2\alpha\mu) < 1 \Longleftrightarrow \alpha\left(\alpha L^2 - 2\mu\right) < 0 \Longleftrightarrow \begin{cases} (\alpha L^2 - 2\mu) < 0 \\ \alpha > 0. \end{cases}$$

Furthermore, to ensure the expression inside the square root is positive, we need

$(1+\alpha^2L^2-2\alpha\mu)>0$. This quadratic equation in α is always positive for $\alpha\in\left]0,\frac{2\mu}{L^2}\right[$ because its discriminant is $\Delta=4\left(\mu^2-L^2\right)<0$ since $\mu\leq L$ (a consequence of the assumptions). ■

Remark 3.1.1 *If the strong monotonicity assumption is not satisfied, then Algorithm 1 is likely to diverge for any choice of α, even a variable choice.*

Given that the assumptions of $(StroM)$ and (LC) for the application are considered too strong, it was desirable to have an algorithm that does not require the calculation of these constants, where they have been replaced by the co-coercivity condition.

Recall that F is said to be co-coercive of constant $k>0$ on C if and only if

$$\langle F(x)-F(y),x-y\rangle\geq k\left\|F(x)-F(y)\right\|^2,\forall x,y\in C.$$

In this case, Algorithm 2 (the algorithm of this method) is the same as the basic projection procedure where the choice of the step sequence $\{\alpha_k\}$ is given by the following theorem.

Theorem 3.1.2 *Let $\varnothing\neq C\subseteq\mathbb{R}^n$ a closed convex and F a co-coercive application of constant k on C. Assume that the set S is nonempty.*

If $0\leq inf(\alpha_k)\leq sup(\alpha_k)\leq 2k$, then Algorithm 2 generates a sequence $\{x^k\}$ that converges to a point $\overline{x}\in S$.

3.1.2 Korpolevich method

This is a variant of the basic projection method proposed by Korpolevich in 1976, also known as the extragradient method. This method converges if F is monotone and Lipschitz continuous on C, but it requires a second projection to be calculated at each iteration.

The origin of the name extragradient comes from the case where (VIP) is equivalent to an optimization problem, we know that its optimality condition is

$$\langle \nabla f(\overline{x}), x - \overline{x} \rangle \geq 0, \forall x \in C.$$

Therefore, the extra-evaluation (2 times) of F corresponds to the extra-evaluation of the gradient of f.

Algorithm 3

Initialization: $k = 0, x^0 \in C$ and $\varepsilon > 0$ a given precision.

Stopping criterion: If $\left\| x^k - P_C\left(x^k - \alpha F\left(x^k\right)\right)\right\| \leq \varepsilon$, stop $x^k \in S$.

Iteration: Otherwise, we calculate

$$\begin{cases} y^k = P_C\left(x^k - \alpha F\left(x^k\right)\right) \\ x^{k+1} = P_C\left(x^k - \alpha F\left(y^k\right)\right). \end{cases}$$

We put $k = k + 1$ and repeat.

End of algorithm.

Theorem 3.1.3 *Let $\varnothing \neq C \subseteq \mathbb{R}^n$ a closed convex and F a monotone application and satisfies the Lipschitz condition of constant $L > 0$ on C. If $\alpha \in \left]0, \frac{1}{L}\right[$, then the sequence generated by Algorithm 3 converges to a point $\overline{x} \in S$.*

Proof. We begin the proof by demonstrate that the sequence $\left\{\left\|x^k - \overline{x}\right\|^2, \overline{x} \in S\right\}$ is nonincreasing.

We have the following inequality

$\|P_C(u) - v\|^2 \leq \|u - v\|^2 - \|u - P_C(u)\|^2, \forall u \in \mathbb{R}^n, v \in C.$

Let $u = x^k - \alpha F\left(y^k\right)$, $x^{k+1} = P_C(u)$ and $v = \overline{x} \in S \subset C.$

Therefore,

$$\begin{aligned}\left\|x^{k+1}-\overline{x}\right\|^2 &\leq \left\|x^k-\alpha F\left(y^k\right)-\overline{x}\right\|^2-\left\|x^k-\alpha F\left(y^k\right)-x^{k+1}\right\|^2\\ &\leq \left\|x^k-\overline{x}\right\|^2+\alpha^2\left\|F\left(y^k\right)\right\|^2-2\alpha\left\langle x^k-\overline{x},F\left(y^k\right)\right\rangle-\left\|x^k-x^{k+1}\right\|^2\\ &\quad -\alpha^2\left\|F\left(y^k\right)\right\|^2+2\alpha\left\langle x^k-x^{k+1},F\left(y^k\right)\right\rangle\\ &\leq \left\|x^k-\overline{x}\right\|^2-\left\|x^k-x^{k+1}\right\|^2+2\alpha\left\langle -x^k+\overline{x}+x^k-x^{k+1},F\left(y^k\right)\right\rangle\\ &\leq \left\|x^k-\overline{x}\right\|^2-\left\|x^k-x^{k+1}\right\|^2+2\alpha\left\langle \overline{x}-x^{k+1},F\left(y^k\right)\right\rangle \qquad (*)\end{aligned}$$

On the other hand, we have

$$\left\langle \overline{x}-x^{k+1},F\left(y^k\right)\right\rangle=\left\langle \overline{x}-y^k+y^k-x^{k+1},F\left(y^k\right)\right\rangle=\left\langle \overline{x}-y^k,F\left(y^k\right)\right\rangle+\left\langle y^k-x^{k+1},F\left(y^k\right)\right\rangle.$$

Using the monotonicity of F, we obtain $\langle F(x),x-\overline{x}\rangle\geq 0,\forall x\in C.$

Then, $\left\langle \overline{x}-y^k,F\left(y^k\right)\right\rangle\leq 0$, for $y^k\in C$.

Consequently, $\left\langle \overline{x}-x^{k+1},F\left(y^k\right)\right\rangle\leq\left\langle y^k-x^{k+1},F\left(y^k\right)\right\rangle.$

If we replace inequality $(*)$, we find that

$$\begin{aligned}\left\|x^{k+1}-\overline{x}\right\|^2 &\leq \left\|x^k-\overline{x}\right\|^2-\left\|x^k-x^{k+1}\right\|^2+2\alpha\left\langle y^k-x^{k+1},F\left(y^k\right)\right\rangle\\ &\leq \left\|x^k-\overline{x}\right\|^2-\left\|x^k-y^k+y^k-x^{k+1}\right\|^2+2\alpha\left\langle y^k-x^{k+1},F\left(y^k\right)\right\rangle\\ &\leq \left\|x^k-\overline{x}\right\|^2-\left\|x^k-y^k\right\|^2-\left\|y^k-x^{k+1}\right\|^2-2\left\langle x^k-y^k,y^k-x^{k+1}\right\rangle\\ &\quad +2\alpha\left\langle y^k-x^{k+1},F\left(y^k\right)\right\rangle\\ &\leq \left\|x^k-\overline{x}\right\|^2-\left\|x^k-y^k\right\|^2-\left\|y^k-x^{k+1}\right\|^2\\ &\quad +2\left\langle y^k-x^{k+1},y^k-x^k+\alpha F\left(y^k\right)\right\rangle\\ &\leq \left\|x^k-\overline{x}\right\|^2-\left\|x^k-y^k\right\|^2-\left\|y^k-x^{k+1}\right\|^2\\ &\quad +2\left\langle x^{k+1}-y^k,x^k-y^k-\alpha F\left(y^k\right)\right\rangle \ (**)\end{aligned}$$

We have

$$\begin{aligned}\left\langle x^{k+1}-y^k,x^k-y^k-\alpha F\left(y^k\right)\right\rangle &=\left\langle x^{k+1}-y^k,x^k-y^k-\alpha F\left(y^k\right)+\alpha F\left(x^k\right)-\alpha F\left(x^k\right)\right\rangle\\ &=\left\langle x^{k+1}-y^k,x^k-\alpha F\left(x^k\right)-y^k\right\rangle\\ &\quad +2\left\langle x^{k+1}-y^k,\alpha F\left(x^k\right)-\alpha F\left(y^k\right)\right\rangle.\end{aligned}$$

In addition, using the classical projection inequality, it implies

$\left\langle x^k-\alpha F\left(x^k\right)-y^k,x^{k+1}-y^k\right\rangle\leq 0$, hence $y^k=P_C\left(x^k-\alpha F\left(x^k\right)\right).$

We return back to our inequality $(**)$,

$$\left\|x^{k+1}-\overline{x}\right\|^{2} \leq \left\|x^{k}-\overline{x}\right\|^{2}-\left\|x^{k}-y^{k}\right\|^{2}-\left\|y^{k}-x^{k+1}\right\|^{2}+2\left\langle \alpha\left(F\left(x^{k}\right)-F\left(y^{k}\right)\right), x^{k+1}-y^{k}\right\rangle$$

We also have the following result, $(a-b)^{2} \geq 0 \Longleftrightarrow a^{2}+b^{2} \geq 2ab$, then

$$\begin{aligned}
\left\|x^{k+1}-\overline{x}\right\|^{2} \leq{} & \left\|x^{k}-\overline{x}\right\|^{2}-\left\|x^{k}-y^{k}\right\|^{2}-\left\|y^{k}-x^{k+1}\right\|^{2}+\left\|y^{k}-x^{k+1}\right\|^{2} \\
& +\alpha^{2}\left\|F\left(x^{k}\right)-F\left(y^{k}\right)\right\|^{2} \\
\leq{} & \left\|x^{k}-\overline{x}\right\|^{2}-\left\|x^{k}-y^{k}\right\|^{2}+\alpha^{2}L^{2}\left\|x^{k}-y^{k}\right\|^{2} \\
\leq{} & \left\|x^{k}-\overline{x}\right\|^{2}-\left(1-\alpha^{2}L^{2}\right)\left\|x^{k}-y^{k}\right\|^{2} \\
\leq{} & \left\|x^{k}-\overline{x}\right\|^{2}.
\end{aligned}$$

In other words, the distance between x^k and S decreases at each iteration. This is the Fejér monotonicity of the sequence $\left\{x^{k}\right\}$ with respect to S.

As consequences of Fejér monotonicity, the inequality above, we can conclude:

- The sequence $\left\{\left\|x^{k}-\overline{x}\right\|^{2}, \overline{x}\in S\right\}$ is non-increasing (and hence convergent to some limit $L\left(\overline{x}\right)$) for any $\overline{x}\in S$.
- The sequence $\left\{x^{k}\right\}$ is bounded.
- $\lim\limits_{k\longrightarrow+\infty}\left\|x^{k}-y^{k}\right\|=0.$

Since $\left\{x^{k}\right\}$ is bounded, it has a convergent subsequence $\left\{x^{k_j}\right\}$, that its limit is denote by $\widehat{x}$ (cluster point).

We use the fact that $\lim\limits_{k\longrightarrow+\infty}\left\|x^{k}-y^{k}\right\|=0$ to show that any cluster point $\widehat{x}$ of $\left\{x^{k}\right\}$ is also a cluster point of $\left\{y^{k}\right\}$.

According to the first expression given in Korpolevich's scheme, then passing to the limit along subsequences $\left\{x^{k_j}\right\}, \left\{y^{k_j}\right\}$ and using the continuity of F and projection operator, we obtain

$$\lim_{j\longrightarrow+\infty} y^{k_j} = P_C\left(\lim_{j\longrightarrow+\infty} x^{k_j} - \alpha F\left(\lim_{j\longrightarrow+\infty} x^{k_j}\right)\right)$$
$$\widehat{x} = P_C\left(\widehat{x} - \alpha F\left(\widehat{x}\right)\right),$$

thus, $\widehat{x} \in S$.

Therefore, we conclude if a sequence $\left\{\left\|x^k - \overline{x}\right\|^2, \overline{x} \in S\right\}$ converges, then the limit of every convergent subsequence must be the same as the limit of the whole sequence.

The whole sequence $\left\{\left\|x^k - \widehat{x}\right\|^2\right\}$ converges to $L\left(\widehat{x}\right)$.

The subsequence $\left\{\left\|x^{k_j} - \widehat{x}\right\|^2\right\}$ converges to 0.

Therefore, $L\left(\widehat{x}\right)$ must be equal to 0.

Which is equivalent to

$$\lim_{j\longrightarrow+\infty} x^k = \widehat{x} \in S.$$

■

Remark 3.1.2 *- It should be noted that monotonicity can be replaced by pseudomonotonicity in Algorithm 3.*

- If the Lipschitz continuity assumption is not satisfied in turn, the convergence of the Korpolevich algorithm is not guaranteed. Sometimes it is difficult to verify this property for any mapping, even if the constant L is known, it may be very large, then the interval $]0,(1/L)[$ *will be too small to give suitable values for the step* α *which will lead to a very slow convergence. The following algorithm was proposed by A. N. Iusem in 1994 in order to overcome these obstacles.*

3.1.3 Modified Korpolevich method

This is a variant of Korpolevich's method for which the mapping F is only continuous and monotone on C, and its algorithm is given below.

Algorithm 4

Initialization: $k=0$, $x^0 \in C$, $\varepsilon_1, \varepsilon_2 \in \left]0,1\right[$, $\alpha_{\max} > 0$ and $\varepsilon > 0$ a given precision.

Stopping criterion: Calculate $z^k = P_C\left(x^k - \alpha_{\max} F\left(x^k\right)\right)$

If $\left\|x^k - z^k\right\| \leq \varepsilon$, stop $x^k \in S$.

$$\begin{cases} \text{Else } \left\|F\left(z^k\right) - F\left(x^k\right)\right\| \leq \frac{\left\|z^k - x^k\right\|}{2\alpha_{\max}\left\|F\left(x^k\right)\right\|^2}, \text{ we take } \alpha_k = \alpha_{\max} \\ \text{Else if } \alpha \in \left]0, \alpha_{\max}\right[\text{ such that} \\ \varepsilon_1 \frac{\left\|z^k - x^k\right\|}{2\alpha_{\max}\left\|F\left(x^k\right)\right\|^2} \leq \left\|F\left(P_C\left(x^k - \alpha F\left(x^k\right)\right)\right) - F\left(x^k\right)\right\| \leq \varepsilon_2 \frac{\left\|z^k - x^k\right\|}{2\alpha_{\max}\left\|F\left(x^k\right)\right\|^2} \\ \text{Then, we take } \alpha_k = \alpha \text{ and we calculate } y^k = P_C\left(x^k - \alpha_k F\left(x^k\right)\right). \end{cases}$$

Iteration:

$$\begin{cases} x^{k+1} = P_C\left(x^k - \lambda_k F\left(y^k\right)\right) \\ \text{where } \lambda_k = \frac{\left\langle F\left(y^k\right), x^k - y^k\right\rangle}{\left\|F\left(y^k\right)\right\|^2} \end{cases}.$$

We take $k = k+1$ and repeat.

End of algorithm.

The step α_k is determined by a linear search procedure known as relaxation inspired from optimization methods. Moreover, the expression of the step λ_k ensures that the hyperplane H_k of normal $F\left(y^k\right)$ is passing through y^k separates the iterate x^k from the set S and $x^k - \lambda_k F\left(y^k\right)$ is the projection of x^k onto H_k. The hyperplane H_k is well defined and given by $H_k = \left\{x \in \mathbb{R}^n / \left\langle F\left(y^k\right), x - y^k\right\rangle = 0\right\}$.

Notion of separation

Let be the hyperplane $H = \left\{x \in \mathbb{R}^n / \alpha^T x = \beta\right\}$ ($\alpha \in \mathbb{R}^n_*$ and $\beta \in \mathbb{R}$).

- H separates $\mathbb{R}^n$ into two closed halfspaces $H_+ = \left\{x \in \mathbb{R}^n / \alpha^T x \geq \beta\right\}$ and $H_- = \left\{x \in \mathbb{R}^n / \alpha^T x \leq \beta\right\}$.

- If there are any two sets C_1 and C_2 of $\mathbb{R}^n$ then they are said to be strictly separate if $C_1 \subset H_+$ and $C_2 \subset H_-$.

- Particular case: if C_1 and C_2 are convex.

- Moreover, we can always separate a point y from C such that C is a non-empty closed convex and $y \notin C$.

We now show that we can separate the iterate x^k generated by Algorithm 4 from the set of solutions S.

In fact, S is a closed convex set (according to the Theorem 2.2.3). On the other hand, if we consider $\overline{x}$ to be any solution of (VIP), then we have $\langle F(\overline{x}), x - \overline{x}\rangle \geq 0, \forall x \in C$.

Moreover, applying Theorem 2.2.2, we will have $\langle F(x), x - \overline{x}\rangle \geq 0, \forall x \in C$.

We take $x = y^k \in C$ since $y^k = P_C\left(x^k - \alpha_k F\left(x^k\right)\right)$, then $\left\langle F\left(y^k\right), \overline{x} - y^k\right\rangle \leq 0$.

Now we just need to show that $\left\langle F\left(y^k\right), x^k - y^k\right\rangle > 0$.

For this, we note that $\lambda_k = \frac{\left\langle F\left(y^k\right), x^k - y^k\right\rangle}{\left\|F\left(y^k\right)\right\|^2} > 0$, which implies that $\left\langle F\left(y^k\right), x^k - y^k\right\rangle > 0$.

Or it can be done as follows

$$
\begin{aligned}
P_{H_k}\left(x^k\right) = x^k - \lambda_k F\left(y^k\right) \in H_k \quad &\Longleftrightarrow \left\langle F\left(y^k\right), x^k - \lambda_k F\left(y^k\right) - y^k\right\rangle = 0 \\
&\Longleftrightarrow \left\langle F\left(y^k\right), x^k - y^k\right\rangle = \lambda_k \left\|F\left(y^k\right)\right\|^2 > 0.
\end{aligned}
$$

To find the expression for λ_k , simply apply the projection formula for the point y onto the hyperplane H by taking $y = x^k$, $\alpha = F\left(y^k\right)$ and $\beta = F\left(y^k\right)^T y^k$, we obtain

$$P_{H_k}\left(x^k\right)=x^k-\frac{\left\langle F\left(y^k\right),x^k-y^k\right\rangle}{\left\|F\left(y^k\right)\right\|^2}F\left(y^k\right).$$

Remark 3.1.3 *It is obvious that the numerical behaviour of this type of these methods depends largely on the structure of the set C. With the exception of a few convex sets ($\mathbb{R}_+^n$, boxes, spheres,...), we note that the projections are given by explicit formulae, whereas in practice there are several interesting applications where calculating the projections is very costly. Furthermore, in order to determine the displacement step α_k, we must use a relaxation procedure where it is necessary to evaluate $P_C\left(x^k-\alpha F\left(x^k\right)\right)$. That means, if the procedure at iteration k requires m_k steps, then we need to evaluate m_k projections onto C in order to determine y^k, plus another projection for the calculation of x^{k+1}.*

3.1.4 Double projection method

With the aim of reducing the number of projections required to compute, in 1997 Iusem again proposed a new variant of projection methods for solving monotone (VIP). The strategy of this variant is based, on the one hand, on the introduction of an Armijo-type linear search procedure to determine the displacement step α_k while ensuring the separation of x^k from the set S and, on the other hand, we need to carry out only two projections at each iteration, hence the name of the method: Double projection method.

Algorithm 5

Initialization: $k=0$, $x^0\in C$, $\delta\in(0,1)$, $0<\widehat{\beta}\leq\widetilde{\beta}$, a se sequence $\{\beta_k\}\subset\left[\widehat{\beta},\widetilde{\beta}\right]$ and ϵ a given precision

Calculate $z^k=P_C\left(x^k-\beta_kF\left(x^k\right)\right)$ and $r\left(x^k,\beta_k\right)=x^k-z^k$

Criterion for stopping: If $\left\|\ r\left(x^k,\beta_k\right)\right\|\leq\varepsilon$,, we stop, $x^k\in S$

Otherwise, find j the smallest nonnegative integer such that

$\left\langle F\left(2^{-j}z^{k}+\left(1-2^{-j}\right)x^{k}\right), r\left(x^{k},\beta_{k}\right)\right\rangle \geq \frac{\delta}{\beta_{k}}\left\|r\left(x^{k},\beta_{k}\right)\right\|^{2}$

We take $\alpha_k = 2^{-j}$ and we put $y^k = \alpha_k z^k + (1-\alpha_k)\, x^k$

Iteration: $x^{k+1} = P_C\left(x^k - \lambda_k F\left(y^k\right)\right)$

with $\lambda_k = \left\langle F\left(y^k\right), x^k - y^k\right\rangle / \left\|F\left(y^k\right)\right\|^2$

Let $k = k+1$, and repeat

End of algorithm.

We give the following results which show that Algorithm 5 is well defined and the sequence generated by it converges globally to a solution of (VIP).

Proposition 3.1.3 *Let $\left\{x^k\right\}$ be the sequence generated by Algorithm 5 and suppose that the set S is non-empty, then*

i) The sequence $\left\{\left\|x^k - \overline{x}\right\|\right\}$ is decreasing for all $\overline{x} \in S$.

ii) The sequence $\left\{x^k\right\}$ is bounded.

iii) $\lim\limits_{k\to\infty}\left\langle F\left(y^k\right), x^k - y^k\right\rangle = 0.$

iv) If an accumulation point of the sequence $\left\{x^k\right\}$ belongs to S, then $\left\{x^k\right\}$ converges to a solution in S.

Theorem 3.1.4 *Consider $VIP(F,C)$, a variational problem, assuming that F is continuous monotone application and S is a non-empty set. Let $\left\{x^k\right\}$ be the sequence generated by Algorithm 5, then it converges globally to a solution in S.*

For the separation of the iterate x^k from the set S, we have the same hyperplane H_k and according to Theorem 2.2.2, we have

$\langle F(\overline{x}), x - \overline{x}\rangle \geq 0, \forall x \in C.$

Furthermore, $y^k = \alpha_k z^k + (1-\alpha_k)\, x^k \in C$ because $\alpha_k \in \left]0,1\right[$ and C is convex.

Then, $\left\langle F\left(y^k\right), y^k - \overline{x}\right\rangle \geq 0 \Longrightarrow \left\langle F\left(y^k\right), \overline{x} - y^k\right\rangle \leq 0.$

To show that $\left\langle F\left(y^k\right), x^k - y^k\right\rangle > 0$, we use the same argument as before or we use the formula for the linear search procedure as follows.

We have,

$$\begin{aligned}
\left\langle F\left(y^k\right), x^k - y^k\right\rangle &= \left\langle F\left(y^k\right), x^k - \alpha_k z^k + (1-\alpha_k)\, x^k\right\rangle \\
&= \alpha_k \left\langle F\left(y^k\right), z^k - x^k\right\rangle \\
&\geq \tfrac{\delta}{\beta_k} \left\|r\left(x^k, \beta_k\right)\right\|^2 \\
&> 0 \quad (\text{because } x^k \notin S).
\end{aligned}$$

According to the numerical tests carried out by applying the double projection method, the results obtained were not really encouraging and this method seemed to be the worst when compared with the others. However, theoretically it is considered to be the origin of all subsequent developments in projection methods.

Exercises

Exercise 3.1.1 *Consider the following* $VIP\,(F,C)$ *where* $F\,(x) = Ax$ *with* $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $C = \mathbb{R}^2$.

1. *Does* F *satisfy the conditions of strong monotonicity and Lipschitz continuity?*

2. *Study the nature of sequence* $\left\{\left\|x^k - \overline{x}\right\|^2, \overline{x} \in S\right\}$, *where* $\left\{x^k\right\}$ *is generated by basic projection procedure.*

Solution 3.1.1 *We check the Lipschitz continuity for* F :

F is Lipschitz continuous on $\mathbb{R}^2$ if there exists $L > 0$ such that:

$$\|F\,(x) - F\,(y)\| \leq L\,\|x - y\|\,, \forall x, y \in \mathbb{R}^2$$

$$\begin{aligned} \|F\,(x) - F\,(y)\| &= \|A\,(x-y)\| \\ &\leq \|A\|\,\|x-y\| \\ &\leq 1.\,\|x-y\|\,. \end{aligned}$$

We have take $\|A\| = \max\limits_{j=1:n} \sum\limits_{i=1}^{n} |a_{ij}|$. *Then, F is Lipschitz continuous on $\mathbb{R}^2$ for all $L \geq 1$. We check the strong monotonicity for F : F is strongly monotone on $\mathbb{R}^2$ if there exists $\mu > 0$ such that:*

$$\langle F\,(x) - F\,(y)\,, x - y\rangle \geq \mu\,\|x - y\|^2\,, \forall x, y \in \mathbb{R}^2.$$

$$\begin{aligned} \langle F\,(x) - F\,(y)\,, x - y\rangle &= \left\langle \begin{pmatrix} x_2 - y_2 \\ -\,(x_1 - y_1) \end{pmatrix}, \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\rangle \\ &= (x_2 - y_2)\,(x_1 - y_1) - (x_2 - y_2)\,(x_1 - y_1) \\ &= 0. \end{aligned}$$

In the case where $x \neq y$, we see that F can not be strongly monotone.Nature of sequence $\left\{\left\|x^k-\overline{x}\right\|^2, \overline{x}\in S\right\}$, where $\left\{x^k\right\}$ is generated by basic projection procedure and knowing $\overline{x}=(0,0)^t$:

$$\begin{aligned}
\left\|x^{k+1}-\overline{x}\right\|^2 &= \left\|x^{k+1}\right\|^2 \\
&= \left\|P_C\left(x^k-\alpha F\left(x^k\right)\right)\right\|^2 \\
&= \left\|x^k-\alpha F\left(x^k\right)\right\|^2 \\
&= \left\|x^k-\alpha A x^k\right\|^2 \\
&= \left\|\begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix} - \alpha \begin{pmatrix} x_2^k \\ -x_1^k \end{pmatrix}\right\|^2 \\
&= \left\|\begin{pmatrix} x_1^k-\alpha x_2^k \\ x_2^k+\alpha x_1^k \end{pmatrix}\right\|^2 \\
&= \left(x_1^k-\alpha x_2^k\right)^2+\left(x_2^k+\alpha x_1^k\right)^2 \\
&= \left(x_1^k\right)^2+\left(\alpha x_2^k\right)^2-2\alpha x_1^k x_2^k+\left(x_1^k\right)^2+\left(\alpha x_2^k\right)^2+2\alpha x_1^k x_2^k \\
&= \left(1+\alpha^2\right)\left(\left(x_1^k\right)^2+\left(x_2^k\right)^2\right) \\
&= \left(1+\alpha^2\right)\left\|x^k\right\|^2 \\
&\geq \left\|x^k\right\|^2 \quad (\textit{Because } \alpha>0).
\end{aligned}$$

Therefore, we deduce the sequence $\left\{\left\|x^k-\overline{x}\right\|^2\right\}$ is not decreasing, then the algorithm of basic projection procedure diverges for anu choice for the initial point differente from $\overline{x}$.

1. **Exercise 3.1.2** *Let be the problem $VIP(F,C)$ such that $F(x)=\begin{pmatrix} x_1+1 \\ 2x_2+1 \end{pmatrix}$, $C=\mathbb{R}_+^2$.*

   1. *Are the assumptions of the basic projection procedure satisfied for the given F.*

   2. *If so, give the inverval of suitable values for the displacement steps associated to the basic projection procedure.*

3. *Calculate up until x^3 using this procedure where $x^0 = (1,1)^T$ and $\alpha = \frac{1}{4}$ and test the stopping criterion at each iteration.*

Solution 3.1.2

$$F(x) = \begin{pmatrix} x_1 + 1 \\ 2x_2 + 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{matrix} 1 \\ 1 \end{matrix} = Ax + b, \ C = \mathbb{R}^2_+.$$

1. *a) F is strongly monotone on C if there exists a constant $\mu > 0$*

$$\langle F(x) - F(y), x - y \rangle \geq \mu \|x - y\|^2, \forall x, y \in C.$$

We have

$$\begin{aligned} \langle F(x) - F(y), x - y \rangle &= \left\langle \begin{pmatrix} x_1 - y_1 \\ 2(x_2 - y_2) \end{pmatrix}, \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\rangle \\ &= (x_1 - y_1)^2 + 2(x_2 - y_2)^2 \\ &= \|x - y\|^2 + (x_2 - y_2)^2 \\ &> \|x - y\|^2. \end{aligned}$$

We can choose any value for μ in $]0, 1]$.

b) F is Lipschitz continuous on C if there exists $L > 0$ such that:

$$\|F(x) - F(y)\| \leq L \|x - y\|, \forall x, y \in C.$$

$$\begin{aligned} \|F(x) - F(y)\| &= \|A(x - y)\| \\ &\leq \|A\| \|x - y\| \\ &\leq 2. \|x - y\|. \end{aligned}$$

We can choose any value for L in $[2, \infty+[$.

Therefore, the assumptions of the basic projection procedure satisfied for the given F.

2. *The inverval of suitable values for the displacement steps associated to the basic projection procedure is* $\alpha \in \left]0, \frac{2\mu}{L^2}\right[= \left]0, \frac{1}{2}\right[$ *for* $\mu = 1$ *and* $L = 2$.

3. *Application of basic projection procedure until* $k = 3$ *taking* $x^0 = (1,1)^T$ *and* $\alpha = \frac{1}{4}$*:*

$$\begin{cases} \triangle_k = \left\| x^k - P_C\left(x^k - \alpha F\left(x^k\right)\right) \right\| \\ x^{k+1} = P_C\left(x^k - \alpha F\left(x^k\right)\right). \end{cases}$$

$\triangle_0 = \|x^0 - P_C\left(x^0 - \alpha F\left(x^0\right)\right)\| = 0.9014.$

$x^1 = P_C\left(x^0 - \alpha F\left(x^0\right)\right) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$

$\triangle_1 = \|x^1 - P_C\left(x^1 - \alpha F\left(x^1\right)\right)\| = 0.4507$

$x^2 = P_C\left(x^1 - \alpha F\left(x^1\right)\right) = \begin{pmatrix} \frac{1}{8} \\ 0 \end{pmatrix}.$

$\triangle_2 = \|x^2 - P_C\left(x^2 - \alpha F\left(x^2\right)\right)\| = 0.128$

$x^3 = P_C\left(x^2 - \alpha F\left(x^2\right)\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

$\triangle_3 = \|x^3 - P_C\left(x^3 - \alpha F\left(x^3\right)\right)\| = 0.$

We remark that we obtain the exact solution by the basic projection procedure after only 3 iterations.

Exercise 3.1.3 *Given* F *a mapping* $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n, C$ *a non-empty closed convex of* $\mathbb{R}^n$ *and we consider a* $VIP(F,C)$.

Let $A \in \mathbb{R}^{n\times n}$ *a be a symmetric positive definite matrix in what follows.*

Consider the space $\mathbb{R}^n$ *with* $\|.\|_A$*the norm induced by the matrix* A *and the inner product* $\langle x, Ax\rangle = \|x\|_A^2$.

We give the following results

- If A is a symmetric positive definite matrix, then A^{-1} is also.

- If $\lambda \in Spec(A)$, then $\lambda^{-1} \in Spec(A^{-1})$.

- $\|x-y\|_A^2 = \|x\|_A^2 + \|y\|_A^2 - 2\langle x, Ay\rangle$.

We have,

$$\forall x \in \mathbb{R}^n, \lambda_{\min}\|x\|_2^2 \leq \|x\|_A^2 \leq \lambda_{\max}\|x\|_2^2 \tag{(*)}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of the matrix A.

P_{CA} denotes the orthogonal projection application on C associated with the norm $\|.\|_A$.

We have also, P_{CA} is a non-expensive application, i.e. $\|P_{CA}(x) - P_{CA}(y)\|_A \leq \|x-y\|_A, \forall x, y \in \mathbb{R}^n$.

In addition, $\overline{x} = P_{CA}(x)$ if and only if $\langle \overline{x} - x, A(y-\overline{x})\rangle \geq 0$, $\forall y \in C, \forall x \in \mathbb{R}^n$.

1. *Show that $\overline{x}$ is a solution of $VIP(F,C)$ if and only if $\overline{x} = P_{CA}(\overline{x} - \alpha A^{-1}F(\overline{x}))$. Write the iterative diagram associated with the basic projection procedure in this case.*

2. *Show that $\|A^{-1}x\|_A^2 = \|x\|_{A^{-1}}^2$.*

3. *Assume that F is strongly monotone of constant μ and Lipschitz con of continuous of constant L on C for the norm $\|.\|_2$, then show that the interval of approximate values of the displacement step α_k corresponding to this procedure so that the sequence ¨$\left\{\|x^k - \overline{x}\|_A^2\right\}$ is decreasing ($\overline{x}$ is solution $VIP(F,C)$) is given by $\left]0, \frac{2\mu\lambda_{\min}^2}{L^2\lambda_{\max}}\right[$.*

1. We show that that $\overline{x}$ is a fixed point of the application

   $P_{CA}(I - \alpha A^{-1}F), \forall\alpha > 0$:

$$\begin{aligned}
\overline{x} \in S \quad & \Longleftrightarrow \langle F(\overline{x}), x-\overline{x}\rangle \geq 0, \forall x \in C \\
& \Longleftrightarrow \langle F(\overline{x}), A(x-\overline{x})\rangle \geq 0, \forall x \in C \\
& \Longleftrightarrow \langle A^{-1}F(\overline{x}), A(x-\overline{x})\rangle \geq 0, \forall x \in C \\
& \Longleftrightarrow \langle \alpha A^{-1}F(\overline{x}), A(x-\overline{x})\rangle \geq 0, \forall x \in C, \alpha \succ 0 \\
& \Longleftrightarrow \langle \overline{x}-(\overline{x}-\alpha A^{-1}F(\overline{x})), A(x-\overline{x})\rangle \geq 0, \forall x \in C, \alpha \succ 0
\end{aligned}$$

By identification with the classical projection inequality, we get that $\overline{x}$ is a fixed point of the application $P_{CA}\left(I-\alpha A^{-1}F\right), \forall \alpha > 0$, then

$$\overline{x} = P_{CA}\left(\overline{x}-\alpha A^{-1}F(\overline{x})\right), \forall \alpha > 0.$$

2. Iterative scheme of the basic projection method associated to the norm $\|.\|_A$:

$$x^{k+1} = P_{CA}\left(x^k-\alpha A^{-1}F\left(x^k\right)\right).$$

3. Values of α for wich the sequence $\left\{\left\|x^k-\overline{x}\right\|_A^2\right\}$ is nonincreasing, where the sequence$\left\{x^k\right\}$ is generated by the basic projection algorithm associated to the norm $\|.\|_A$:

$$\begin{aligned}
\left\|x^{k+1}-\overline{x}\right\|_A^2 \quad & = \left\|P_{CA}\left(x^k-\alpha A^{-1}F\left(x^k\right)\right)-P_{CA}\left(\overline{x}-\alpha A^{-1}F(\overline{x})\right)\right\|_A^2 \\
& \leq \left\|x^k-\alpha A^{-1}F\left(x^k\right)-\overline{x}-\alpha A^{-1}F(\overline{x})\right\|_A^2 \\
& = \left\|x^k-\overline{x}\right\|_A^2+\alpha^2\left\|A^{-1}\left(F\left(x^k\right)-F(\overline{x})\right)\right\|_A^2 \\
& -2\alpha\left\langle x^k-\overline{x}, AA^{-1}\left(F\left(x^k\right)-F(\overline{x})\right)\right\rangle \\
& = \left\|x^k-\overline{x}\right\|_A^2+\alpha^2\left\|\left(F\left(x^k\right)-F(\overline{x})\right)\right\|_{A^{-1}}^2 \\
& -2\alpha\left\langle x^k-\overline{x}, F\left(x^k\right)-F(\overline{x})\right\rangle.
\end{aligned}$$

We have that F is strongly monotone on C if there is $\mu > 0$ such that

$$\langle x-y, F(x)-F(y)\rangle \geq \mu\|x-y\|_2^2, \forall x, y \in C.$$

We apply it for $x^k \in C$ and $\overline{x} \in C$, we get

$$\left\langle x^k - \overline{x}, F\left(x^k\right) - F\left(\overline{x}\right)\right\rangle \geq \mu \left\|x^k - \overline{x}\right\|_2^2.$$

Using $(*)$, yields

$$\left\langle x^k - \overline{x}, F\left(x^k\right) - F\left(\overline{x}\right)\right\rangle \geq \mu \left\|x^k - \overline{x}\right\|_2^2 \geq \frac{\mu}{\left(\lambda_{\max}\right)_A} \left\|x^k - \overline{x}\right\|_A^2.$$

Therefore,

$$-2\alpha \left\langle x^k - \overline{x}, F\left(x^k\right) - F\left(\overline{x}\right)\right\rangle \leq \frac{-2\alpha\mu}{\left(\lambda_{\max}\right)_A} \left\|x^k - \overline{x}\right\|_A^2.$$

F is Lipschitz continuous on C if there is $L \geq 0$ such that

$$\left\|F(x) - F(y)\right\|_2 \leq L \left\|x - y\right\|_2, \forall x, y \in C.$$

We apply it for $x^k \in C$ and $\overline{x} \in C$, we get

$$\left\|F\left(x^k\right) - F\left(\overline{x}\right)\right\|_2^2 \leq L^2 \left\|x^k - \overline{x}\right\|_2^2.$$

Using $(*)$, we obtain

$$\left(\lambda_{\min}\right)_{A^{-1}} \left\|F(x) - F(y)\right\|_2^2 \leq \left\|F(x) - F(y)\right\|_{A^{-1}}^2 \leq \left(\lambda_{\max}\right)_{A^{-1}} \left\|F(x) - F(y)\right\|_2^2.$$

Then,

$$\tfrac{1}{\left(\lambda_{\max}\right)_{A^{-1}}} \left\|F(x) - F(y)\right\|_{A^{-1}}^2 \quad \left\|F(x) - F(y)\right\|_2^2 \leq L^2 \left\|x^k - \overline{x}\right\|_2^2.$$

Using that if $\lambda \in spec\,(A)$ implies that $\lambda^{-1} \in spec\,(A^{-1})$, then

$(\lambda_{\max})_{A^{-1}} = \frac{1}{(\lambda_{\min})_A}$ and replacing in the last inequality, we obtain

$$\begin{aligned}
\|F(x) - F(y)\|_{A^{-1}}^2 &\le \tfrac{L^2}{(\lambda_{\min})_A} \left\|x^k - \overline{x}\right\|_2^2 . \\
&\le \tfrac{L^2}{(\lambda_{\min})_A^2} \left\|x^k - \overline{x}\right\|_A^2 \\
\alpha^2 \|F(x) - F(y)\|_{A^{-1}}^2 &\le \tfrac{\alpha^2 L^2}{(\lambda_{\min})_A^2} \left\|x^k - \overline{x}\right\|_A^2 .
\end{aligned}$$

Subsiting () and () in (), we get

$$\left\|x^{k+1} - \overline{x}\right\|_A^2 \le \left(1 + \frac{\alpha^2 L^2}{(\lambda_{\min})_A^2} - \frac{2\alpha\mu}{(\lambda_{\max})_A}\right) \left\|x^k - \overline{x}\right\|_A^2 .$$

We conclude that $\left\{\left\|x^k - \overline{x}\right\|_A^2\right\}$ is nonincreasing if

$0 < \left(1 + \frac{\alpha^2 L^2}{(\lambda_{\min})_A^2} - \frac{2\alpha\mu}{(\lambda_{\max})_A}\right) < 1$, then α must belong to $\left]0, \frac{2\mu(\lambda_{\min})_A^2}{L^2(\lambda_{\max})_A}\right[$.

As special case if $A = I_n$ (Identity matrix), then the norm $\|.\|_A$ becomes $\|.\|_2$. Hence the values of α for wich the sequence $\left\{\left\|x^k - \overline{x}\right\|_2^2\right\}$ is nonincreasing, where the sequence $\left\{x^k\right\}$ is generated by the basic projection algorithm associated to the norm $\|.\|_2$ are $\left]0, \frac{2\mu}{L^2}\right[$ since $(\lambda_{\max})_A = (\lambda_{\min})_A = 1$.

Exercise 3.1.4 *Let $VIP(F,C)$ where F is given by $F(x) = \begin{pmatrix} x_1 + 1 \\ 2x_2 + 1 \end{pmatrix}$ and the set $C = \mathbb{R}_+^2$. We will use in this exercise the norm associated to symmetric positive definite matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $\|x\|_A^2 = \langle x, Ax \rangle$.*

1. *Is the basic projection procedure applicable for the given problem?*

2. *Does $\alpha = 0.2$ ensure the convergence of the basic procedure applied for this problem?*

3. *Give the iterative formula of this procedure and determine the expression of the projection on C associated in this case.*

4. *Apply this procedure to the iterate x^3 taking as initial point $x^0 = (1,1)^T$ and the given displacement step α.*

Exercise 3.1.5 *Repeat the exercise 3.1.2 with Korpolevich method by taking $\alpha = \frac{1}{4}$ and compare the obtained results.*

Exercise 3.1.6 *Let $VIP(F,C)$ where F is given by* $F(x) = \begin{pmatrix} 3x_1 + 1 \\ x_2 + 2 \end{pmatrix}$ *and the set $C = [1,\ 2] \times [1,\ 3]$.*

1. *Verify that $\overline{x} = (1,1)^T$ is a solution for the given problem.*

2. *Show that the basic projection procedure is applicable for the given problem.*

3. *Give the interval of suitable values of the displacement step α for this procedure.*

4. *Apply this procedure to the iterate x^2, taking as initial point $x^0 = (2,2)^T$ and the displacement step $\alpha = 0.1$.*

5. *Does $\alpha = 0.1$ ensure the convergence of the Korpolevich method for this problem.*

6. *Repeat question 4 for the Korpolevich method and compare the obtained results with those obtained by the basic procedure. What do you deduce?*

Chapter 4

Projection and contraction methods for solving *(VIP)*

This chapter introduces a novel class of methods for solving monotone variational Inequalities problems, building upon the foundational work of projection and contraction PC methods introduced by He, Solodov, and Tseng in the early 1990s [3, 12, 13]. Renowned for their simplicity, robustness, and strong global convergence properties, PC methods are distinguished by their ability to converge from any arbitrary starting point. Our study of these methods class begins by constructing a suitable descent direction, which guarantees progress toward the solution by ensuring a sufficient decrease in the function value at each iteration. Subsequently, we derive an effective displacement step to update the iterates. The theoretical framework will be concretized by presenting corresponding iterative algorithms, which we develop and analyze for two distinct cases: first, for (VIP) where the underlying mapping is general (linear or nonlinear), and second, the case where the mapping is only linear

.

4.1 General principle of a PC methods

From any point in $\mathbb{R}^n$, the general iterative scheme associated to these methods is as follows

$$x^{k+1} = x^k - \alpha_k d(x^k).$$

It can also be written in the following form

$$x^{k+1} = P_C(x^k - \alpha_k d(x^k)),$$

The fundamental principle of a projection and contraction method is to minimise the error given by the function

$$\Psi(x) = \frac{1}{2} \|x - \overline{x}\|^2, \ \overline{x} \in S, \ x \in \mathbb{R}^n.$$

If we take $-d(x^k)$ as descent direction for Ψ, then at each iteration this direction must verify

$$\left\langle d(x^k), \nabla\Psi(x) \right\rangle = \left\langle d(x^k), x - \overline{x} \right\rangle \geq \varphi(x) \geq 0, \tag{4.1}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ is a continuous function.

In many works, authors have described $d(x^k)$ as a profitable direction and φ as an error measure function. In what follows, we will define these two terms.

Definition 4.1.1 *Let $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ be a continuous function and $h > 0$ a constant. We call φ an error measure function for (VIP) if it satisfies*

$$\begin{aligned} &\varphi(x) \geq h \|e(x)\|^2 \\ \textit{and} \;\; &\varphi(x) = 0 \Longleftrightarrow e(x) = 0 \Longleftrightarrow \;\; x \in S, \\ \textit{where } &e(x) = x - P_C\left(x - F(x)\right). \end{aligned}$$

Definition 4.1.2 *Let φ be an error measure function for (VIP) and $d : \mathbb{R}^n \to \mathbb{R}^n$ a function. We call d a profitable direction for (VIP) if*

$$\langle d(x), x - \overline{x} \rangle \geq \varphi(x) \geq 0.$$

From these two tools, we can define the displacement step for a PC method by the expression given below

$$\alpha_k = \frac{\varphi(x^k)}{\|d(x^k)\|^2}. \tag{4.2}$$

The computation of this step comes from the following idea, it is clear that the displacement step is obtained through an exact line search applied to the function Ψ.

Indeed, we have

$$\alpha_{optimal} = \arg\min_{\alpha \geq 0} (f(\alpha) = \frac{1}{2}(\left\|x^{k+1} - \overline{x}\right\|^2).$$

In addition, the sequence $\left\{x^k\right\}$ is generated by

$$x^{k+1} = x^k - \alpha_k d(x^k).$$

By simple calculations, we get

$$f(\alpha) = \frac{1}{2}\alpha^2 \left\|d(x^k)\right\|^2 - \alpha \left\langle d(x^k), x^k - \overline{x}\right\rangle + \frac{1}{2}\left\|x^k - \overline{x}\right\|^2$$

$$f'(\alpha) = 0 \Leftrightarrow \alpha \left\|d(x^k)\right\|^2 - \left\langle d(x^k), x^k - \overline{x}\right\rangle = 0 \Leftrightarrow \alpha_{optimal} = \frac{\left\langle d(x^k), x^k - \overline{x}\right\rangle}{\left\|d(x^k)\right\|^2}.$$

Unfortunately, $\alpha_{optimal}$ is given in function of the solution $\overline{x}$ which is unknown. So, from (4.1), (4.2) and since $(\alpha_k \in]0, \alpha_{optimal}])$, we can take

$$0 < \alpha_k = \frac{\varphi(x^k)}{\left\|d(x^k)\right\|^2} \leq \alpha_{optimal}.$$

Based on these ingredients d, φ and α_k, the projection and contraction methods made their first apparition and were subsequently developed by several authors.

Throughout this chapter, we assume that C is a non-empty closed convex set of $\mathbb{R}^n$, and that the projection onto C is easy to compute.

4.2 *PC* method applied to *(V IP)* monotone

4.2.1 Case of a general mapping

In this section, we examine the projection and contraction method, as developed by Bingsheng He in 1997, for solving variational inequalities problems governed by a monotone mapping. The algorithm is iterative and employs fundamental concepts from projection theory to generate a sequence of points that converge to the solution. A core assumption for its convergence is that the underlying mapping F is monotone and continuous, notably, unlike many others.

In his original work, He employs a norm $\|.\|_G$ induced by a positive definite symmetric matrix G. However, to simplify the description of the method and facilitate calculations in the applied exercises, we will instead use the standard Euclidean norm

($G = I_n$,the identity matrix).

To present this method, we need the following fundamental inequalities:

First, we use the classic inequality of the orthogonal projection of a point $v \in \mathbb{R}^n$ on the set C,

$$\langle v - P_C(v), P_C(v) - x \rangle \geq 0 \quad \forall v \in \mathbb{R}^n \quad \forall x \in C, \tag{4.3}$$

and as we have already seen, solving (VIP) is equivalent to find the zero of the equation

$$e(x) = x - P_C\left(x - F(x)\right).$$

We put $v = x - F(x)$ and we replace in (4.3), we get

$$\langle e(x) - F(x), P_C[x - F(x)] - \overline{x} \rangle \geq 0 \quad \forall x \in \mathbb{R}^n \ . \tag{4.4}$$

Now, let $\overline{x} \in S$, for any $x \in \mathbb{R}^n$, $P_C[x - F(x)] \in C$, then

$$\langle F(\overline{x}), P_C[x - F(x)] - \overline{x} \rangle \geq 0 \ \forall x \in \mathbb{R}^n. \tag{4.5}$$

Moreover, by using the assumption that F is monotone on C, so it verifies

$$\langle F(P_C[x - F(x)]) - F(\overline{x}), P_C[x - F(x)] - \overline{x} \rangle \geq 0, \ \forall x \in \mathbb{R}^n \ . \tag{4.6}$$

Using these inequalities, He constructed his projection and contraction method for monotone (VIP) by adding (4.4) + (4.5) + (4.6), then we find

$$\langle e(x) - [F(x) - F(P_C[x - F(x)])], (x - \overline{x}) - e(x) \rangle \geq 0, \quad \forall x \in \mathbb{R}^n. \tag{4.7}$$

We denote by

$$d(x) = \quad e(x) - [F(x) - F(P_C[x - F(x)])].$$

Afterwards, we obtain

$$\langle d(x), x - \overline{x} \rangle \geq \langle d(x), e(x) \rangle . \tag{4.8}$$

Now, we suppose that

$$\langle F(x) - F(y), x - y \rangle \leq (1 - \delta) \|x - y\|^2 , \quad \forall x, y \in \mathbb{R}^n, \ \delta \in (0, 1). \tag{4.9}$$

Under this assumption, we have

$$\langle d(x), e(x) \rangle = \|e(x)\|^2 - \langle F(x) - F(P_C[x - F(x)], e(x) \rangle \geq \delta \|e(x)\|^2 .$$

In other words

$$\langle d(x), x - \overline{x} \rangle \geq \langle d(x), e(x) \rangle \geq \delta \|e(x)\|^2 \geq 0. \tag{4.10}$$

Based on (4.10) and the two definitions (Def 4.1.1) and (Def 4.1.2), we can take d as a profitable direction and $\varphi(x) = \langle d(x), e(x) \rangle$ to construct the following PC method.

Algorithme 1

Initialization: $x^0 \in \mathbb{R}^n$ and ε a given precision, $k = 0$.

Stopping criterion: While $\left\|e(x^k)\right\| > \varepsilon$, do

Iteration:

$$x^{k+1} = x^k - \alpha_k d(x^k), \tag{4.11}$$

where

$$d(x^k) = \quad e(x^k) - [F(x^k) - F(P_C[x^k - F(x^k)])], \tag{4.12}$$

and

$$\alpha_k = \frac{\left\langle d(x^k), e(x^k) \right\rangle}{\|d(x^k)\|^2}. \tag{4.13}$$

Put $k = k + 1$ and repeat

End of algorithm.

Theorem 4.2.1 *The sequence* $\{x^k\}$ *generated by Algorithm 1 of* (PC) *methods for* (VIP) *satisfies*

$$\left\|x^{k+1}-\overline{x}\right\|^2 \leq \left\|x^k-\overline{x}\right\|^2 - \alpha_k \left\langle d(x^k), e(x^k)\right\rangle, \quad for\ all\ \overline{x} \in S. \tag{4.14}$$

Proof. Indeed,

$$\begin{aligned}
\left\|x^{k+1}-\overline{x}\right\|^2 &= \left\|x^k-\overline{x}-\alpha_k d(x^k)\right\|^2 \\
&= \left\|x^k-\overline{x}\right\|^2 + \alpha_k^2 \left\|d(x^k)\right\|^2 - 2\alpha_k \left\langle x^k-\overline{x}, d(x^k)\right\rangle \\
&= \left\|x^k-\overline{x}\right\|^2 + \alpha_k \left\langle d(x^k), e(x^k)\right\rangle - 2\alpha_k \left\langle x^k-\overline{x}, d(x^k)\right\rangle \\
&\leq \left\|x^k-\overline{x}\right\|^2 + \alpha_k \left\langle d(x^k), e(x^k)\right\rangle - 2\alpha_k \left\langle d(x^k), e(x^k)\right\rangle \\
&= \left\|x^k-\overline{x}\right\|^2 - \alpha_k \left\langle d(x), e(x^k)\right\rangle, \qquad \forall\ \overline{x} \in S.
\end{aligned}$$

Note that it is possible to write that the sequence $\{x^k\}$ generated by Algorithm 1 also verifies

$$\left\|x^{k+1}-\overline{x}\right\|^2 \leq \left\|x^k-\overline{x}\right\|^2 - \alpha_k\delta\ \left\|e(x^k)\right\|^2, \forall\ \overline{x} \in S. \tag{4.15}$$

■

The inequality (4.15) indicates that we obtain a large decrease at iteration $(k+1)$ if $\left\|e(x^k)\right\|$ is big. Conversely, if $\left\|e(x^k)\right\|$ is already very small, then we obtain also a very small decrease in the same iteration, which means that x^{k+1} is a good enough approximation to $\overline{x} \in S$.

Theorem 4.2.2 *Let* $\{x^k\}$ *the sequence generated by algorithm 1 of* PC *methods for* (VIP) *converges globally to the solution* $\overline{x}$.

Proof. The fundamental inequality (4.15) shows that the sequence $\left\{x^k\right\}$ is Fejér monotone with respect to the solution set S. This means

- The sequence $\left\{\left\|x^k-\overline{x}\right\|^2, \overline{x}\in S\right\}$ is non-increasing, hence it is convergent.
- The sequence $\left\{x^k\right\}$ is bounded.

Summing the inequality (4.15) over iterations, we get

$$\alpha_k\delta\sum_{k=0}^{+\infty}\left\|e(x^k)\right\|^2\leq\left\|x^0-\overline{x}\right\|^2<+\infty.$$

Therefore, the series on the left must converge, which forces its terms to go to zero

$$\lim_{k\longrightarrow+\infty}\left\|e(x^k)\right\|^2=0.$$

That we can write

$$\lim_{k\longrightarrow+\infty}\left\|x^k-w^k\right\|^2=0$$

where

$$w^k=P_C\left(x^k-F(x^k)\right)$$

Since $\left\{x^k\right\}$ is bounded, it has a convergent subsequence $\left\{x^{k_j}\right\}$, that its limit is denoted by $\overline{x}$ (cluster point).

According to (), we show that any cluster point $\widehat{x}$ of $\left\{x^k\right\}$ is also a cluster point of $\left\{w^k\right\}$. By the continuity of F and projection operator, taking limit in the expression of w^k, we conclude that $\overline{x}$ is a solution

$$\overline{x}=P_C\left(\overline{x}-F(\overline{x})\right).$$

If a sequence $\left\{\left\|x^k-\overline{x}\right\|^2, \overline{x}\in S\right\}$ converges, then the limit of every convergent subsequence must be the same as the limit of the whole sequence. ■

Unfortunately, the hypothesis (4.9) may not hold for general variational inequalities problems. To address this limitation, the same author introduced an Armijo-type line search procedure to dynamically determine an appropriate step size in each iteration. Following the same foundational steps, a modified algorithm was developed. However, this modified version incurs a higher computational cost, as the incorporated line search requires multiple projections onto the feasible set C per iteration.

4.2.2 Linear case

Before the study presented at the beginning of this chapter, the same author has already proposed another work concerning a PC method applied to the monotone and linear class of (VIP).

Let us now consider the class of linear variational inequalities that the associated mapping is supposed linear monotone.

$$(LVIP)\begin{cases} Find \ \ \overline{x} \in C \ \ \ such \ that \\ \langle M\overline{x} + q, x - \overline{x} \rangle \geq 0, \ \forall x \in C, \end{cases} \tag{4.16}$$

where $M \in \mathbb{R}^{n\times\ n}$ is a positive semi-definite matrix (not necessarily symmetrical) and $q \in \mathbb{R}^n$.

For this method, we proceed in an almost the same way as the general case.

By adding together (4.4) and (4.5), we obtain

$$\langle M\overline{x} - Mx + e(x), x - \overline{x} - e(x) \rangle \geq 0$$

$$\langle e(x) - M(x - \overline{x}), x - \overline{x} - e(x) \rangle \geq 0. \tag{4.17}$$

According (3.14), we have

$$\left\langle (I + M^T)e(x^k), (x - \overline{x}) \right\rangle \geq \|e(x)\|^2 + (x - \overline{x})^T M(x - \overline{x}),$$

and since M is positive semi-definite, then

$$\left\langle (I+M^T)e(x),(x-\overline{x})\right\rangle \geq \|e(x)\|^2 \geq 0. \tag{4.18}$$

We take $d=(I+M^T)e(x)$ as the profitable direction for the function Ψ and $\varphi(x)=\|e(x)\|^2$.

In this case, the algorithm associated to this method will be as follows.

Algorithme 2

Initialization: $x^0\in\mathbb{R}^n$, ε a given precision, $k=0$

Stopping criterion: While $\left\|e(x^k)\right\|>\varepsilon$,

Iteration:

$$x^{k+1}=x^k-\alpha_k d(x^k), \tag{4.19}$$

where

$$d(x^k)=(I+M^T)e(x^k) \tag{4.20}$$

$$\alpha_k=\frac{\left\|e(x^k)\right\|^2}{\left\|d(x^k)\right\|^2} \tag{4.21}$$

Put $k=k+1$ and repeat

End of algorithm.

Theorem 4.2.3 *The sequence $\{x^k\}$ generated by Algorithm 2 of (PC) methods for a (VIP) with F is linear and monotone satisfies*

$$\left\|x^{k+1}-\overline{x}\right\|^2 \leq \left\|x^k-\overline{x}\right\|^2-\alpha_k\left\|e(x^k)\right\|^2, for\ \ all\ \overline{x}\in S. \tag{4.22}$$

Proof.

$$\begin{aligned}\left\|x^{k+1}-\overline{x}\right\|^2 &= \left\|x^k-\overline{x}-\alpha_k d(x^k)\right\|^2\\ &= \left\|x^k-\overline{x}\right\|^2-2\alpha_k\left\langle x^k-\overline{x}, d(x^k)\right\rangle+\alpha_k^2\left\|d(x^k)\right\|^2\\ &= \left\|x^k-\overline{x}\right\|^2-2\alpha_k\left\langle x^k-\overline{x}, d(x^k)\right\rangle+\alpha_k\left\|e(x^k)\right\|^2\end{aligned}$$

According to (4.15),

$$\left\|x^{k+1}-\overline{x}\right\|^2 \quad \leq \left\|x^k-\overline{x}\right\|^2-\alpha_k\left\|e(x^k)\right\|^2.$$

Furthermore, using the expression for α_k given in Algorithm 2, we find that the latter is bounded below

$$\alpha_k=\frac{\left\|e(x^k)\right\|^2}{\|(M^T+I)e(x^k)\|^2}\geq\frac{\left\|e(x^k)\right\|^2}{\|(M^T+I)\|^2\left\|e(x^k)\right\|^2},$$

in other words

$$\alpha_k \geq \frac{1}{\|(M^T+I)\|^2}=c>0$$

Afterwards,

$$\left\|x^{k+1}-\overline{x}\right\|^2 \leq \left\|x^k-\overline{x}\right\|^2-c\left\|e(x^k)\right\|^2, \quad \forall \;\; \overline{x}\in S. \tag{4.23}$$

■

Theorem 4.2.4 *Let $\{x^k\}$ the sequence generated by Algorithm 2 of PC methods for (VIP). Suppose that F is linear monotone, then $\{x^k\}$ converges globally to the solution $\overline{x}$.*

Proof. For the demonstration, we follow the same steps presented in the proof of the Theorem (4.2.2). ■

Exercises

Exercise 4.2.1 *Repeat exercise* (3.1.2) *applying Algorithms 1 and 2 of the PC methods and compare the results obtained.*

Solution 4.2.1 *Let be the problem* $VIP\,(F,C)$ *such that*

$$F\,(x) = \begin{pmatrix} x_1+1 \\ 2x_2+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{matrix} 1 \\ 1 \end{matrix} = Ax+b,\ C=\mathbb{R}^2_+.$$

Taking $x^0 = (1,1)^t$.

1- *Application of Algorithm 1*:

$$Alg1 \quad \begin{cases} While\ \left\|e(x^k)\right\| > 0 \quad do \\ x^{k+1} = x^k - \alpha_k d(x^k),\ where \\ d(x^k) = \quad e(x^k) - [F(x^k) - F(P_C[x^k - F(x^k)])],\ and \\ \alpha_k = \frac{\left\langle d(x^k), e(x^k)\right\rangle}{\left\|d(x^k)\right\|^2}. \end{cases}$$

$\triangle_0 = \|e(x^0)\| = \sqrt{2}$.

$d(x^0) = \quad e(x^0) - [F(x^0) - F(P_C[x^0 - F(x^0)])] = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$

$\alpha_0 = \frac{\left\langle d(x^0), e(x^0)\right\rangle}{\|d(x^0)\|^2} = -1.$

We have obtain $\alpha_0 < 0$.

$x^1 = x^0 - \alpha_0 d(x^0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$

$\triangle_1 = \|e(x^1)\| = 1.$

$d(x^1) = \quad e(x^1) - [F(x^1) - F(P_C[x^1 - F(x^1)])] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

We have $\|d(x^1)\| = 0$, *so* α_1 *is undefined.*

This means the algorithm stops or fails at $k = 1$ *because the search direction is zero.*

But $e(x^1) \neq 0$, *so* x^1*is not a solution.*

We remark that Algorithm 1 did not reach an approximate solution, it may be because the condition (4.9) *is not satisfied by* F.

Indeed,

$$\langle F(x) - F(y), x - y\rangle \leq (1-\delta)\left\|x-y\right\|^2, \qquad \forall x, y \in \mathbb{R}^2, \delta \in (0,1).$$

$$\begin{aligned}
\langle F(x) - F(y), x - y\rangle &= \left\langle \begin{pmatrix} x_1 - y_1 \\ 2\left(x_2 - y_2\right) \end{pmatrix}, \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\rangle \\
&= \left(x_1 - y_1\right)^2 + 2\left(x_2 - y_2\right)^2 \\
&= \left\|x-y\right\|^2 + \left(x_2 - y_2\right)^2 \\
&> \left\|x-y\right\|^2.
\end{aligned}$$

In the case where $x \neq y$, *It does not exist* $\delta \in (0,1)$ *such that*

$\langle F(x) - F(y), x - y\rangle \leq (1-\delta)\left\|x-y\right\|^2$.

2- *Application of Algorithm 2:*

$$Alg2 \quad \begin{cases} While \ \left\|e(x^k)\right\| > 0 \ \ do \\ x^{k+1} = x^k - \alpha_k d(x^k), \ where \\ d(x^k) = (I + M^T)e(x^k), \ and \\ \alpha_k = \frac{\left\|e(x^k)\right\|^2}{\left\|d(x^k)\right\|^2}. \end{cases}$$

$\triangle_0 = \|e(x^0)\| = \sqrt{2}$.

$d(x^0) = \quad (I + M^T)e(x^0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

$\alpha_0 = \frac{\left\|e(x^0)\right\|^2}{\|d(x^0)\|^2} = \frac{2}{13}.$

$$x^1 = x^0 - \alpha_0 d(x^0) = \begin{pmatrix} 9/13 \\ 7/13 \end{pmatrix}.$$

$\triangle_1 = \|e(x^1)\| = 0.8771.$

$$d(x^1) = \quad (I + M^T)e(x^1) = \begin{pmatrix} 18/13 \\ 21/13 \end{pmatrix}.$$

$\alpha_1 = \frac{\|e(x^1)\|^2}{\|d(x^1)\|^2} = 26/153.$

$x^2 = x^1 - \alpha_1 d(x^1) = (101/221, 175/663).$

$\triangle_2 = \|e(x^2)\| = 0.5277.$

$$d(x^2) = \quad (I + M^T)e(x^2) = \begin{pmatrix} 0.9140 \\ 0.7820 \end{pmatrix}.$$

$\alpha_2 = \frac{\|e(x^1)\|^2}{\|d(x^1)\|^2} = 0.1904.$

$x^3 = x^2 - \alpha_2 d(x^2) = (0.2830, 0.1132).$

$\triangle_2 = \|e(x^2)\| = 0.3048.$

Exercise 4.2.2 *Consider the following non-linear $VIP\,(F, C)$*

$$F\,(x_1, x_2) = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} \text{ and } C = [0,\ 0.4] \times [0,\ 0.4]\,.$$

1. *Show that F is monotone on C.*
2. *Does F satisfy the condition (4.9)?*
3. *Check that $\overline{x} = (0,0)^T$ is a solution for this (VIP).*
4. *Apply Algorithm 1 and the algorithm proposed in the previous exercise up to, taking $x^0 = (0.4, 0.4)^T$ as the initial point and the stopping test $\left\|x^k - \overline{x}\right\|$, then compare the obtained results.*

Exercise 4.2.3 *We want to propose a method of (PC) for a general monotone differentiable (VIP).*

Let be a (VIP) with F a monotone differentiable mapping and C a non-empty closed convex subset of $\mathbb{R}^n$.

1. Considering the linear part of the Taylor development of the mapping F in the neighborhood of the iterate x^k and following the same steps as in Algorithm 2, determine the profitable direction and the displacement step corresponding to this method.

2. Write the algorithm associated with this method.

Exercise 4.2.4 *Let a (VIP) with F is a monotone and Lipschitz continuous operator on C a non-empty closed convex set of $\mathbb{R}^n$. We consider the iterative scheme of the Korpolevich method applied to this (VIP).*

$$x^{k+1} = P_C\left(x^k - \alpha F\left(P_C\left(x^k - \alpha F\left(x^k\right)\right)\right)\right), \text{ such that } L < 1 \text{ and } \alpha = 1.$$

We want to demonstrate that the Korpolevich direction $F\left(P_C\left(x^k - F\left(x^k\right)\right)\right)$, $\forall x^k \in C$ is a profitable direction for this (VIP). Recall that

$$e\left(x^k\right) = x^k - P_C\left(x^k - F\left(x^k\right)\right).$$

1. *Using the classical projection inequality, show that*

$$\left\langle e\left(x^k\right), F\left(x^k\right)\right\rangle \geq \left\|e\left(x^k\right)\right\|^2.$$

2. *Using the monotonicity hypothesis of the application F, show that*

$$\left\langle x^k - \overline{x}, F\left(P_C\left(x^k - F\left(x^k\right)\right)\right)\right\rangle \geq \left\langle e\left(x^k\right), F\left(P_C\left(x^k - F\left(x^k\right)\right)\right)\right\rangle, \quad \forall \overline{x} \in S.$$

3. *From the first two questions, the Cauchy Schwarz inequality and the Lipschitz continuity of F, show that $F\left(P_C\left(x^k - F\left(x^k\right)\right)\right)$ is a profitable direction for (VIP).*

4. *Taking the error measure function $\varphi(x) = \|e(x)\|^2$, the Korpolevich direction and introducing the P_C operator on the iterative scheme, write the corresponding projection and contraction algorithm.*

5. *Let the following (VIP) with $F(x_1, x_2) = (0.5x_1 + 0.2x_2 + 1, 0.5x_2)$ and $C = \mathbb{R}^n_+$.*

   (a) *Show that F is a monotone and Lipschitz continuous application on C and give the interval of convenable values of the step of the Korpolevich algorithm.*

   (b) *Verify that $\overline{x} = (0,0)^T$ is a solution for this (VIP).*

   (c) *Apply the algorithm of question 4 and that of Korpolevich to $k = 2$ by taking the initial point $x^0 = (1,1)^T$ and the stopping test $\left\|x^k - \overline{x}\right\|$, then compare the obtained results.*

Exercise 4.2.5 *We assume that we have a sequence generated by $x^{k+1} = P_C\left(x^k + \alpha_k d^k\right)$ associated with a projection and contraction type algorithm to solve the $VIP(F,C)$ with F monotone on C and C a non-empty closed convex of $\mathbb{R}^n$. We give*

- $y^k = P_C\left(x^k - F\left(x^k\right)\right),\ e\left(x^k\right) = x^k - y^k.$

- $\langle F(x) - F(y), x - y\rangle \leq \sigma \|x - y\|^2, \forall x, y \in C$ *and* $\sigma \in]0,1[.$

- $d^k = \left(e\left(x^k\right) + F\left(y^k\right)\right).$

Let $\overline{x}$ be a solution of $VIP(F,C)$

1. *Using the classical projection inequality for $x^k, y^k, \overline{x}$, show that*

   $\left\langle e\left(x^k\right), x^k - \overline{x}\right\rangle \geq \left\langle e\left(x^k\right), e\left(x^k\right) - F\left(x^k\right)\right\rangle.$

2. *Using that F is monotone on C, show that* $\left\langle F\left(y^k\right), x^k - \overline{x}\right\rangle \geq \left\langle e\left(x^k\right), F\left(y^k\right)\right\rangle$.

3. *From the first two questions, deduce that d^k is a profitable direction and determine the corresponding displacement step α_k.*

Exercise 4.2.6 *Consider the problem $VIP(F, C)$ with the mapping F and C are given by*

$$F\left(x_1, x_2\right) = \begin{pmatrix} \frac{-1}{x_1^3} \\ \frac{-2}{x_2^2} \end{pmatrix}, \quad C = [1, 2] \times [1, 3].$$

1. Is F strongly monotone and Lipschitz continuous on C?

2. Determine the solution of this $VIP(F, C)$.

3. Determine the multipliers of (KKT) associated with this $VIP(F, C)$.

4. Apply 2 iterations from $x^0 = (1, 1)^T$, the basic procedure algorithm taking $\alpha = 0.2$ and He's Algorithm 2 taking at each iteration k, the matrix $M = J_F\left(x^k\right)$. Compare the results obtained.

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