People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

Setif 1 University-Ferhat Abbas
Faculty of Sciences
Department of Mathematics

جامعـة سطيف 1 فرحـــات عباس
كليـــة العلـــــوم
قســـــم الرياضيـــــات

Doctoral Thesis

Submitted in Fulfillment of the Requirements
for the Degree of Doctor in Mathematics

Option : Applied Mathematics

Theme :

NEW APPROACHES FOR SOLVING SYSTEMS OF TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

Presented by

Mme. **Nedjoua DRIAI**

Supervisor : **Dr. Ali KHALOUTA**

Thesis defended on **May 23, 2026**, in front of the jury composed of:

Mr.	Hamid BENSERIDI	Prof	Setif 1 University-Ferhat Abbas	President
Mr.	Ali KHALOUTA	MCA	Setif 1 University-Ferhat Abbas	Supervisor
Mme.	Aziza BACHMAR	MCA	Setif 1 University-Ferhat Abbas	Examiner
Mme.	Amel BOUDIAF	MCA	Setif 1 University-Ferhat Abbas	Examiner
Mr.	Zoheir CHEBEL	MCA	University Center of Barika	Examiner
Mme.	Leila AIT-KAKI	MCA	ENS El Katiba Assia Djebar of Constantine	Examiner

2025/2026


بسم الله الرحمن الرحيم

شكر وتقدير

الحمد لله الذي بنعمته تتم الصالحات.

أولا أشكر الله العلي القدير إذ وهبني الصحة والقدرة على إتمام هذا العمل ووفقني في كل خطوه خطوتها، فله الحمد وله الشكر دوما وأبدا.

ثم الشكر الجزيل إلى **الأستاذ القدير الدكتور خطوة علي** المشرف على هذا العمل حيث كان قدوة عالية في العلم والأخلاق، فله الشكر على المتابعة الدقيقة والاهتمام البالغ بالعمل في كل مراحله، وله الشكر على نبل معاملته وجميل إحسانه.

والشكر أيضا موصول لأعضاء لجنه المناقشة على قبولهم قراءة العمل وتقييمه وتقويمه.

ولله الحمد والشكر في البدء والختام.

Acknowledgements

First, I thank God Almighty for granting me the health and ability to complete this work and guiding me in every step I took. I thank him forever and ever.

Then, I would like to express my deepest and sincere gratitude to ***Dr. Ali Khalouta,*** *the supervisor of this work, who has been a great role model in terms of knowledge and ethics. I thank him for his close follow-up and keen interest in the work at all stages, and for his noble treatment and kind generosity.*

Thanks also go to the members of the discussion committee for accepting to read, evaluate, and assess this work.

Praise and thanks be to God in the beginning and in the end.

إهداء

إلى والديّ الكريمين:

أبي العزيز، رمز الاجتهاد والإخلاص، سندي وقدوتي في الحياة.

أمي الغالية، نبع الحنان والأمان.

إلى إخوتي وأخواتي الأعزاء.

إلى عائلتي الصغيرة: زوجي الكريم، وأبنائي الأبرياء، قرة عيني وبهجتي.

إلى كل طالب يضحي من أجل العلم.

إلى كل معلم وأستاذ ينير دروب طلبة العلم.

أهدي هذا العمل.

Dedication

To my dear parents:

My beloved father, a symbol of diligence and sincerity, my support and role model in life.

My dear mother, a source of tenderness and security.

To my dear brothers and sisters.

To my little family: my dear husband and my righteous sons, my joy and delight, the apple of my eye.

To every student who sacrifices for the sake of knowledge.

To every teacher and professor who enlightens the path of students.

I dedicate this work.

Table of contents

Introduction **iii**

1 Preliminaries and notations **1**

1.1 Theory of fractional calculus 1

1.1.1 L^P Spaces 2

1.1.2 Space of absolutely continuous functions 2

1.1.3 Special functions 3

1.1.4 Riemann-Liouville fractional integral and derivative 4

1.1.5 Caputo fractional derivative 8

1.2 Fractional differential equations via Caputo fractional derivative 13

1.2.1 The results of existence and uniqueness 13

1.3 Applications of fractional calculus 16

2 Decomposition approaches for classical nonlinear partial differential equations **18**

2.1 Adomian decomposition approach (ADA) 18

2.1.1 Analysis of the ADA 19

2.2 Homotopy analysis approach (HAA) 22

2.2.1 Analysis of the HAA 23

2.3 Variational iteration approach (VIA) 26

2.3.1 Analysis of the VIA 27

2.4 Reduced differential transform approach (RDTA) 31

2.4.1 Analysis of the RDTA 33

3 Finding solutions to systems of nonlinear time-fractional partial differential equations **37**

3.1 The ADA procedure 38

3.2 The HAA procedure 45

3.3 The VIA procedure 52

3.4 The RDTA procedure 57

4 A new mixed approach for resolving the system of nonlinear time-fractional Burgers equations **63**

4.1 History of the nonlinear time-fractional Burgers equation 63

4.2 KhT definitions and results 64

4.3 Khalouta-Daftardar-Jafari approach (KhDJA) 67

4.4 Convergence and absolute error analysis of KhDJA 70

4.5 Numerical applications 73

4.6 Numerical simulations and discussion 77

Conclusions and future research directions **80**

Bibliography **80**

Introduction

Various natural phenomena that appear in different fields of science and engineering can be successfully described using the concept of fractional calculus. The concept was developed to find an answer to the question that prompted L' Hospital in 1695 to ask Leibnitz about the scenario of $n = \frac{1}{2}$ for the derivative operator of order n i.e. $\frac{d^n}{dt^n}$. Fractional calculus plays an important role in the mathematical modeling of real-world natural phenomena, e.g., reaction–diffusion processes, decentralized wireless networks, water wave motion, signal processing, population growth, earthquake design, traffic models with fractional derivatives, diffusion processes , fractional stochastic systems, control processing, medical science and many more domains that have emerged in recent decades.

Systems of nonlinear time-fractional partial differential equations (SNTFPDEs) have played an important role in various fields such as mathematics, physics, chemistry, economics, medicine, biology, geology, and engineering, due to the nonlocal nature of their power-law structure with arbitrary positive order. Nowadays, it is impossible to model many real-world problems without using SNTFPDEs. Fractional calculus can be called the calculus of the century because of its diverse applications in all disciplines of applied sciences, technology, and engineering.

In applied mathematics and many other scientific fields, it is important to obtain accurate solutions to SNTFPDEs. In general, SNTFPDEs do not have exact or closed-form solutions for most problems, and it is very difficult to obtain exact solutions. For this reason, various approximate and numerical approaches have been developed to resolve SNTFPDEs. For instance, Ali Khalouta **[21]** proposed two different approaches known as the homotopy perturbation transform approach and the variational iteration transform approach to

resolve the Caputo time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas. Singh and Kumar **[31]** introduced a new approximate solution of the time-fractional order multi-dimensional Navier–Stokes equation by adopting the fractional reduced differential transform approach. Singh et al. **[5]** announced the Adomian decomposition approach to find an approximate solution for the fractional-order new coupled Korteweg-de Vries system. Jaradat et al. **[17]** presented the residual power series approach for time-fractional Drinfeld-Sokolov-Wilson system.

This thesis aims to suggest a new mixed approach, called Khalouta-Daftardar-Jafari approach (KhDJA). This approach is used to provide analytical solutions for a system of nonlinear time-fractional Burgers equations and to the assist mathematicians in resolving numerous systems related to physics and engineering in the future.

The system of nonlinear time-fractional Burgers equations takes the following form

$$\begin{cases} {}^{C}D_t^{\alpha}u(x,t) = \pm u_{xx}(x,t) \pm 2u(x,t)u_x(x,t) - (u(x,t)v(x,t))_x \\ {}^{C}D_t^{\beta}v(x,t) = \pm v_{xx}(x,t) \pm 2v(x,t)v_x(x,t) - (u(x,t)v(x,t))_x \end{cases},$$

under the initial conditions

$$\begin{cases} u(x,0) = f(x) \\ v(x,0) = g(x) \end{cases}.$$

Here ${}^{C}D_t^{\alpha}$ and ${}^{C}D_t^{\beta}$ signify the Caputo time-fractional derivative operators of the functions $u(x,t)$ and $v(x,t)$ of order α and β with $0 < \alpha \leq 1$ and $0 < \beta \leq 1$, respectively, $x \in \mathbb{R}$ signify the space variable and $t \in [0,T]$, $T > 0$ signify the time variable, and $f(x)$ and $g(x)$ are two well-known real functions.

The system of nonlinear time-fractional Burgers equations is a system of nonlinear time-fractional partial differential equations that modifies the classical system of nonlinear Burgers equations by replacing the integer-order derivative with a fractional-order derivative, usually the Caputo derivative. This generalization allows the modeling of non-Newtonian fluids, anomalous diffusion, and other phenomena exhibiting memory effects and non-local behavior in areas such as fluid dynamics, combustion, and nonlinear acoustics.

The remaining thesis is organized in the following way:

Chapter 1, entitled "Preliminaries and notations" details the concepts related to fractional integrals, derivatives, and certain results concerning the existence and uniqueness of the solution of fractional differential equations.

Chapter 2, entitled "Decomposition approaches for classical nonlinear partial differential equations" presents four decomposition approaches for finding approximate analytical solutions to classical nonlinear partial differential equations. Several numerical examples are given to illustrate the validity and effectiveness of the presented approaches.

Chapter 3, entitled "Finding solutions to systems of nonlinear time-fractional partial differential equations" focuses on adapting the decomposition approaches introduced in Chapter 2 to resolve systems of nonlinear time-fractional partial differential equations involving the Caputo time-fractional derivative. The accuracy and reliability of these approaches are demonstrated by resolving various types of such systems.

Chapter 4, entitled "A new mixed approach for resolving the system of nonlinear time-fractional Burgers equations" introduces the mathematical formulation of KhDJA to resolve systems of nonlinear time-fractional partial differential equations, specifically the system of nonlinear time-fractional Burgers equations. It also presents the convergence of the proposed approach in the sense of Banach space and numerical simulations of some applications of the proposed problem.

Finally, the conclusions of this work are presented in the last paragraph entitled "Conclusions and future research directions".

Chapter 1

Preliminaries and notations

In the current chapter, we will cover the main concepts and findings associated with the theory of fractional calculus, which we will use extensively in this thesis. In addition, we will give the necessary conditions for the existence and uniqueness of the solution to the fractional differential equations. We conclude this chapter with some applications of fractional calculus in various scientific fields.

1.1 Theory of fractional calculus

Fractional calculus is a generalization of ordinary derivation and integration to an arbitrary order. Non-integer derivatives are widely used in many areas of science and engineering, such as probability, viscoelasticity, electronics, economics, mechanics and biology, and others. A special interest in fractional derivatives is related to the mechanical modeling of rubbers and erasers. In short, all types of materials that retain the memory of their previous deformations, especially those of a viscoelastic nature. In fact, the fractional derivative appears automatically. In this paragraph, we will present the fractional integral operator as well as the two most common definitions of fractional derivatives, those of the Riemann-Liouville and Caputo, presenting the most important properties of these concepts.

1.1.1 L^P Spaces

Definition 1.1.1 ***[9]*** *Let $[a,b]$ $(-\infty \leq a \leq b \leq \infty)$ a finite or infinite interval of $\mathbb{R}$ and $p \in \mathbb{R}$ with $1 \leq p < \infty$. We signify by $L^p([a,b])$ the space of equivalence classes of $p-$integrable power functions on $[a,b]$ with values in $\mathbb{C}$*

$$L^p([a,b]) = \{u : [a,b] \to \mathbb{C},\ u \text{ measurable and } \|u\|_{L^p} < \infty\},$$

with

$$\|u\|_{L^p([a,b])} = \left(\int_a^b |u(t)|^p\, dt\right)^{\frac{1}{p}}.$$

The space $L^p([a,b])$ equipped with the norm $\|.\|_{L^p}$ is a Banach space.

If $p = 2$, then $L^2([a,b])$ is the space of equivalence classes of square-integrable measurable functions on $[a,b]$.

The space $L^2([a,b])$ provided by the norm

$$\|u\|_{L^2([a,b])} = \left(\int_a^b |u(t)|^2\, dt\right)^{\frac{1}{2}},$$

is a Hilbert space.

1.1.2 Space of absolutely continuous functions

Definition 1.1.2 ***[9]*** *Let $[a,b]$ $(-\infty \leq a \leq b \leq \infty)$ a finite or infinite interval of $\mathbb{R}$. We signify by $AC([a,b])$ the space of absolutely continuous functions on $[a,b]$ consisting of functions u which are primitives of Lebesgue-integrable functions i.e.*

$$u \in AC([a,b]) \Longleftrightarrow \exists \phi \in L^1([a,b]) \text{ such that } u(t) = u(a) + \int_a^t \phi(s)ds, \text{ for all } t \in [a,b].$$

Definition 1.1.3 ***[9]*** *We signify by $AC^n([a,b])$, $n \in \mathbb{N}^*$, the space of functions u defined on $[a,b]$ with values in $\mathbb{C}$ which have continuous derivatives on $[a,b]$ up to order $n-1$ and such that $u^{(n-1)} \in AC([a,b])$ i.e.*

$$AC^n([a,b]) = \left\{u : [a,b] \longrightarrow \mathbb{C} : u^{(k)} \in C([a,b]), k = 0...n-1, u^{(n-1)} \in AC([a,b])\right\}.$$

Remark 1.1.1 *For $n = 1$, we have*

$$AC^1([a,b]) = AC([a,b]).$$

A characterization of the functions of the space $AC^n([a,b])$ is given by the following lemma.

Lemma 1.1.1 *A function $u \in AC^n([a,b]), n \in \mathbb{N}^*$ if and only if it is given by the following relation*

$$u(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}u^{(n)}(s)ds + \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k.$$

1.1.3 Special functions

In this paragraph, we present the Gamma function, the Beta function, and the Mittag-Leffler function, which will be used later. These three functions play an important role in the theory of fractional differential equations.

Definition 1.1.4 ***[25]*** *The Gamma function is a function that naturally extends the factorial to real numbers and even to complex numbers. For $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, we define the gamma function by*

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt. \tag{1.1.1}$$

By integrating by parts in (1.1.1), we show that

$$\Gamma(z+1) = z\Gamma(z), \mathrm{Re}(z) > 0. \tag{1.1.2}$$

Relation (1.1.2) makes it possible to establish that

$$\Gamma(n+1) = n!, \forall n \in \mathbb{N}.$$

Definition 1.1.5 ***[25]*** *The Beta function is a type of Euler integral defined for complexe numbers z and w by*

$$\beta(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0. \tag{1.1.3}$$

The Beta function is related to the gamma function by the following relation

$$\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0. \tag{1.1.4}$$

It follows from (1.1.4) that

$$\beta(z,w) = \beta(w,z), \ \ \mathrm{Re}(z) > 0, \ \ \mathrm{Re}(w) > 0.$$

Definition 1.1.6 *[25] The Mittag-Leffler function of two parameters α and β is expressed as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ z \in \mathbb{C},\ \alpha, \beta > 0. \tag{1.1.5}$$

For $\beta = 1$, we have the Mittag-Leffler function of one parameter α expressed as

$$E_{\alpha,1}(z) = E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},\ z \in \mathbb{C},\ \alpha > 0. \tag{1.1.6}$$

For $\alpha = \beta = 1$, we have

$$E_{1,1}(z) = e^z.$$

1.1.4 Riemann-Liouville fractional integral and derivative

According to the Riemann-Liouville approach to fractional calculus, the concept of a fractional integral of order $\alpha, (\alpha > 0)$ generalizes the famous formula of integrals repeated $n-$times

$$\begin{aligned} I_a^n u(t) &= \int_a^t ds_1 \int_a^{s_1} ds_2 ... \int_a^{s_{n-1}} u(s_n) ds_n \\ &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} u(s) ds, n \in \mathbb{N}^*, \end{aligned}$$

which reduces the calculation of the nth primitive of a continuous function u on an interval $[a, b]$ to a single convolution-type integral.

Let us signify by $D^n, n \in \mathbb{N}$, the derivation operator of order n, then we have

$$D^n I_a^n = I_d, I_a^n D^n \neq I_d,$$

where I_d is the identity operator.

Definition 1.1.7 *[25] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u \in L^1([a, b])$ is expressed by*

$$I_a^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, t > a. \tag{1.1.7}$$

Definition 1.1.8 *[22] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function u defined on an interval $[a, b]$, is expressed by*

$$\begin{aligned} D_a^\alpha u(t) &= D^n I_a^{n-\alpha} u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s) ds, t > a, \end{aligned} \tag{1.1.8}$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

In particular, if $\alpha = 0$, then

$$D_a^0 u(t) = I_a^0 u(t) = u(t).$$

If $\alpha = n \in \mathbb{N}$, then

$$D_a^n u(t) = u^{(n)}(t).$$

Moreover, if $0 < \alpha < 1$, then $n = 1$, so

$$D_a^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} u(s) ds, t > a.$$

Remark 1.1.2 *The definition of fractional-order derivation is based on that of fractional-order integration, a fractional-order derivation has a global character, unlike an integer derivation. It turns out that the fractional-order derivative of a function requires knowledge of $u(t)$ over the entire interval $]a, t[$, where, as in the integer case, only local knowledge of u around t is necessary. This property allows fractional-order systems to be interpreted as long-memory systems, and thus the entire set of systems to be interpreted as short-memory systems.*

Example 1.1.1 *Let $\alpha > 0, \gamma > -1$, and $u(t) = (t-a)^\gamma$, then*

$$\begin{aligned} (1)\ I_a^\alpha u(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha}, \\ (2)\ D_a^\alpha u(t) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}. \end{aligned}$$

Indeed,

$$I_a^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (s-a)^\gamma ds. \tag{1.1.9}$$

By making the following variable change

$$s - a = \theta(t - a) \Rightarrow ds = (t - a)d\theta, \tag{1.1.10}$$

then, (1.1.9) becomes

$$\begin{aligned} I_a^\alpha u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t ((t-a) - \theta(t-a))^{\alpha-1} (\theta(t-a))^\gamma (t-a) d\theta \\ &= \frac{(t-a)^{\alpha+\gamma}}{\Gamma(\alpha)} \int_0^1 \theta^\gamma (1-\theta)^{\alpha-1} d\theta. \end{aligned}$$

Employing (1.1.3) and the property (1.1.4) we attain to

$$\begin{aligned} I_a^\alpha u(t) &= \frac{\beta(\gamma+1,\alpha)}{\Gamma(\alpha)} (t-a)^{\alpha+\gamma} \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\alpha+\gamma}. \end{aligned}$$

Similarly, we can prove (2).

In particular, if $\gamma = 0$ and $\alpha > 0$, then the Riemann-Liouville fractional derivative of a constant is generally non-zero. In fact, we have

$$D_a^\alpha C = \frac{C}{\Gamma(1-\alpha)} (t-a)^{-\alpha}, \alpha > 0.$$

It is easy to establish the following result.

Proposition 1.1.1 *Let $\alpha > 0$ and $n = [\alpha] + 1$, then*

$$D_a^\alpha u(t) = 0 \Longleftrightarrow u(t) = \sum_{j=1}^{n} C_j (t-a)^{\alpha-j}, \forall C_1, ...C_n \in \mathbb{R}.$$

In particular, if $0 < \alpha < 1$, then

$$D_a^\alpha u(t) = 0 \Longleftrightarrow x(t) = C(t-a)^{\alpha-1}, \forall C \in \mathbb{R}.$$

Let us now state some properties of the operators D_a^α and I_a^α.

Lemma 1.1.2 ***[25]*** *The Riemann-Liouville fractional integral operator $I_a^\alpha, \alpha > 0$ is linear and bounded from the space $L^p([a,b])$ $(1 \leq p \leq \infty)$ in itself, i.e. $\exists k > 0$ such that*

$$\left\| I_a^\alpha u \right\|_p \leq k \left\| u \right\|_p \forall u \in L^p([a,b]), \text{ where } k = \frac{(b-a)^\alpha}{\alpha \left| \Gamma(\alpha) \right|}.$$

The following result characterizes the necessary conditions for the existence of the Riemann-Liouville fractional derivative D_a^α.

Lemma 1.1.3 *[25] Let $\alpha > 0$ and $n = [\alpha] + 1$. If $u \in AC^n([a,b])$, then the Riemann-Liouville fractional derivative $D_a^\alpha u$ exists almost everywhere on $[a,b]$ and is expressed by the formula*

$$D_a^\alpha u(t) = \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-a} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds.$$

An important property of the Riemann-Liouville fractional integral operator I_a^α called the property of semigroup, is expressed by the following lemma.

Lemma 1.1.4 *If $\alpha > 0$ and $\beta > 0$, then the relation*

$$I_a^\alpha I_a^\beta u(t) = I_a^{\alpha+\beta} u(t), \tag{1.1.11}$$

is satisfied almost everywhere on $[a,b]$ for all $u \in L^p([a,b]), 1 \leq p < \infty$.

Proof. Let $u \in L^p([a,b])$, then according to Fubini's theorem, we have

$$\begin{aligned} I_a^\alpha I_a^\beta u(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} I_a^\beta u(s) ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-s)^{\alpha-1} ds \int_a^s (s-\xi)^{\beta-1} u(\xi) d\xi \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t u(\xi) d\xi \int_\tau^t (t-s)^{\beta-1} ds \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-\xi)^{\alpha+\beta-1} u(\xi) d\xi \\ &= I_a^{\alpha+\beta} u(t). \end{aligned}$$

Thus, the goal was achieved. ■

One of the most important properties that connect the Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral is the following.

Lemma 1.1.5 *For $\alpha > 0$ and $u \in L^1([a,b])$, we have*

$$D_a^\alpha I_a^\alpha u(t) = u(t), \ \ a.e. \ on \ [a,b]. \tag{1.1.12}$$

Proof. According to (1.1.8) and the property (1.1.11), we have for $n = [\alpha] + 1$

$$D_a^\alpha I_a^\alpha u(t) = D^n I_a^{n-\alpha} I_a^\alpha u(t) = D^n I_a^n u(t) = u(t), \text{ a.e. on } [a, b],$$

which proves the result. ■

Remark 1.1.3 *Property (1.1.12) means that the Riemann-Liouville fractional derivative operator is the left inverse of the Riemann-Liouville fractional integral operator of the same order.*

1.1.5 Caputo fractional derivative

The Riemann-Liouville fractional derivative has some drawbacks for modeling real phenomena. The problems studied require a definition of fractional derivatives that allows the use of physically interpretable initial conditions, including $u(0), u'(0), u''(0)$, etc. These failures led to an alternative definition of fractional derivatives meeting these requirements in the late 1960s, introduced by Caputo **[11]**. In fact, Caputo and Mainardi **[12]** used this definition in their work on viscoelasticity.

In this portion, we give the definition of the fractional derivative in the sense of Caputo as well as some essential properties.

Let $[a, b]$ be a finite interval of $\mathbb{R}$ and let I_a^α and D_a^α be the Riemann-Liouville fractional integral and derivative operators given by (1.1.7) and (1.1.8).

Definition 1.1.9 *[22] The Caputo fractional derivative of order $\alpha > 0$ of a function u defined on $[a, b]$ is expressed by*

$$^C D_a^\alpha u(t) = D_a^\alpha \left(u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (t-a)^k \right), \tag{1.1.13}$$

where

$$n = [\alpha] + 1, \text{ if } \alpha \notin \mathbb{N}, \text{ and } n = \alpha \text{ if } \alpha \in \mathbb{N}^*. \tag{1.1.14}$$

If $\alpha = 0$, then

$$^C D_a^0 u(t) = u(t).$$

In particular, when $0 < \alpha < 1$, the relation (1.1.13) takes the form

$$ {}^{C}D_a^{\alpha}u(t) = D_a^{\alpha}\left(|u(t) - u(a)|\right). $$

Thus, if $\alpha \notin \mathbb{N}$ and u is a function for which the Caputo fractional derivative (1.1.13) and the Riemann-Liouville fractional derivative (1.1.8) exist, then they are related to each other by the relation

$$ {}^{C}D_a^{\alpha}u(t) = D_a^{\alpha}u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-a}, n = [\alpha] + 1. \tag{1.1.15} $$

In particular, if $0 < \alpha < 1$, then we have

$$ {}^{C}D_a^{\alpha}u(t) = D_a^{\alpha}u(t) - \frac{u(a)}{\Gamma(\alpha-1)}(t-a)^{-a}. $$

According to relation (1.1.15), if $\alpha \notin \mathbb{N}$, the Caputo fractional derivative (1.1.13) is identical to the Riemann-Liouville fractional derivative (1.1.8) if the function u and all its derivatives up to order $n-1$ ($n = [\alpha] + 1$) vanish at the point a, i.e.

$$ {}^{C}D_a^{\alpha}u(t) = D_a^{\alpha}u(t) \Longleftrightarrow u(a) = u^{'}(a) = ... = u^{(n-1)}(a) = 0. $$

If $\alpha = n \in \mathbb{N}$ and the usual derivative $u^{(n)}(t)$ exists, then ${}^{C}D_a^{\alpha}u(t)$ coincides with $u^{(n)}(t)$, i.e.

$$ {}^{C}D_a^{\alpha}u(t) = u^{(n)}(t). \tag{1.1.16} $$

The Caputo fractional derivative (1.1.13) is defined for functions $u(t)$ for which the Riemann-Louville derivative (1.1.8) exists, in particular for functions $u(t) \in AC^n\left([a,b]\right)$ we have the following theorem.

Theorem 1.1.1 *[22] Let $\alpha > 0$, and n be given by (1.1.14). If $u \in AC^n\left([a,b]\right)$, then the Caputo fractional derivative ${}^{C}D_a^{\alpha}u(t)$ exists almost everywhere on $[a,b]$.*

i) If $\alpha \notin \mathbb{N}$, then ${}^{C}D_a^{\alpha}u(t)$ is expressed by

$$ \begin{aligned} {}^{C}D_a^{\alpha}u(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds, \\ &= I^{n-\alpha}D^n u(t). \end{aligned} \tag{1.1.17} $$

In particular, when $0 < \alpha < 1$ *and* $u \in AC([a,b])$, *then*

$$\begin{aligned} {}^{C}D_a^{\alpha}u(t) &= \frac{1}{\Gamma(n-1)}\int_a^t (t-s)^{-\alpha}u^{'}(s)ds, \\ &= I^{1-\alpha}u^{'}(t). \end{aligned} \tag{1.1.18}$$

ii) If $\alpha = n$, *then*

$${}^{C}D_a^{n}u(t) = u^{(n)}(t).$$

Proof. According to the definition of the Caputo fractional derivative (1.1.13), we have

$$\begin{aligned} {}^{C}D_a^{\alpha}u(t) &= D_a^{\alpha}\left(u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right) \\ &= D^n I_a^{n-\alpha}\left(u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right). \end{aligned}$$

We put

$$D(t) = I_a^{n-\alpha}\left(u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right).$$

From (1.1.7), we have

$$D(t) = \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left(u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k\right)ds.$$

Integrating by parts, we will have

$$\begin{aligned} D(t) &= \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left(u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k\right)ds \\ &= -\frac{(t-s)^{n-\alpha}}{\Gamma(n-\alpha+1)}\left(u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k\right)_{s=a}^{s=t} \\ &\quad + \frac{(t-s)^{n-\alpha}}{\Gamma(n-\alpha+1)}\int_a^t (t-s)^{n-\alpha}D\left(u(s) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(s-a)^k\right)ds \\ &= I_a^{n-\alpha+1}D\left(u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right). \end{aligned}$$

By repeating this process n-times, we get

$$\begin{aligned} D(t) &= I_a^{n-\alpha+n}D^n\left(u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right) \\ &= I_a^{n}I^{n-\alpha}D^n\left(u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k\right). \end{aligned}$$

But

$$\sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(t-a)^k,$$

is a polynomial of degree $n-1$, therefore

$$D(t) = I_a^n I_a^{n-\alpha} D^n u(t).$$

So,

$$\begin{aligned} {}^C D_a^\alpha u(t) &= D^n D(t) \\ &= D^n I_a^n I_a^{n-\alpha} D^n u(t) \\ &= I_a^{n-\alpha} D^n u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds. \end{aligned}$$

Thus, the goal was achieved. ■

Theorem 1.1.2 *[22] Let $\alpha > 0$, n expressed by (1.1.14) and $u \in C^n([a,b])$. Then the Caputo fractional derivative ${}^C D_a^\alpha u(t)$ is continuous on $[a,b]$.*

i) If $\alpha \notin \mathbb{N}$, then ${}^C D_a^\alpha u(t)$ is expressed by (1.1.17) and takes the forme (1.1.18) for $0 < \alpha < 1$.

ii) If $\alpha = n \in \mathbb{N}$, then

$${}^C D_a^n u(t) = u^{(n)}(t).$$

Example 1.1.2 *Let $0 < \alpha < 1$ and $u(t) = (t-a)^\gamma, \gamma > -1$. Then*

$${}^C D_a^\alpha u(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha}.$$

Indeed,

$$\begin{aligned} {}^C D_a^\alpha u(t) &= I_a^{1-\alpha} u'(t) \\ &= \frac{\gamma}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha}(s-a)^{\gamma-1} ds. \end{aligned}$$

By making the variable change (1.1.10), we will have

$$\begin{aligned} {}^C D_a^\alpha u(t) &= \frac{\gamma}{\Gamma(1-\alpha)}(t-a)^{\gamma-\alpha} \int_0^1 \theta^{\gamma-1}(1-\theta)^{-\alpha} d\theta \\ &= \frac{\gamma}{\Gamma(1-\alpha)}(t-a)^{\gamma-\alpha}\beta(\gamma, 1-\alpha) \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha}. \end{aligned}$$

In particular, the Caputo fractional derivative of a constant is zero

$$^{C}D_{a}^{\alpha}C=0,\forall C\in\mathbb{R}.$$

The Caputo fractional derivative, like the Riemann-Liouville fractional derivative, represents the left inverse operation of the fractional integral I_a^α.

Lemma 1.1.6 *Let $\alpha>0$ and $u\in L^{\infty}([a,b])$, then*

$$^{C}D_{a}^{\alpha}I_{a}^{\alpha}u(t)=u(t).$$

Lemma 1.1.7 *Let $\alpha>0$ and $u\in L^{\infty}([a,b])$, then*

$$I_{a}^{\alpha\,C}D_{a}^{\alpha}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}t^{k}, \tag{1.1.19}$$

for certain constants $u^{(k)}(0)\in\mathbb{R}$, $k\in\{0,1,...,n-1\}$, $n=[\alpha]+1$.

We defined the Caputo fractional derivative of a function u on a finite interval $[a,b]$ by (1.1.15), and we saw in Theorem 1.1.1 that it can be represented by (1.1.17) and (1.1.18) provided that $u\in AC^{n}([a,b])$. In fact, formula (1.1.17) allows us to define the Caputo fractional derivative on the semi-axis $\mathbb{R}^{+}$. Indeed, the Caputo fractional derivative of a function $u\in AC^{n}([a,\infty))$ on the semi-axis $\mathbb{R}^{+}$ is given by

$$^{C}D_{0^{+}}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}u^{(n)}(s)ds,t>0.$$

In particular, if $0<\alpha<1$ and $u\in AC^{1}([a,\infty))$, then

$$^{C}D_{0^{+}}^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-s)^{-\alpha}u^{'}(s)ds,t>0.$$

Remark 1.1.4 *If we have $u=\{u(X,t):X=(x_1,x_2,...,x_n)\in\mathbb{R}^n$ and $t\in\mathbb{R}^*\}$ is a continuous function, then the Riemann-Liouville fractional integral operatr I_t^α and the Caputo time-fractional derivative operator $^{C}D_t^\alpha$ of the function $u(X,t)$ are expressed respectively by*

$$I_{t}^{\alpha}u(X,t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{0}^{t}(t-s)^{\alpha-1}u(X,s)ds,\alpha>0,\\ u(X,t),\qquad\qquad\alpha=0.\end{cases} \tag{1.1.20}$$

and

$$^{C}D_{t}^{\alpha}u(X,t)=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{0}^{t}(t-s)^{n-\alpha-1}u^{(n)}(X,s)ds,n-1<\alpha<n,\\ u^{(n)}(X,t),\qquad\qquad\alpha=n.\end{cases} \tag{1.1.21}$$

1.2 Fractional differential equations via Caputo fractional derivative

Definition 1.2.1 *Let $\alpha > 0, \alpha \notin \mathbb{N}$, $n = [\alpha] + 1$ and $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, then*

$$^{C}D^{\alpha}u(t) = f(t, u(t)),$$

is called a fractional differential equation in the sense of Caputo. We use the initial conditions of this equation as

$$u^{(k)}(0) = u_k, k = 0, 1, 2, ..., n-1,$$

where Ω is an open set of $\mathbb{R}^2$.

1.2.1 The results of existence and uniqueness

In this paragraph, we study the existence and uniqueness of the solution of a Cauchy problem for fractional differential equations in the sense of Caputo and we have the problem in the following form

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t)), \\ u(0) = u_0, \end{cases} \tag{1.2.1}$$

where $t \in [0, T], 0 < \alpha \leq 1, u_0 \in \mathbb{R}$, and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 1.2.1 *[3] (Banach's fixed point theorem)*

Let B be a Banach space and a contracting operator $\mathbb{F} : B \longrightarrow B$. Then $\mathbb{F}$ admits a unique fixed point. i.e $\exists u \in B$ such that $\mathbb{F}(u) = u$.

Theorem 1.2.2 *[3] Let $\mathbb{F}$ be an operator defined on a Banach space B, such that $\mathbb{F}^n$ is contracting on $B, \forall \ \ n \geq 1$. Then $\mathbb{F}$ has a unique fixed point.*

Lemma 1.2.1 *Let $0 < \alpha \leq 1$ and let $h : [0, T] \to \mathbb{R}$ a continuous function. A function u is a solution of the Cauchy problem*

$$\begin{cases} {}^{C}D^{\alpha}u(t) = h(t), t \in [0, T], 0 < \alpha \leq 1, \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{1.2.2}$$

if and only if it is the solution of the integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. \tag{1.2.3}$$

Proof. Applying the operator I^α to Eq. (1.2.2) yields

$$\begin{aligned} I^{\alpha C} D^\alpha u(t) &= I^\alpha h(t) \\ &\Longrightarrow u(t) + C_0 = I^\alpha h(t) \\ &\Longrightarrow u(t) = I^\alpha h(t) - C_0. \end{aligned}$$

The initial condition gives

$$\begin{aligned} u(0) &= I^\alpha h(0) - C_0 = -C_0 \\ &\Longrightarrow C_0 = -u_0. \end{aligned}$$

So

$$\begin{aligned} u(t) &= I^\alpha h(t) - (-u_0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + u_0. \end{aligned}$$

On the other hand we have

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &= I^\alpha h(t) + u_0. \end{aligned}$$

Applying ${}^C D^\alpha$ to Eq. (1.2.3) gives

$$\begin{aligned} {}^C D^\alpha u(t) &= {}^C D^\alpha I^\alpha h(t) + {}^C D^\alpha u_0 \\ &= h(t). \end{aligned}$$

All that remains is to verify that $u(0) = u_0$

$$\begin{aligned} u(0) &= I^\alpha h(0) + u_0 = 0 + u_0 \\ &= u_0. \end{aligned}$$

So, there is a solution to the problem (1.2.2).

Thus, the goal was achieved. ■

Theorem 1.2.3 *Let $0 < \alpha \leq 1$ and $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and verifies the following Lipschitz condition*

$$|f(t,u) - f(t,v)| \leq k\,|u - v|\,, \forall t \in [0,T] \text{ and } u, v \in \mathbb{R}.$$

If

$$\frac{kT^{\alpha}}{\Gamma(\alpha+1)} < 1,$$

then the Cauchy problem (1.2.1) admits a unique solution $u(t) \in C([0,T],\mathbb{R})$.

Proof. First, we transform the problem (1.2.1) into a fixed point problem (Lemma 1.2.1), by considering the operator

$$\begin{aligned} \mathbb{F} : C([0,T],\mathbb{R}) &\longrightarrow C([0,T],\mathbb{R}) \\ u &\longrightarrow \mathbb{F}u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds, \end{aligned}$$

where $C([0,T],\mathbb{R})$ is the Banach space of continuous functions u defined from $[0,T]$ in $\mathbb{R}$, equipped with the norm

$$\|u\| = \sup_{t\in[0,T]} |u(t)|\,.$$

It is clear that the fixed points of operator $\mathbb{F}$ are the solutions of problem (1.2.1). $\mathbb{F}$ is well defined, in fact: if $u(t) \in C([0,T],\mathbb{R})$, then $\mathbb{F}u(t) \in C([0,T],\mathbb{R})$.

To show that $\mathbb{F}$ admits a fixed point, it suffices to show that $\mathbb{F}$ is a contraction, in fact if $u, v \in C([0,T],\mathbb{R}), t \in [0,T]$, using the Lipschitiz condition we obtain

$$\begin{aligned} |\mathbb{F}u(t) - \mathbb{F}v(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(f(s,u(s)) - f(s,v(s))\right)ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t |f(s,u(s)) - f(s,v(s))|\,(t-s)^{\alpha-1}ds \\ &\leq \frac{k}{\Gamma(\alpha)}\int_0^t |u(s) - v(s)|\,(t-s)^{\alpha-1}ds \\ &\leq \frac{k}{\Gamma(\alpha)}\,|u(s) - v(s)|\int_0^t (t-s)^{\alpha-1}ds \\ &\leq \frac{kT^{\alpha}}{\alpha\Gamma(\alpha)}\,|u(s) - v(s)|\,. \end{aligned}$$

So,

$$\|\mathbb{F}u - \mathbb{F}v\| \leq \frac{kT^{\alpha}}{\Gamma(\alpha+1)}\,\|u - v\|\,.$$

On account of $\frac{kT^{\alpha}}{\Gamma(\alpha+1)} < 1$, we can conclude that $\mathbb{F}$ is a contraction and according to Banach's fixed point theorem 1.2.1, $\mathbb{F}$ admits a unique fixed point which is a solution to the Cauchy problem (1.2.1).

Thus, the goal was achieved. ■

1.3 Applications of fractional calculus

Fractional calculus is increasingly used in various areas of research. However, the growing interest in this concept lies in its applications in basic and applied sciences. It should be noted that in most of the fields mentioned below, fractional operators are used to account for memory effects. Let us mention the works **[27],[32]** which bring together various applications of fractional calculus.

In automatic:

In automatic model, it was only in the early 1990s that the CRONE regulator (non-integer order robust control) was proposed by A. Oustaloup **[30]**. By exploiting the advantages of the properties of fractional order systems, this regulator ensures the robustness of the control in a given frequency band. The success of this approach was enormous, several variants of this order emerged (1st, 2nd and 3rd generations). Since then, fractional order has attracted the attention of many researchers. In 1999, I. Podlubny **[26]** proposed the $I^{\alpha}D^{\beta}$ regulator, which includes a fractional integration of order α and a fractional derivative of order β, thus expanding the application of fractional calculus to control theory.

In physics:

One of the most remarkable applications of fractional calculus in physics is in the context of classical mechanics. F. Riewe **[29]** showed that the Lagrangian containing the time derivatives of fractional orders leads to an equation of motion with non-conservative forces such as friction. This result is remarkable because frictional forces and nonconservative forces are essential for the treatment of usual macroscopic variations, and consequently, for the most common methodological advances in classical mechanics. F. Riewe generalized the

usual calculus of variations to the Lagrangian dependent on fractional derivatives [**28**] in order to treat the usual nonconservative forces. Furthermore, several approaches have been developed to generalize the principle of least action and the Euler-Lagrange equation to the case of fractional derivatives [**2**],[**4**].

In continuous media mechanics:

The deformation of continuous media (solid or fluid) is often described using two tensors, one for deformations indicate ε_{ij}, and the other for stresses σ_{ij}. Some materials, such as polymers (glue, rubber, etc.), exhibit behavior intermediate between viscous and elastic, described as viscoelastic. These systems can be modeled using the following relationship between the two tensors:

$$\sigma_{ij} = E\varepsilon_{ij}(t) + \eta D^{\alpha}\varepsilon_{ij}(t), 0 < \alpha < 1.$$

This law is justified by R.L. Bagley and P.J. Torvik in [**6**],[**7**] (for $\alpha = \frac{1}{2}$). In [**24**], the introduction of fractional derivatives in the case of polymers is motivated by the following analysis: due to the length of the fibers, the applied deformations take time to communicate step by step (the length of the wound fibers being much greater than the geometric distance). They gradually dampen and induce memory effects (the state at time t will depend on the previous states). If the stress decreases when $t^{-(1+\alpha)}$, it can induce a fractional derivative of order α. This operator thus makes it possible to give a simple macroscopic description (requiring only a few parameters) of complex microscopic phenomena.

In acoustics:

For some wind musical instruments, visco-thermal losses can be modeled effectively using time-fractional derivatives [**16**].

Chapter 2

Decomposition approaches for classical nonlinear partial differential equations

In the current chapter, we provide four decomposition approaches to find the approximate analytical solutions of classical nonlinear partial differential equations. Several numerical examples are given to illustrate the accuracy and efficiency of the presented approaches.

2.1 Adomian decomposition approach (ADA)

The ADA is a very effective technique for resolving large classes of nonlinear ordinary and partial differential equations, with essential applications in different fields of applied mathematics, engineering, physics, and biology. George Adomian [**1**] first introduced it to find an approximate solution to a differential equation. One of the advantages of the ADA is that it can provide an analytical approximation to a relatively large class of nonlinear equations without linearization, perturbation, closure approximations, or discretization methods, which entail massive numerical computations. Thanks to the presented decomposition approach, the problem structure will not be changed and no linearization will be performed.

2.1.1 Analysis of the ADA

To clarify the fundamental aspects of the ADA, let us consider the following functional equation

$$\mathcal{F}\left(u(x,t)\right) = f(x,t), \tag{2.1.1}$$

where $u = \left\{u(x,t), (x,t) \in \mathbb{R} \times \mathbb{R}^{+}\right\}$ signify the unknown function, $\mathcal{F}$ is a nonlinear ordinary or partial differential operator comprising linear and nonlinear terms and f is a known function. The linear part is generally decomposed into $\mathcal{L} + \mathcal{R}$, where $\mathcal{L}$ is an invertible differential operator and $\mathcal{R}$ is the rest of the linear operator. Under these conditions, Eq. (2.1.1) can be written as follows

$$\mathcal{L}\left(u(x,t)\right) + \mathcal{R}\left(u(x,t)\right) + \mathcal{N}\left(u(x,t)\right) = f(x,t), \tag{2.1.2}$$

where $\mathcal{N}$ signify a nonlinear differential operator.

Eq. (2.1.2) reduced to

$$\mathcal{L}\left(u(x,t)\right) = f(x,t) - \mathcal{R}\left(u(x,t)\right) - \mathcal{N}\left(u(x,t)\right). \tag{2.1.3}$$

By applying the inverse operator $\mathcal{L}_t^{-1}(.)$ to Eq. (2.1.3), we get

$$\mathcal{L}_t^{-1}(\mathcal{L}\left(u(x,t)\right)) = \mathcal{L}_t^{-1}\left(f(x,t)\right) - \mathcal{L}_t^{-1}\left(\mathcal{R}\left(u(x,t)\right)\right) - \mathcal{L}_t^{-1}\left(\mathcal{N}\left(u(x,t)\right)\right), \tag{2.1.4}$$

where $\mathcal{L}_t^{-1}(.) = \int\int ... \int (.)\,(dt)^m$ signify the inverse operator of $\mathcal{L}(.)$.

Since

$$\mathcal{L}_t^{-1}(\mathcal{L}\left(u(x,t)\right)) = u(x,t) - \phi(x),$$

where $\phi(x)$ signify the constant of integration.

Consequently, Eq. (2.1.4) becomes

$$u(x,t) = \phi(x) + \mathcal{L}_t^{-1}\left(f(x,t)\right) - \mathcal{L}_t^{-1}\left(\mathcal{R}\left(u(x,t)\right)\right) - \mathcal{L}_t^{-1}\left(\mathcal{N}\left(u(x,t)\right)\right). \tag{2.1.5}$$

The ADA provides the solution in the form of an infinite series as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{2.1.6}$$

and the nonlinear term $\mathcal{N}(u(x,t))$, can be decomposed as

$$\mathcal{N}(u(x,t)) = \sum_{n=0}^{\infty} \mathcal{A}_n(u_n(x,t)). \tag{2.1.7}$$

Here, $\mathcal{A}_n(u_n(x,t)), n \geq 0$ are called the Adomian polynomials of the nonlinear term $\mathcal{N}(u(x,t))$, and its values can be obtained from the following algorithm

$$\mathcal{A}_n(u_n(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[\mathcal{N}\left(\sum_{k=0}^{n} \lambda^k u_k(x,t)\right)\right]_{\lambda=0}, n \geq 0, \tag{2.1.8}$$

where λ is a real parameter.

Substituting Eqs. (2.1.5) and (2.1.6) into Eq. (2.1.7), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = \phi(x) + \mathcal{L}_t^{-1}(f(x,t)) - \mathcal{L}_t^{-1}\mathcal{R}\left(\sum_{n=0}^{\infty} u_n(x,t)\right) - \mathcal{L}_t^{-1}\left(\sum_{n=0}^{\infty} \mathcal{A}_n(u_n(x,t))\right). \tag{2.1.9}$$

Comparing the two sides of Eq. (2.1.9), we obtain the values of the terms $u_n(x,t), n \geq 0$ as follows

$$\begin{aligned}
u_0(x,t) &= \phi(x) + \mathcal{L}_t^{-1}(f(x,t)), \\
u_1(x,t) &= -\mathcal{L}_t^{-1}\mathcal{R}(u_0(x,t)) - \mathcal{L}_t^{-1}(\mathcal{A}_0(u_0(x,t))), \\
u_2(x,t) &= -\mathcal{L}_t^{-1}\mathcal{R}(u_1(x,t)) - \mathcal{L}_t^{-1}(\mathcal{A}_1(u_1(x,t)), \\
u_3(x,t) &= -\mathcal{L}_t^{-1}\mathcal{R}(u_2(x,t)) - \mathcal{L}_t^{-1}(\mathcal{A}_2(u_2(x,t)), \\
&\vdots
\end{aligned}$$

and so on.

In general, the recurrence relation is expressed by

$$\begin{cases} u_0(x,t) = \phi(x) + \mathcal{L}_t^{-1}(f(x,t)), \\ u_{n+1}(x,t) = -\mathcal{L}_t^{-1}\mathcal{R}(u_n(x,t)) - \mathcal{L}_t^{-1}(\mathcal{A}_n(u_n(x,t)). \end{cases} \tag{2.1.10}$$

Finally, we find the solution to Eq. (2.1.1) by calculating the terms $u_0(x,t), u_1(x,t), u_2(x,t), ...$ from the recurrence relation (2.1.10) and using Eq. (2.1.6).

Example 2.1.1 *Considering the nonlinear Burger equation*

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\frac{\partial u(x,t)}{\partial x} = 0, \tag{2.1.11}$$

under the initial condition

$$u(x,0) = x. \tag{2.1.12}$$

Before starting the resolution steps, let us define the following operators

$$\mathcal{L}\left(u(x,t)\right) = \frac{\partial u(x,t)}{\partial t}, \mathcal{R}\left(u(x,t)\right) = -\frac{\partial^2 u(x,t)}{\partial x^2},\ \mathcal{N}\left(u(x,t)\right) = u(x,t)\frac{\partial u(x,t)}{\partial x},\ \textit{and}\ f(x,t) = 0.$$

Let's symbolize

$$\mathcal{L}_t(.) = \frac{\partial(.)}{\partial t}\ \textit{and}\ \mathcal{L}_t^{-1}(.) = \int_0^t (.)dt.$$

Applying the inverse operator $\mathcal{L}_t^{-1}(.)$ to Eq. (2.1.11), we get

$$u(x,t) = u(x,0) + \mathcal{L}_t^{-1}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) - \mathcal{L}_t^{-1}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right).$$

Replacing the initial condition (2.1.12), we get

$$u(x,t) = x + \mathcal{L}_t^{-1}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) - \mathcal{L}_t^{-1}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right). \tag{2.1.13}$$

Considering the solution $u(x,t)$ as an infinite series expressed by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.1.14}$$

Substituting Eq. (2.1.14) into Eq. (2.1.13), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x + \mathcal{L}_t^{-1}\left(\sum_{n=0}^{\infty}\frac{\partial^2 u_n(x,t)}{\partial x^2}\right) - \mathcal{L}_t^{-1}\left(\sum_{n=0}^{\infty}\mathcal{A}_n(u_n(x,t))\right), \tag{2.1.15}$$

where $\mathcal{A}_n(u_n(x,t)), n \geq 0$ signify the Adomian polynomials of the nonlinear term

$$\mathcal{N}\left(u(x,t)\right) = u(x,t)\frac{\partial u(x,t)}{\partial x}.$$

Using the formula (2.1.8), the first terms of $\mathcal{A}_n(u_n(x,t))$ are given by

$$\begin{aligned}
&\mathcal{A}_0(u_0(x,t)) = u_0(x,t)\frac{\partial u_0(x,t)}{\partial x},\\
&\mathcal{A}_1(u_1(x,t)) = u_1(x,t)\frac{\partial u_0(x,t)}{\partial x} + u_0(x,t)\frac{\partial u_1(x,t)}{\partial x},\\
&\mathcal{A}_2(u_2(x,t)) = u_0(x,t)\frac{\partial u_2(x,t)}{\partial x} + u_1(x,t)\frac{\partial u_1(x,t)}{\partial x} + u_2(x,t)\frac{\partial u_0(x,t)}{\partial x},\\
&\qquad\vdots
\end{aligned}$$

By identifying term by term in Eq. (2.1.15), we obtain

$$\begin{aligned}
u_0(x,t) &= x,\\
u_1(x,t) &= \mathcal{L}_t^{-1}\left(\frac{\partial u_0(x,t)}{\partial x}\right) - \mathcal{L}_t^{-1}(\mathcal{A}_0(u_0(x,t)))\\
&= \int_0^t (0)dt - \int_0^t x dt\\
&= -xt,\\
u_2(x,t) &= \mathcal{L}_t^{-1}\left(\frac{\partial^2 u_1}{\partial x^2}\right) - \mathcal{L}_t^{-1}(\mathcal{A}_1(u_1(x,t)))\\
&= \int_0^t (0)dt - \int_0^t (-2xt)\, dt\\
&= xt^2,\\
u_3(x,t) &= \mathcal{L}_t^{-1}\left(\frac{\partial^2 u_2}{\partial x^2}\right) - \mathcal{L}_t^{-1}(\mathcal{A}_1(u_1(x,t)))\\
&= \int_0^t (0)dt - \int_0^t 3xt^2 dt\\
&= -xt^3,\\
&\vdots\\
u_n(x,t) &= (-1)^n x t^n.
\end{aligned}$$

Finally, we find the solution using Eq. (2.1.14) in this way

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...\\
&= x(1 - t + t^2 - t^3 + ...)\\
&= x(-1)^n t^n\\
&= \frac{x}{1+t}, |t| < 1,
\end{aligned}$$

which is the exact solution of Eq. (2.1.11) under the initial condition (2.1.12).

2.2 Homotopy analysis approach (HAA)

The HAA, presented by Professor Shijun Liao [**23**] in 1992, is a powerful mathematical tool for resolving nonlinear problems, including nonlinear ordinary/partial differential equations.

The HAA uses the concept of homotopy from topology to generate a series solution by transforming the nonlinear problem into an infinite series of linear subproblems. One of its most important characteristic is the use of the convergence control parameter (h), which provides flexibility to ensure and control the convergence of the series solution, a capability often lacking in other semi-analytical approaches for highly nonlinear systems.

2.2.1 Analysis of the HAA

To explain the basic idea of the HAA, let us consider the nonlinear differential equation

$$\mathcal{N}\left[u(x,t)\right] = 0, \tag{2.2.1}$$

where $\mathcal{N}$ is a nonlinear operator and $u(x,t)$ is an unknown function, and x and t are independent variables.

In accordance with the HAA, we construct the so-called zeroth-order deformation equation as

$$(1-q)\mathcal{L}\left[\phi(x,t,q) - u_0(x,t)\right] = qhH(x,t)\mathcal{N}\left[\phi(x,t,q)\right], \tag{2.2.2}$$

where $q \in [0,1]$ is an embedding parameter, $h \neq 0$ is an auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $H(x,t) \neq 0$ is an auxiliary function, $u_0(x,t)$ is an initial guess of $u(x,t)$, $\phi(x,t,q)$ is an unknown function.

It is clear that when the embedding parameter $q = 0$ and $q = 1$, Eq. (2.2.2) becomes

$$\phi(x,t,0) = u_0(x,t), \;\; \phi(x,t,1) = u(x,t),$$

respectively.

Thus, as q increases from 0 to 1, the solution $\phi(x,t,q)$ various from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$. Expanding $\phi(x,t,q)$ in the Taylor series with respect to the embedding parameter q, we obtain

$$\phi(x,t,q) = u_0(x,t) + \sum_{n=1}^{+\infty} q^n u_n(x,t), \tag{2.2.3}$$

where

$$u_n(x,t) = \frac{1}{n!}\left.\frac{\partial^n \phi(x,t,q)}{\partial q^n}\right|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are correctly chosen, the series (2.2.3) converges at $q = 1$, then we have

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{+\infty} u_n(x,t),$$

which must be one of the solutions of the original nonlinear differental equation (2.2.1).

Let us define the vector

$$\overrightarrow{u_n}(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}.$$

Differentiating the zeroth-order deformation equation (2.2.2) $n-$times with respect to the embedding parameter q, and then dividing them by $n!$ and finally setting $q = 0$, we get the following n^{th}-order deformation equation

$$\mathcal{L}\left[u_n(x,t) - \chi_n u_{n-1}(x,t)\right] = hH(x,t)\mathcal{R}_n\left[\overrightarrow{u_{n-1}}(x,t)\right], \tag{2.2.4}$$

where

$$\mathcal{R}_n\left[\overrightarrow{u_{n-1}}(x,t)\right] = \frac{1}{(n-1)!}\left.\frac{\partial^{n-1}\mathcal{N}\left[\phi(x,t,q)\right]}{\partial q^{n-1}}\right|_{q=0},$$

and

$$\chi_n = \begin{cases} 0, n \leq 1 \\ 1, n > 1 \end{cases}.$$

Taking the inverse operator $\mathcal{L}^{-1}$ on both sides of Eq. (2.2.4), we get

$$u_n(x,t) = \chi_n u_{n-1}(x,t) + \mathcal{L}^{-1}\left[hH(x,t)\mathcal{R}_n\left[\overrightarrow{u_{n-1}}(x,t)\right]\right]. \tag{2.2.5}$$

The nth deformation equation (2.2.5) is a linear that can be easily solved.

Consequently, the approximate solution can be written in this way

$$u(x,t) = \sum_{n=0}^{N} u_n(x,t).$$

When $N \to \infty$, we obtain an accurate approximation solution to Eq. (2.2.1) expressed as

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t).$$

Example 2.2.1 *Considering the nonlinear gas dynamic equation*

$$\frac{\partial u(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t)(1-u(x,t)) = 0, \tag{2.2.6}$$

under the initial condition

$$u(x,0) = e^{-x}. \tag{2.2.7}$$

where $(x,t) \in [0,1] \times \mathbb{R}^+$.

For resolving Eq. (2.2.6) using HAA, according to the initial conditions expressed in Eq. (2.2.7), it is natural to choose

$$\mathcal{L}\left[\phi(x,t,q)\right] = \frac{\partial \phi(x,t,q)}{\partial t},$$

with the property $\mathcal{L}[C] = 0$, *where* C *is real constant.*

We now define a nonlinear operator as

$$\mathcal{N}\left(\phi(x,t,q)\right) = \frac{\partial \phi(x,t,q)}{\partial t} + \frac{1}{2}\frac{\partial^2 \phi(x,t,q)}{\partial x^2} - \phi(x,t,q)(1-\phi(x,t,q)).$$

With hypothesis $H(x,t) = 1$, *we construct the zeroth-order deformation equation as*

$$(1-q)\mathcal{L}\left[\phi(x,t,q) - u_0(x,t)\right] = qh\mathcal{N}\left[\phi(x,t,q)\right].$$

When $q = 0$ *and* $q = 1$, *we get*

$$\begin{cases} \phi(x,t,0) = u_0(x,t), \\ \phi(x,t,1) = u(x,t). \end{cases}$$

Thus, we obtain the nth*-order deformation equation*

$$\mathcal{L}\left[u_n(x,t) - \chi_n u_{n-1}(x,t)\right] = h\mathcal{R}_n\left[\overrightarrow{u}_{n-1}(x,t)\right], \tag{2.2.8}$$

where

$$\mathcal{R}_n\left[\overrightarrow{u}_{n-1}(x,t)\right] = \frac{\partial u_{n-1}(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 u_n(x,t)}{\partial x^2} + \sum_{j=0}^{n-1} u_j(x,t)u_{n-1-j}(x,t) - u_{n-1}(x,t).$$

Taking the inverse operator $\mathcal{L}^{-1}$ *on both sides of Eq. (2.2.8), we get the* n^{th} *deformation equation as in*

$$u_n(x,t) = \chi_n u_{n-1}(x,t) + \mathcal{L}^{-1}\left[h\mathcal{R}_n\left[\overrightarrow{u}_{n-1}(x,t)\right]\right]. \tag{2.2.9}$$

Using the initial condition (2.2.7) and resolving Eq. (2.2.9) for $n = 1, 2, 3, ...$yields

$$\begin{aligned} u_0(x,t) &= e^{-x} \\ u_1(x,t) &= -hte^{-x}, \\ u_2(x,t) &= -h(1+h)te^{-x} + \frac{1}{2}h^2t^2e^{-x}, \\ u_3(x,t) &= -h(1+h)^2te^{-x} + h^2(1+h)t^2e^{-x} - \frac{1}{6}h^3t^3e^{-x}, \\ &\vdots \end{aligned}$$

and so on.

Similarly, $u_4(x,t), u_5(x,t), ...$ are calculated.

Finally, the approximate solution is expressed as

$$u(x,t) = \left(1 - \left(3h\left(1+h\right) + h^3\right)t + \frac{1}{2}h^2\left(3+2h\right)t^2 - \frac{1}{6}h^3t^3\right)e^{-x} + ... \tag{2.2.10}$$

Taking $h = -1$, Eq. (2.2.10) can be expressed as

$$\begin{aligned} u(x,t) &= \left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + ...\right)e^{-x} \\ &= \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)e^{-x} \\ &= e^{t-x}, \end{aligned}$$

which is the exact solution of Eq. (2.2.6) under the initial condition (2.2.7).

2.3 Variational iteration approach (VIA)

The VIA is a semi-analytical technique, developed by Ji-Huan He **[15]** in the late 1990s, for resolving linear and nonlinear differential equations without linearization or perturbation. It has proven effective in analytically addressing many nonlinear problems. The fundamental principle of the VIA is to find a correction functional, which is added to the initial condition of the solution to obtain a better approximation. The correction functional is determined by resolving a linear variational equation, which is obtained by taking the variational derivative of an appropriate functional with respect to the unknown solution. This technique requires determining a general Lagrange multiplier λ, which can be determined using the variational

theory. The resulting successive approximations are expected to converge to the exact solution of the original equation.

2.3.1 Analysis of the VIA

To explain the principle of the VIA, let us consider the general nonlinear partial differential equation

$$\mathcal{L}\left(u(x,t)\right)+\mathcal{N}\left(u(x,t)\right)=f(x,t), \tag{2.3.1}$$

under the initial conditions

$$u^{(k)}(x,0)=u_k(x), k=0,1,...,m-1, \tag{2.3.2}$$

where $(x,t)\in\mathbb{R}\times\mathbb{R}^{+}$ signify the spatial and time variables, respectively, $\mathcal{L}=\dfrac{\partial^m}{\partial t^m}, m\in\mathbb{N}$ signify a linear differential operator, $\mathcal{N}$ signify a nonlinear operator, and f signify a non-homogeneous term.

According to the VIA, the correction functional formula for (2.3.1) can be constructed as follows

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda(x,\tau)\left[\mathcal{L}\left(u_n(x,\tau)\right)+\mathcal{N}\left(u_n(x,\tau)\right)-f(x,\tau)\right]d\tau, \tag{2.3.3}$$

where $u_n(x,t)$ is the nth approximate solution, $\lambda(x,\tau)$ is a general Lagrange multiplier that can be identified optimally via the variational theory.

We apply restricted variations to the nonlinear term $\mathcal{N}\left(u_n(x,t)\right)$, i.e, $\delta\mathcal{N}\left(u_n(x,t)\right)=0$.

By taking the variation to the correction functional (2.3.3) and making it stationary, i.e, $\delta u_{n+1}(x,t)=0$, we get

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t\lambda(x,\tau)\left[\mathcal{L}\left(u_n(x,\tau)\right)-f(x,\tau)\right]d\tau=0. \tag{2.3.4}$$

Notice that $\delta\ f(x,\tau)=0$, then Eq. (2.3.4) reduces to

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t)+\int_0^t\lambda(x,\tau)\left[\delta\mathcal{L}\left(u_n(x,\tau)\right)-\delta f(x,\tau)\right]d\tau\\ 0 &= \delta u_n(x,t)+\int_0^t\lambda(x,\tau)\delta\mathcal{L}\left(u_n(x,\tau)\right)d\tau.\end{aligned}$$

Integrating the integral by parts and using $\delta u_n(x,0) = 0$, we obtain the following Lagrange multipliers

$$\begin{aligned} \lambda(x,\tau) &= -1 \text{ for } m = 1, \\ \lambda(x,\tau) &= \tau - t \text{ for } m = 2, \end{aligned}$$

and in general, we have

$$\lambda(x,\tau) = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \text{ for } m \geq 1. \tag{2.3.5}$$

As a result, by substituting the value of $\lambda(x,\tau)$ found in Eq. (2.3.5) into Eq. (2.3.3), we obtain the following iteration relation

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left[\mathcal{L}\left(u_n(x,\tau)\right) + \mathcal{N}\left(u_n(x,\tau)\right) - f(x,\tau)\right] d\tau,$$

Now we define the operator $\mathcal{O}\left(u(x,t)\right)$ as

$$\mathcal{O}\left(u(x,t)\right) = \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left[\mathcal{L}\left(u_n(x,\tau)\right) + \mathcal{N}\left(u_n(x,\tau)\right) - f(x,\tau)\right] d\tau, \tag{2.3.6}$$

and we define the components $v_n(x,t), n = 0, 1, 2, ...$, as follows

$$\begin{cases} v_0(x,t) = u_0(x,t), \\ v_1(x,t) = \mathcal{O}\left(v_0(x,t)\right), \\ v_2(x,t) = \mathcal{O}\left(v_0(x,t) + v_1(x,t)\right), \\ \vdots \\ v_{n+1}(x,t) = \mathcal{O}\left(v_0(x,t) + v_1(x,t) + ... + v_n(x,t)\right). \end{cases} . \tag{2.3.7}$$

Then, we have

$$u(x,t) = \lim_{n \longrightarrow \infty} u_n(x,t) = \sum_{k=0}^{\infty} v_n(x,t).$$

Finally, the solution of Eq. (2.3.1) under the the initial conditions (2.3.2) can be deduced using (2.3.6) and (2.3.7) in the form of a series as

$$u(x,t) = \sum_{n=0}^{\infty} v_n(x,t).$$

The initial approximation $v_0(x,t) = u_0(x,t)$ can be chosen freely if it satisfies the initial conditions of the problem. The success of the technique depends on the correct choice

of the initial approximation $v_0(x,t)$. In this alternative approach, we choose the initial approximation as follows

$$v_0(x,t) = \sum_{k=0}^{m-1} \frac{u_k(x)}{k!} t^k.$$

Example 2.3.1 *Considering the nonlinear Newell-Whitehead-Segel equation*

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2u(x,t) - 3u^2(x,t), \tag{2.3.8}$$

under the initial condition

$$u(x,0) = 1. \tag{2.3.9}$$

Constructing a correction functional for Eq. (2.3.8) with using the Lagrange multiplier, we get

$$u(x,t) = u_n(x,t) + \int_0^t \lambda(x,\tau) \left[\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - 2u_n(x,\tau) + 3u_n^2(x,\tau)) \right] d\tau. \tag{2.3.10}$$

Taking the variation with respect to $u_n(x,t)$, we obtain

$$\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta \int_0^t \lambda(x,\tau) \left[\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - 2u_n(x,\tau) + 3u_n^2(x,\tau)) \right] d\tau \\
&= \delta u_n(x,t) + \int_0^t \lambda(x,\tau) \delta \left[\frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - 2u_n(x,\tau) + 3u_n^2(x,\tau)) \right] d\tau \\
&= \delta u_n(x,t) + \int_0^t \lambda(x,\tau) \left[\delta \frac{\partial u_n(x,\tau)}{\partial \tau} - \delta \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - 2\delta u_n(x,\tau) + 3\delta u_n^2(x,\tau) \right] d\tau.
\end{aligned}$$

Note that $\dfrac{\partial^2 u_n(x,\tau)}{\partial x^2}$ and $u_n^2(x,\tau)$ are restricted variables, i.e.

$$\begin{aligned}
\delta \frac{\partial^2 u_n(x,\tau)}{\partial x^2} &= 0, \\
\delta u_n^2(x,\tau) &= 0.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \int_0^t \lambda(x,\tau) \left[\delta \frac{\partial u_n(x,\tau)}{\partial \tau} - 2\delta u_n(x,\tau) \right] d\tau \\
&= \delta u_n(x,t) + \int_0^t \lambda(x,\tau) \delta \frac{\partial u_n(x,\tau)}{\partial \tau} d\tau - 2 \int_0^t \lambda(x,\tau) \delta u_n(x,\tau) d\tau.
\end{aligned}$$

Integrating the first integral by parts, we get

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \lambda(x,\tau)\delta u_n(x,\tau)|_0^t - \int_0^t \frac{\partial\lambda(x,\tau)}{\partial\tau}\delta u_n(x,\tau)d\tau \\ &\quad -2\int_0^t \lambda(x,\tau)\delta u_n(x,\tau)d\tau \\ &= (1+\lambda(x,t))\,\delta u_n(x,t) + \int_0^t \left(-\frac{\partial\lambda(x,\tau)}{\partial\tau} - 2\lambda(x,\tau)\right)\delta u_n(x,\tau)d\tau.\end{aligned}$$

Thus we have this first order partial differential equation

$$-\frac{\partial\lambda(x,\tau)}{\partial\tau} - 2\lambda(x,\tau) = 0,$$

with condition

$$1+\lambda(x,t) = 0.$$

Thus, we have

$$\lambda(x,t) = -e^{-2(t-\tau)},$$

and the following iteration formula (2.3.10) becomes as

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{-2(t-\tau)}\left[\frac{\partial u_n(x,\tau)}{\partial\tau} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - 2u_n(x,\tau) + 3u_n^2(x,\tau))\right]d\tau.$$

With the given initial condition in Eq. (2.3.9), we can use it as an initial approximation as $u_0(x,t) = u(x,0) = 1$ *and hence we can obtain the following successive approximations*

$$\begin{aligned}u_0(x,t) &= 1, \\ u_1(x,t) &= u_0(x,t) - \int_0^t e^{-2(t-\tau)}\left[\frac{\partial u_0(x,\tau)}{\partial\tau} - \frac{\partial^2 u_0(x,\tau)}{\partial x^2} - 2u_0(x,\tau) + 3u_0^2(x,\tau))\right]d\tau \\ &= 1 - t + t^2 - \frac{2}{3}t^3 + \frac{1}{3}t^4 - \frac{2}{15}t^5 + \frac{2}{45}t^6 + ..., \\ u_2(x,t) &= u_1(x,t) - \int_0^t e^{-2(t-\tau)}\left[\frac{\partial u_1(x,\tau)}{\partial\tau} - \frac{\partial^2 u_1(x,\tau)}{\partial x^2} - 2u_1(x,\tau) + 3u_1^2(x,\tau))\right]d\tau \\ &= 1 - t + 2t^2 - \frac{9}{3}t^3 + \frac{10}{3}t^4 - \frac{43}{15}t^5 + \frac{90}{45}t^6 + ..., \\ u_3(x,t) &= u_2(x,t) - \int_0^t e^{-2(t-\tau)}\left[\frac{\partial u_2(x,\tau)}{\partial\tau} - \frac{\partial^2 u_2(x,\tau)}{\partial x^2} - 2u_2(x,\tau) + 3u_2^2(x,\tau))\right]d\tau \\ &= 1 - t + 2t^2 - \frac{11}{3}t^3 + \frac{19}{3}t^4 - \frac{148}{15}t^5 + \frac{611}{45}t^6 + ...,\end{aligned}$$

$$\begin{aligned} u_4(x,t) &= u_3(x,t) - \int_0^t e^{-2(t-\tau)} \left[\frac{\partial u_3(x,\tau)}{\partial \tau} - \frac{\partial^2 u_3(x,\tau)}{\partial x^2} - 2u_3(x,\tau) + 3u_3^2(x,\tau)) \right] d\tau \\ &= 1 - t + 2t^2 - \frac{11}{3}t^3 + \frac{20}{3}t^4 - \frac{180}{15}t^5 + \frac{943}{45}t^6 + ..., \\ &\vdots \end{aligned}$$

and so on.

As a result, the successive approximation solution can be constructed as

$$u_n(x,t) = 1 - t + 2t^2 - \frac{11}{3}t^3 + \frac{20}{3}t^4 - \frac{182}{15}t^5 + \frac{994}{45}t^6 - \frac{12667}{315}t^7 + \frac{4612}{63}t^8 - \frac{377822}{2835}t^9 + ...$$

Then the exact solution of Eq. (2.3.8) under the initial condition (2.3.9) can be obtained by

$$\begin{aligned} u(x,t) &= \lim_{n \longrightarrow \infty} u_n(x,t) \\ &= \frac{-\frac{2}{3}e^{2t}}{\frac{1}{3} - e^{2t}}. \end{aligned}$$

2.4 Reduced differential transform approach (RDTA)

The RDTA is an analytical-numerical technique developed by by the Turkish mathematicians Y. Keskin and G. Oturanc [18] for resolving various linear and nonlinear differential equations, including partial differential equations. This technique provides rapidly convergent power series solutions by using Taylor series expansions. RDTA is employed directly without requiring parameters, discretization, linearization or small perturbations, which reduces computational cost and facilitates its use. RDTA is an efficient and effective technique that simplifies complex calculations and can yield exact solutions in many cases, making it a powerful tool for studying various problems in scientific fields, including engineering, physics, and finance.

In this paragraph, we give the basic definitions and fundamental theorems of RDTA, which is used to find approximate analytical solutions to nonlinear partial differential equations.

Consider an analytic two-variable function $u(x,t)$ and suppose that it can be expressed as the product of two functions of one variable, i.e. $u(x,t) = f(x)g(t)$. The function $u(x,t)$

can be described by the properties of the $(1+1)-$dimensional differential transform as follows

$$\begin{aligned} u(x,t) &= \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) \\ &= \sum_{k=0}^{\infty} U_k(x)t^k, \end{aligned}$$

where $U_k(x)$ is called $t-$dimensional spectrum function of the original function $u(x,t)$.

Definition 2.4.1 *[18] The reduced differential transform of $u(x,t)$ is expressed by*

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{2.4.1}$$

where $u(x,t)$ is the original function which is analytic and $k-$times continuously differentiable with respect to time t and space x in the domain of interest and $U_k(x)$ is the transformed function.

Definition 2.4.2 *[18] The reduced differential inverse transform of $U_k(x)$ is expressed by*

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{2.4.2}$$

From Eqs. (2.4.1) and (2.4.2), we can conclude that

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k.$$

The fundamental mathematical operations performed by the RDTA are summarized in the following theorem

Theorem 2.4.1 *[18] Suppose that $U_k(x), V_k(x)$ and $W_k(x)$ be reduced differential transforms of $u(x,t), v(x,t)$ and $w(x,t)$ respectively.*

(1) If

$$w(x,t) = \lambda u(x,t) + \mu v(x,t),$$

then

$$W_k(x) = \lambda U_k(x) + \mu V_k(x), \lambda, \mu \in \mathbb{R}.$$

(2) If

$$w(x,t) = u(x,t)v(x,t),$$

then

$$W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x).$$

(3) If

$$w(x,t) = u^1(x,t)u^2(x,t) \times ... \times u^{n-1}(x,t)u^n(x,t),$$

then

$$W_k(x) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U_{k_1}^1(x)U_{k_2-k_1}^2(x) \times ... \times U_{k_{n-1}-k_{n-2}}^{n-1}(x)U_{k-k_{n-1}}^n(x).$$

(4) If

$$w(x,t) = \frac{\partial^n u(x,t)}{\partial t^n},$$

then

$$W_k(x) = \frac{(k+n)!}{k!}U_{k+n}(x).$$

(5) If

$$w(x,t) = x^m t^n,$$

then

$$W_k(x) = x^m\delta(k-n), \ \textit{where } \delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}.$$

2.4.1 Analysis of the RDTA

Now, to clarify the RDTA, we begin by considering a nonlinear partial differential equation in the formal form

$$\mathcal{L}\left(u(x,t)\right) + \mathcal{N}\left(u(x,t)\right) = f(x,t), \tag{2.4.3}$$

under the initial conditions

$$u^{(k)}(x,0) = u_k(x), k = 0, 1, ..., m-1, \tag{2.4.4}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ are the spatial and time variables, respectively, $\mathcal{L} = \dfrac{\partial^m}{\partial t^m}, m \in \mathbb{N}$ is a linear operator which has partial derivatives, $\mathcal{N}$ is a nonlinear operator, and $f(x,t)$ is a known analytical function.

Conforming to the RDTA and Theorem 2.4.1, we can construct the following iteration formula

$$U_{k+n}(x) = \frac{k!}{(k+n)!}\left[-R_k(x) + F_k(x)\right], \tag{2.4.5}$$

where $R_k(x)$ and $F_k(x)$ are the transformed form of the nonlinear term $\mathcal{N}(u(x,t))$ and $f(x,t)$, rerspectively.

From the initial condition (2.4.4), we write

$$U_0(x) = u_k(x). \tag{2.4.6}$$

Substituting Eq. (2.4.6) into Eq. (2.4.5) and using simple iterative calculations, we obtain the following $U_k(x)$ values for $k = 1, 2, ..., n$. The inverse transformation of the set of $\{U_k(x)\}_{k=0}^n$ values gives the following approximate solution as

$$\widetilde{u_n}(x,t) = \sum_{k=0}^{n} U_k(x)t^k,$$

where n is the order of the approximate solution.

Therefore, the exact solution of Eq. (2.4.3) under the initial conditions (2.4.4) is expressed as

$$u(x,t) = \lim_{n\to\infty} \widetilde{u_n}(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k,$$

Example 2.4.1 *Considering the nonlinear wave-like equation with variable coefficients*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = x^2\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2}\right) - x^2\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 - u(x,t), \tag{2.4.7}$$

under the initial conditions

$$u(x,0) = 0, \ \frac{\partial u(x,0)}{\partial t} = x^2, \tag{2.4.8}$$

where $(x,t) \in]0,1[\times \mathbb{R}^+$.

Applying the RDTA to Eqs. (2.4.7) and (2.4.8), we obtain the following recurrence relation formula

$$\begin{aligned} U_0(x) &= 0, U_1(x) = x^2, \\ U_{k+2}(x) &= \frac{k!}{(k+n)!}\left[x^2\frac{\partial}{\partial x}A_k(x) - x^2 B_k(x) - U_k(x)\right], \end{aligned} \tag{2.4.9}$$

where $A_k(x)$ *and* $B_k(x)$ *are transformed form of the nonlinear terms* $\dfrac{\partial u(x,t)}{\partial x}\dfrac{\partial^2 u(x,t)}{\partial x^2}$ *and* $\left(\dfrac{\partial^2 u(x,t)}{\partial x^2}\right)^2$.

For the convenience of the reader, the first few nonlinear terms are as follows

$$\begin{aligned} A_0(x) &= \frac{\partial U_0(x,t)}{\partial x}\frac{\partial^2 U_0(x,t)}{\partial x^2}, \\ A_1(x) &= \frac{\partial U_0(x,t)}{\partial x}\frac{\partial^2 U_1(x,t)}{\partial x^2} + \frac{\partial U_1(x,t)}{\partial x}\frac{\partial^2 U_0(x,t)}{\partial x^2}, \\ A_2(x) &= \frac{\partial U_0(x,t)}{\partial x}\frac{\partial^2 U_2(x,t)}{\partial x^2} + \frac{\partial U_1(x,t)}{\partial x}\frac{\partial^2 U_1(x,t)}{\partial x^2} + \frac{\partial U_2(x,t)}{\partial x}\frac{\partial^2 U_0(x,t)}{\partial x^2}, \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} B_0(x) &= \left(\frac{\partial^2 U_0(x,t)}{\partial x^2}\right)^2, \\ B_1(x) &= 2\frac{\partial^2 U_0(x,t)}{\partial x^2}\frac{\partial^2 U_1(x,t)}{\partial x^2}, \\ B_2(x) &= 2\frac{\partial^2 U_0(x,t)}{\partial x^2}\frac{\partial^2 U_2(x,t)}{\partial x^2} + \left(\frac{\partial^2 U_1(x,t)}{\partial x^2}\right)^2, \\ &\vdots \end{aligned}$$

From the recurrence relation (2.4.9), we obtain

$$\begin{aligned} U_0(x) &= 0, \\ U_1(x) &= x^2, \\ U_2(x) &= 0, \\ U_3(x) &= -\frac{1}{3!}x^2, \\ U_4(x) &= 0, \\ U_5(x) &= \frac{1}{5!}x^2, \\ &\vdots \end{aligned}$$

Then the approximate solution can be expressed as

$$\begin{aligned}
\widetilde{u_n}(x,t) &= \sum_{k=0}^{n} U_k(x)t^k \\
&= tx^2 - \frac{x^2}{3!}t^3 + \frac{x^2}{5!}t^5 + ... \\
&= x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + ...\right). \\
&= x^2\sum_{k=0}^{n}\frac{(-1)^k t^{2k+1}}{(2k+1)!}.
\end{aligned}$$

So, the exact solution of Eq. (2.4.7) under the initial conditions (2.4.8) is expressed as

$$\begin{aligned}
u(x,t) &= \lim_{n\to\infty}\widetilde{u_n}(x,t) \\
&= \lim_{n\to\infty} x^2\sum_{k=0}^{n}\frac{(-1)^k t^{2k+1}}{(2k+1)!} \\
&= x^2\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{(2k+1)!} \\
&= x^2\sin t.
\end{aligned}$$

Chapter 3

Finding solutions to systems of nonlinear time-fractional partial differential equations

In the current chapter, we focus on adapting the ADA, HAA, VIA, and RDTA for resolving systems of nonlinear time-fractional partial differential equations involving the Caputo time-fractional derivative. The accuracy and reliability of these approaches are demonstrated by resolving various types of these systems.

Let's suppose that the general form of system of nonlinear time-fractional partial differential equations in the Caputo sense is given by the following formula

$$ {}^{C}D_t^{\alpha} u_i(X,t) + \mathcal{L}_i\left(u_i(X,t)\right) + \mathcal{N}_i\left(u_i(X,t)\right) = f_i(X,t), i = 1,2,...,j, \tag{3.1.1} $$

under the initial conditions

$$ u_i^{(k)}(X,0) = g_{i,k}(x), k = 0,1,2,...,m-1. \tag{3.1.2} $$

Here, the symbol ${}^{C}D_t^{\alpha} = \dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ signify the Caputo fractional derivative operator with respect to time for the function $u_i(X,t)$ of order α with $m-1 < \alpha \leq m, m \in \mathbb{N}^*$, $\{u_i(X,t)\}_{i=1}^{j}$ are unknown functions with $X = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$ signify the space variables and $t \in \mathbb{R}^+$ signify the time variable, $\mathcal{L}_i$ and $\mathcal{N}_i$ signify the linear and nonlinear operators, respectively, $f_i(X,t)$ signify the non-homogeneous terms, and $g_{i,k}(x)$ signify the real functions.

3.1 The ADA procedure

For resolving the system (3.1.1) under the initial conditions (3.1.2) using the ADA, we begin by executing the Riemann-Liouville fractional integral operator I_t^α announced in Eq. (1.1.20) to Eq. (3.1.1), which yields the following result

$$I_t^\alpha \left[{}^C D_t^\alpha u_i(X,t)\right] + I_t^\alpha \left[\mathcal{L}_i\left(u_i(X,t)\right) + \mathcal{N}_i\left(u_i(X,t)\right)\right] = I_t^\alpha \left[f_i(X,t)\right],\ i = 1,2,...,j.$$

Additionally, using relation (1.1.19) with the specified initial conditions, we get

$$u_i(X,t) = \sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!} t^k + I_t^\alpha \left[f_i(X,t)\right] - I_t^\alpha \left[\mathcal{L}_i\left(u_i(X,t)\right) + \mathcal{N}_i\left(u_i(X,t)\right)\right],\ i = 1,2,...,j. \tag{3.1.3}$$

The ADA consists of decomposing the unknown functions $u_i(X,t)$ into a sum of components defined by the decomposition series

$$u_i(X,t) = \sum_{n=0}^{\infty} u_{i,n}(X,t),\ i = 1,2,...,j. \tag{3.1.4}$$

The nonlinear terms $\mathcal{N}_i\left(u_i(X,t)\right)$ can be expanded using Adomian polynomials $(\mathcal{A}_{i,n})$ as follows

$$\mathcal{N}_i\left(u_i(X,t)\right) = \sum_{n=0}^{\infty} \mathcal{A}_{i,n}(u_{i,n}(x,t)),\ i = 1,2,...,j. \tag{3.1.5}$$

These Adomian polynomials can be calculated for all types of nonlinearity using the algorithm (2.1.8).

For $i = 1,2,...,j$, we substitute Eqs. (3.1.4) and (3.1.5) into Eq. (3.1.3) to obtain the following relationship

$$\sum_{n=0}^{\infty} u_{i,n}(X,t) = \sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!} t^k + I_t^\alpha \left[f_i(X,t)\right] - I_t^\alpha \left[\mathcal{L}_i\left(\sum_{n=0}^{\infty} u_{i,n}(X,t)\right) + \sum_{n=0}^{\infty} \mathcal{A}_{i,n}(u_{i,n}(x,t))\right]. \tag{3.1.6}$$

For $i = 1, 2, ..., j$, Eq. (3.1.6) leads to the recurrence relation through the following iterations

$$\begin{aligned}
u_{i,0}(X,t) &= \sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!} t^k + I_t^\alpha \left[f_i(X,t) \right], \\
u_{i,1}(X,t) &= -I_t^\alpha \left[\mathcal{L}_i \left(u_{i,0}(X,t) \right) + \mathcal{A}_{i,0}(u_{i,0}(x,t)) \right], \\
u_{i,2}(X,t) &= -I_t^\alpha \left[\mathcal{L}_i \left(u_{i,1}(X,t) \right) + \mathcal{A}_{i,1}(u_{i,1}(x,t)) \right], \\
u_{i,3}(X,t) &= -I_t^\alpha \left[\mathcal{L}_i \left(u_{i,2}(X,t) \right) + \mathcal{A}_{i,2}(u_{i,2}(x,t)) \right], \\
&\vdots \\
u_{i,n}(X,t) &= -I_t^\alpha \left[\mathcal{L}_i \left(u_{i,n-1}(X,t) \right) + \mathcal{A}_{i,n-1}(u_{i,n-1}(x,t)) \right].
\end{aligned}$$

Utilizing the terms of the series $u_{i,0}(X,t), u_{i,2}(X,t), u_{i,2}(X,t), u_{i,3}(X,t), \ldots$ to obtain the solution using the approximation

$$u_i(X,t) = \lim_{n \to \infty} \Psi_{i,n}, \ i = 1, 2, ..., j.$$

where

$$\Psi_{i,n} = \sum_{k=0}^{n} u_{i,k}(X,t), \ i = 1, 2, ..., j.$$

Example 3.1.1 *We take into consideration the system of nonlinear time-fractional partial differential equations in two unknowns*

$$\begin{cases} {}^C D_t^\alpha u(x,t) + v(x,t) \dfrac{\partial u(x,t)}{\partial x} + u(x,t) = 1 \\ {}^C D_t^\alpha v(x,t) - u(x,t) \dfrac{\partial v(x,t)}{\partial x} - v(x,t) = 1 \end{cases}, \tag{3.1.7}$$

under the initial conditions

$$\begin{cases} u(x,0) = e^x \\ v(x,0) = e^{-x} \end{cases}, \tag{3.1.8}$$

where ${}^C D_t^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha}$ *signify the Caputo time-fractional derivative operator of order* $0 < \alpha \leq 1$.

Using the same ADA procedure described in paragraph 3.1, we attain

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} u_n(x,t) = e^x + I_t^\alpha [1] - I_t^\alpha \left[\sum_{n=0}^{\infty} \mathcal{A}_n(u_n(x,t), v_n(x,t)) + \sum_{n=0}^{\infty} u_n(x,t) \right] \\ \displaystyle\sum_{n=0}^{\infty} v_n(x,t) = e^{-x} + I_t^\alpha [1] + I_t^\alpha \left[\sum_{n=0}^{\infty} \mathcal{B}_n(u_n(x,t), v_n(x,t)) + \sum_{n=0}^{\infty} v_n(x,t) \right] \end{cases}, \tag{3.1.9}$$

where $\mathcal{A}_n(u_n(x,t), v_n(x,t))$ *and* $\mathcal{B}_n(u_n(x,t), v_n(x,t))$ *are the Adomian polynomials describing the nonlinear terms* $v(x,t)\dfrac{\partial u(x,t)}{\partial x}$ *and* $u(x,t)\dfrac{\partial v(x,t)}{\partial x}$, *respectively.*

The first Adomian polynomials are identified by

$$\begin{aligned}
\mathcal{A}_0(u_0(x,t), v_0(x,t)) &= v_0(x,t)\frac{\partial u_0(x,t)}{\partial x},\\
\mathcal{A}_1(u_1(x,t), v_1(x,t)) &= v_0(x,t)\frac{\partial u_1(x,t)}{\partial x} + v_1(x,t)\frac{\partial u_0(x,t)}{\partial x},\\
\mathcal{A}_2(u_2(x,t), v_2(x,t)) &= v_0(x,t)\frac{\partial u_2(x,t)}{\partial x} + v_1(x,t)\frac{\partial u_1(x,t)}{\partial x} + v_2(x,t)\frac{\partial u_0(x,t)}{\partial x},\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_0(u_0(x,t), v_0(x,t)) &= u_0(x,t)\frac{\partial v_0(x,t)}{\partial x},\\
\mathcal{B}_1(u_1(x,t), v_1(x,t)) &= u_0(x,t)\frac{\partial v_1(x,t)}{\partial x} + u_1(x,t)\frac{\partial v_0(x,t)}{\partial x},\\
\mathcal{B}_2(u_2(x,t), v_2(x,t)) &= u_0(x,t)\frac{\partial v_2(x,t)}{\partial x} + u_1(x,t)\frac{\partial v_1(x,t)}{\partial x} + u_2(x,t)\frac{\partial v_0(x,t)}{\partial x},\\
&\vdots
\end{aligned}$$

By equating the terms of the series on both sides of (3.1.9), we attain

$$\begin{cases} u(x,0) = e^x \\ v(x,0) = e^{-x} \end{cases},$$

$$\begin{cases} u_1(x,t) = -e^x \dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ v_1(x,t) = e^{-x} \dfrac{t^\alpha}{\Gamma(\alpha+1)} \end{cases},$$

$$\begin{cases} u_2(x,t) = e^x \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ v_2(x,t) = e^{-x} \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases},$$

$$\begin{cases} u_3(x,t) = -e^x \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ v_3(x,t) = e^{-x} \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{cases},$$

$$\vdots$$

and so on.

Then, the solution in series form can be written as

$$\begin{cases} u(x,t) = e^x \left(1 - \dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = e^x E_\alpha(-t^\alpha) \\ v(x,t) = e^{-x} \left(1 + \dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = e^{-x} E_\alpha(t^\alpha) \end{cases}, \quad (3.1.10)$$

where $E_\alpha(-t^\alpha)$ and $E_\alpha(t^\alpha)$ are the Mittag-Leffler functions of one parameter α defined by Eq. (1.1.6).

For the particular case $\alpha = 1$, by simplifying (3.1.10), we obtain the exact solution of system (3.1.7) as

$$\begin{cases} u(x,t) = e^{x-t} \\ v(x,t) = e^{-x+t} \end{cases}.$$

Example 3.1.2 *We take into consideration the system of nonlinear time-fractional Burgers equations in three unknowns*

$$\begin{cases} {}^C D_t^\alpha u(x,y,t) + \dfrac{\partial v(x,y,t)}{\partial x}\dfrac{\partial w(x,y,t)}{\partial y} - \dfrac{\partial v(x,y,t)}{\partial y}\dfrac{\partial w(x,y,t)}{\partial x} + u(x,y,t) = 0 \\ {}^C D_t^\alpha v(x,y,t) + \dfrac{\partial u(x,y,t)}{\partial x}\dfrac{\partial w(x,y,t)}{\partial y} + \dfrac{\partial u(x,y,t)}{\partial y}\dfrac{\partial w(x,y,t)}{\partial x} - v(x,y,t) = 0 \\ {}^C D_t^\alpha w(x,y,t) + \dfrac{\partial u(x,y,t)}{\partial x}\dfrac{\partial v(x,y,t)}{\partial y} + \dfrac{\partial u(x,y,t)}{\partial y}\dfrac{\partial v(x,y,t)}{\partial x} - w(x,y,t) = 0 \end{cases}, \quad (3.1.11)$$

under the initial conditions

$$\begin{cases} u(x,y,0) = e^{x+y} \\ v(x,y,0) = e^{x-y} \\ w(x,y,0) = e^{-x+y} \end{cases}, \quad (3.1.12)$$

where ${}^C D_t^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha}$ signify the Caputo time-fractional derivative operator of order $0 < \alpha \leq 1$.

Using the same ADA procedure described in paragraph 3.1, we attain

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} u_n(x,t) = e^{x+y} + I_t^{\alpha}\left[\begin{array}{c} -\displaystyle\sum_{n=0}^{\infty} \mathcal{A}_n(v_n(x,y,t), w_n(x,y,t)) + \sum_{n=0}^{\infty} \mathcal{B}_n(v_n(x,y,t), w_n(x,y,t)) \\ -\displaystyle\sum_{n=0}^{\infty} u_n(x,y,t) \end{array} \right] \\ \displaystyle\sum_{n=0}^{\infty} v_n(x,t) = e^{x-y} + I_t^{\alpha}\left[\begin{array}{c} -\displaystyle\sum_{n=0}^{\infty} \mathcal{C}_n(u_n(x,y,t), w_n(x,y,t)) - \sum_{n=0}^{\infty} \mathcal{D}_n(u_n(x,y,t), w_n(x,y,t)) \\ +\displaystyle\sum_{n=0}^{\infty} v_n(x,y,t) \end{array} \right] \\ \displaystyle\sum_{n=0}^{\infty} w_n(x,t) = e^{-x+y} + I_t^{\alpha}\left[\begin{array}{c} -\displaystyle\sum_{n=0}^{\infty} \mathcal{E}_n(u_n(x,y,t), v_n(x,y,t)) - \sum_{n=0}^{\infty} \mathcal{F}_n(u_n(x,y,t), v_n(x,y,t)) \\ +\displaystyle\sum_{n=0}^{\infty} w_n(x,y,t) \end{array} \right] \end{cases}, \tag{3.1.13}$$

where $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$, $\mathcal{D}_n$, $\mathcal{E}_n$, *and* $\mathcal{F}_n$ *are the Adomian polynomials describing the nonlinear terms* $\frac{\partial v(x,y,t)}{\partial x}\frac{\partial w(x,y,t)}{\partial y}$, $\frac{\partial v(x,y,t)}{\partial y}\frac{\partial w(x,y,t)}{\partial x}$, $\frac{\partial u(x,y,t)}{\partial x}\frac{\partial w(x,y,t)}{\partial y}$, $\frac{\partial u(x,y,t)}{\partial y}\frac{\partial w(x,y,t)}{\partial x}$, $\frac{\partial u(x,y,t)}{\partial x}\frac{\partial v(x,y,t)}{\partial y}$, *and* $\frac{\partial u(x,y,t)}{\partial y}\frac{\partial v(x,y,t)}{\partial x}$, *respectively.*

The first Adomian polynomials are identified by

$$\begin{aligned} \mathcal{A}_0 &= \frac{\partial v_0(x,y,t)}{\partial x}\frac{\partial w_0(x,y,t)}{\partial y}, \\ \mathcal{A}_1 &= \frac{\partial v_0(x,y,t)}{\partial x}\frac{\partial w_1(x,y,t)}{\partial y} + \frac{\partial v_1(x,y,t)}{\partial x}\frac{\partial w_0(x,y,t)}{\partial y}, \\ \mathcal{A}_2 &= \frac{\partial v_0(x,y,t)}{\partial x}\frac{\partial w_2(x,y,t)}{\partial y} + \frac{\partial v_1(x,y,t)}{\partial x}\frac{\partial w_1(x,y,t)}{\partial y} + \frac{\partial v_2(x,y,t)}{\partial x}\frac{\partial w_0(x,y,t)}{\partial y}, \\ &\vdots \end{aligned}$$

$$\begin{aligned} \mathcal{B}_0 &= \frac{\partial v_0(x,y,t)}{\partial y}\frac{\partial w_0(x,y,t)}{\partial x}, \\ \mathcal{B}_1 &= \frac{\partial v_0(x,y,t)}{\partial y}\frac{\partial w_1(x,y,t)}{\partial x} + \frac{\partial v_1(x,y,t)}{\partial y}\frac{\partial w_0(x,y,t)}{\partial x}, \\ \mathcal{B}_2 &= \frac{\partial v_0(x,y,t)}{\partial y}\frac{\partial w_2(x,y,t)}{\partial x} + \frac{\partial v_1(x,y,t)}{\partial y}\frac{\partial w_1(x,y,t)}{\partial x} + \frac{\partial v_2(x,y,t)}{\partial y}\frac{\partial w_0(x,y,t)}{\partial x}, \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
\mathcal{C}_0 &= \frac{\partial u_0(x,y,t)}{\partial x}\frac{\partial w_0(x,y,t)}{\partial y},\\
\mathcal{C}_1 &= \frac{\partial u_0(x,y,t)}{\partial x}\frac{\partial w_1(x,y,t)}{\partial y}+\frac{\partial u_1(x,y,t)}{\partial x}\frac{\partial w_0(x,y,t)}{\partial y},\\
\mathcal{C}_2 &= \frac{\partial u_0(x,y,t)}{\partial x}\frac{\partial w_2(x,y,t)}{\partial y}+\frac{\partial u_1(x,y,t)}{\partial x}\frac{\partial w_1(x,y,t)}{\partial y}+\frac{\partial u_2(x,y,t)}{\partial x}\frac{\partial w_0(x,y,t)}{\partial y},\\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_0 &= \frac{\partial u_0(x,y,t)}{\partial y}\frac{\partial w_0(x,y,t)}{\partial x},\\
\mathcal{D}_1 &= \frac{\partial u_0(x,y,t)}{\partial y}\frac{\partial w_1(x,y,t)}{\partial x}+\frac{\partial u_1(x,y,t)}{\partial y}\frac{\partial w_0(x,y,t)}{\partial x},\\
\mathcal{D}_2 &= \frac{\partial u_0(x,y,t)}{\partial y}\frac{\partial w_2(x,y,t)}{\partial x}+\frac{\partial u_1(x,y,t)}{\partial y}\frac{\partial w_1(x,y,t)}{\partial x}+\frac{\partial u_2(x,y,t)}{\partial y}\frac{\partial w_0(x,y,t)}{\partial x},\\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_0 &= \frac{\partial u_0(x,y,t)}{\partial x}\frac{\partial v_0(x,y,t)}{\partial y},\\
\mathcal{E}_1 &= \frac{\partial u_0(x,y,t)}{\partial x}\frac{\partial v_1(x,y,t)}{\partial y}+\frac{\partial u_1(x,y,t)}{\partial x}\frac{\partial v_0(x,y,t)}{\partial y},\\
\mathcal{E}_2 &= \frac{\partial u_0(x,y,t)}{\partial x}\frac{\partial v_2(x,y,t)}{\partial y}+\frac{\partial u_1(x,y,t)}{\partial x}\frac{\partial v_1(x,y,t)}{\partial y}+\frac{\partial u_2(x,y,t)}{\partial x}\frac{\partial v_0(x,y,t)}{\partial y},\\
&\vdots
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}_0 &= \frac{\partial u_0(x,y,t)}{\partial y}\frac{\partial v_0(x,y,t)}{\partial x},\\
\mathcal{F}_1 &= \frac{\partial u_0(x,y,t)}{\partial y}\frac{\partial v_1(x,y,t)}{\partial x}+\frac{\partial u_1(x,y,t)}{\partial y}\frac{\partial v_0(x,y,t)}{\partial x},\\
\mathcal{F}_2 &= \frac{\partial u_0(x,y,t)}{\partial y}\frac{\partial v_2(x,y,t)}{\partial x}+\frac{\partial u_1(x,y,t)}{\partial y}\frac{\partial v_1(x,y,t)}{\partial x}+\frac{\partial u_2(x,y,t)}{\partial y}\frac{\partial v_0(x,y,t)}{\partial x},\\
&\vdots
\end{aligned}
$$

By equating the terms of the series on both sides of (3.1.13), we attain

$$
\begin{cases} u(x,y,0) = e^{x+y} \\ v(x,y,0) = e^{x-y} \\ w(x,y,0) = e^{-x+y} \end{cases},
$$

$$
\begin{cases} u_1(x,y,t) = -e^{x+y}\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ v_1(x,y,t) = e^{x-y}\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ w_1(x,y,t) = e^{-x+y}\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \end{cases},
$$

$$
\begin{cases} u_2(x,y,t) = e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ v_2(x,y,t) = e^{x-y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ w_2(x,y,t) = e^{-x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases},
$$

$$
\begin{cases} u_3(x,y,t) = -e^{x+y}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ v_3(x,y,t) = e^{x-y}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ w_3(x,y,t) = e^{-x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases},
$$

$$
\vdots
$$

and so on.

Then, the solution in series form can be written as

$$
\begin{cases} u(x,y,t) = e^{x+y}\left(1 - \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = e^{x+y}E_{\alpha}(-t^{\alpha}) \\ v(x,y,t) = e^{x-y}\left(1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = e^{x-y}E_{\alpha}(t^{\alpha}) \\ w(x,y,t) = e^{-x+y}\left(1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = e^{-x+y}E_{\alpha}(t^{\alpha}) \end{cases}, \tag{3.1.14}
$$

where $E_{\alpha}(-t^{\alpha})$ and $E_{\alpha}(t^{\alpha})$ are the Mittag-Leffler functions of one parameter α defined by Eq. (1.1.6).

For the particular case $\alpha = 1$, by simplifying (3.1.14), we obtain the exact solution of system (3.1.11) as

$$\begin{cases} u(x,y,t) = e^{x+y-t} \\ v(x,y,t) = e^{x-y+t} \\ w(x,y,t) = e^{-x+y+t} \end{cases}.$$

3.2 The HAA procedure

For resolving the system (3.1.1) under the initial conditions (3.1.2) by means of HAA, we follow these steps.

For $i = 1, 2, ..., m$, let $\mathcal{J}_i$ be the auxiliary linear operators, and $u_{i,0}(X,t)$ the initial approximations of $u_i(X,t)$ that satisfy the initial conditions (3.1.2). It should be noted that the auxiliary linear operators $\mathcal{J}_i$ are different from the linear operators $\mathcal{L}_i$ in system (3.1.1).

Note that the original system (3.1.1) contains the linear operator ${}^{C}D_t^{\alpha}$. Therefore, it is easy for us to choose the auxiliary linear operators for $i = 1, 2, ..., j$

$$\mathcal{J}_i\left[\phi_i(X,t,q)\right] = {}^{C}\ D_t^{\alpha}\phi_i(X,t,q), \tag{3.2.1}$$

satisfying the property

$$\mathcal{J}_i\left[C_i\right] = 0,$$

where C_i are arbitrary constants.

Conforming to (3.1.3) , we can choose the initial approximations to be

$$u_{i,0}(X,t) = \sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!}t^k + I_t^{\alpha}\left[f_i(X,t)\right], i = 1, 2, ..., j.$$

To simplify and in accordance with system (3.1.1), we define the following nonlinear operators for $i = 1, 2, ..., j$

$$\mathcal{M}_i\left[\phi_i(X,t,q)\right] = {}^{C}\ D_t^{\alpha}\phi_i(X,t,q) + \mathcal{L}_i\left(\phi_i(X,t,q)\right) + \mathcal{N}_i\left(\phi_i(X,t,q)\right) - f_i(X,t). \tag{3.2.2}$$

Using Eq.(3.2.2) and the assumption $H_i(X,t) = 1$ we construct the zero-order deformation equations for

$$(1-q)\mathcal{J}_i\left[\phi_i(X,t,q) - u_{i,0}(X,t)\right] = qh\mathcal{M}_i\left[\phi_i(X,t,q)\right],\ i = 1, 2, ..., j. \tag{3.2.3}$$

Clearly, when $q = 0$ and $q = 1$, we obtain

$$\phi_i(X,t,0) = u_{i,0}(X,t) \text{ and } \phi_i(X,t,1) = u_i(X,t),\ i = 1,2,...,j,$$

respectively.

Using the parameter q, $\phi_i(X,t,q)$ can be expanded in Taylor series as follows

$$\phi_i(X,t,q) = u_{i,0}(X,t) + \sum_{n=1}^{+\infty} q^n u_{i,n}(X,t),\ i = 1,2,...,j, \tag{3.2.4}$$

where

$$u_{i,n}(x,t) = \frac{1}{n!}\left.\frac{\partial^n \phi_i(X,t,q)}{\partial q^n}\right|_{q=0},\ i = 1,2,...,j.$$

For $i = 1,2,...,j$, let's assume that the auxiliary parameter h, the initial approximations $u_{i,0}(X,t)$, and the auxiliary linear operators $\mathcal{J}_i =^C D_t^\alpha$ are chosen appropriately, such that the series in Eq. (3.2.4) converges for $q = 1$. In this case, the series in Eq. (3.2.4) becomes, for $q = 1$

$$u_i(X,t) = \phi_i(X,t,1) = u_{i,0}(X,t) + \sum_{n=1}^{+\infty} u_{i,n}(x,t),\ i = 1,2,...,j.$$

Taking the $n-$times derivative of (3.2.3) with respect to the q and dividing by $n!$. Finally, we derive the deformation equations of $n-$order as follows for $q = 0$, i.e.

$$\mathcal{J}_i\left[u_{i,n}(X,t) - \chi_n u_{i,n-1}(X,t)\right] = h\mathcal{R}_{i,n}\left[\overrightarrow{u_{i,n-1}}(X,t)\right],\ i = 1,2,...,j, \tag{3.2.5}$$

with

$$\overrightarrow{u_{i,n}}(X,t) = \left\{u_{i,0}(X,t), u_{i,1}(X,t), u_{i,2}(X,t), ..., u_{i,n}(X,t)\right\},\ i = 1,2,...,j,$$

and

$$\mathcal{R}_{i,n}\left[\overrightarrow{u_{i,n-1}}(X,t)\right] = \frac{1}{(n-1)!}\left.\frac{\partial^{n-1}\mathcal{M}_i\left[\phi_i(X,t,q)\right]}{\partial q^{n-1}}\right|_{q=0},\ i = 1,2,...,j,$$

and

$$\chi_n = \begin{cases} 0, n \leq 1 \\ 1, n > 1 \end{cases},.$$

According to Eq. (3.2.1), we apply the Riemann-Liouville fractional integral operator I_t^α given by Eq. (1.1.20) on both sides of Eq.(3.2.5), we have

$$I_t^{\alpha C} D_t^\alpha\left[u_{i,n}(X,t) - \chi_n u_{i,n-1}(X,t)\right] = hI_t^\alpha \mathcal{R}_{i,n}\left[\overrightarrow{u}_{i,n-1}(X,t)\right].,\ i = 1,2,...,j.$$

Using relation (1.1.19) , we have

$$u_{i,n}(X,t) = \chi_n u_{i,n-1}(X,t) + hI_t^{\alpha}\mathcal{R}_{i,n}\left[\overrightarrow{u}_{i,n-1}(X,t)\right],\ i = 1,2,...,j. \tag{3.2.6}$$

Thus, for $i = 1,2,...,j$, we obtain $u_{i,1}(X,t), u_{i,2}(X,t), u_{i,3}(X,t), ...$ by means of Eq.(3.2.6). The Mth-order approximations of $u_{i,n}(X,t)$ are given by

$$u_i(X,t) = \sum_{n=0}^{M} u_{i,n}(X,t),\ i = 1,2,...,j,$$

and for $M \longrightarrow \infty$, we get an accurate approximation of system (3.1.1) under the initial conditions (3.1.2) as

$$u_i(X,t) = \sum_{n=0}^{+\infty} u_{i,n}(X,t),\ i = 1,2,...,j. \tag{3.2.7}$$

Example 3.2.1 *Let us consider the system of nonlinear time-frational Navier-Stocks equations in two unknowns*

$$\begin{cases} {}^{C}D_t^{\alpha}u(x,y,t) + u(x,y,t)\dfrac{\partial u(x,y,t)}{\partial x} + v(x,y,t)\dfrac{\partial u(x,y,t)}{\partial y} = \dfrac{\partial^2 u(x,y,t)}{\partial x^2} + \dfrac{\partial^2 u(x,y,t)}{\partial y^2} \\ {}^{C}D_t^{\alpha}v(x,y,t) + u(x,y,t)\dfrac{\partial v(x,y,t)}{\partial x} + v(x,y,t)\dfrac{\partial v(x,y,t)}{\partial y} = \dfrac{\partial^2 v(x,y,t)}{\partial x^2} + \dfrac{\partial^2 v(x,y,t)}{\partial y^2} \end{cases}, \tag{3.2.8}$$

under the initial conditions

$$\begin{cases} u(x,y,0) = -\sin(x+y) \\ v(x,y,0) = \sin(x+y) \end{cases}, \tag{3.2.9}$$

where ${}^{C}D_t^{\alpha} = \dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ *signify the Caputo time-fractional derivative operator of order* $0 < \alpha \leq 1$.

By following the same steps of the HAA procedure described in paragraph 3.2.

First, we define nonlinear operators $\mathcal{M}_i\left[\phi_i(x,,t,q)\right]$ *for* $i = 1,2,$ *as follows*

$$\begin{cases} \mathcal{M}_1\left[\phi_1(x,y,t,q)\right] = {}^{C}D_t^{\alpha}\phi_1(x,y,t,q) + \phi_1(x,y,t,q)\dfrac{\partial\phi_1(x,y,t,q)}{\partial x} + \phi_2(x,y,t,q)\dfrac{\partial\phi_1(x,y,t,q)}{\partial y} \\ \qquad\qquad - \dfrac{\partial^2\phi_1(x,y,t,q)}{\partial x^2} - \dfrac{\partial^2\phi_1(x,y,t,q)}{\partial y^2} \\ \mathcal{M}_2\left[\phi_2(x,y,t,q)\right] = {}^{C}D_t^{\alpha}\phi_2(x,y,t,q) + \phi_1(x,y,t,q)\dfrac{\partial\phi_2(x,y,t,q)}{\partial x} + \phi_2(x,y,t,q)\dfrac{\partial\phi_2(x,y,t,q)}{\partial y} \\ \qquad\qquad - \dfrac{\partial^2\phi_2(x,y,t,q)}{\partial x^2} - \dfrac{\partial^2\phi_2(x,y,t,q)}{\partial y^2} \end{cases}.$$

Using the recursive formula (3.2.6), we obtain

$$\begin{cases} u_n(x,y,t) = \chi_n u_{n-1}(x,y,t) + hI_t^{\alpha}\mathcal{R}_{1,n}\left[\overrightarrow{u}_{n-1}(x,y,t), \overrightarrow{v}_{n-1}(x,y,t)\right] \\ v_n(x,y,t) = \chi_n v_{n-1}(x,y,t) + hI_t^{\alpha}\mathcal{R}_{2,n}\left[\overrightarrow{u}_{n-1}(x,y,t), \overrightarrow{v}_{n-1}(x,y,t)\right] \end{cases}, \tag{3.2.10}$$

where

$$\begin{cases} \mathcal{R}_{1,n}\left[\overrightarrow{u}_{n-1}(x,y,t), \overrightarrow{v}_{n-1}(x,y,t)\right] =^{C} D_t^{\alpha}u_{n-1}(x,y,t) + \sum_{j=0}^{n-1} u_j(x,y,t)\dfrac{\partial u_{n-1-j}(x,y,t)}{\partial x} \\ \qquad + \sum_{j=0}^{n-1} v_j(x,y,t)\dfrac{\partial u_{n-1-j}(x,y,t)}{\partial x} - \dfrac{\partial^2 u_{n-1}(x,y,t)}{\partial x^2} - \dfrac{\partial^2 u_{n-1}(x,y,t)}{\partial y^2} \\ \mathcal{R}_{2,n}\left[\overrightarrow{u}_{n-1}(x,y,,t), \overrightarrow{v}_{n-1}(x,y,,t)\right] =^{C} D_t^{\alpha}v_{n-1}(x,y,t) + \sum_{j=0}^{n-1} u_j(x,y,t)\dfrac{\partial v_{n-1-j}(x,y,t)}{\partial x} \\ \qquad + \sum_{j=0}^{n-1} v_j(x,y,t)\dfrac{\partial v_{n-1-j}(x,y,t)}{\partial x} - \dfrac{\partial^2 v_{n-1}(x,y,t)}{\partial x^2} - \dfrac{\partial^2 v_{n-1}(x,y,t)}{\partial y^2} \end{cases} \tag{3.2.11}$$

We define the the initial approximations

$$\begin{cases} u_0(x,y,t) = -\sin(x+y) \\ v_0(x,y,t) = \sin(x+y) \end{cases}.$$

By combining the recursive formula (3.2.10) with (3.2.11), we obtain

$$\begin{cases} u_1(x,y,t) = -h\sin(x+y)\dfrac{2t^{\alpha}}{\Gamma(\alpha+1)} \\ v_1(x,y,t) = h\sin(x+y)\dfrac{2t^{\alpha}}{\Gamma(\alpha+1)} \end{cases},$$

$$\begin{cases} u_2(x,y,t) = -h(1+h)\sin(x+y)\dfrac{2t^{\alpha}}{\Gamma(\alpha+1)} - h^2\sin(x+y)\dfrac{4t^{2\alpha}}{\Gamma(2\alpha+1)} \\ v_2(x,y,t) = h(1+h)(\sin(x+y)\dfrac{2t^{\alpha}}{\Gamma(\alpha+1)} + h^2\sin(x+y)\dfrac{4t^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases},$$

$$\begin{cases} u_3(x,y,t) = -h(1+h)^2\sin(x+y)\dfrac{2t^{\alpha}}{\Gamma(\alpha+1)} - h^2(1+h)\sin(x+y)\dfrac{8t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \qquad -h^3\sin(x+y)\dfrac{8t^{3\alpha}}{\Gamma(3\alpha+1)} \\ v_3(x,y,t) = h(1+h)^2(\sin(x+y)\dfrac{2t^{\alpha}}{\Gamma(\alpha+1)} + h^2(1+h)\sin(x+y)\dfrac{8t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \qquad +h^3\sin(x+y)\dfrac{8t^{3\alpha}}{\Gamma(3\alpha+1)} \end{cases},$$

$\vdots$

and so on.

According to Eq. (3.2.7), the expression of the series solutions by HAA can be written as

$$\begin{cases} u(x,y,t) = -\sin(x+y) - (3h+3h^2+h^3)\sin(x+y)\dfrac{2t^\alpha}{\Gamma(\alpha+1)} - (3h^2+2h^3)\sin(x+y)\dfrac{4t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \qquad\qquad -h^3\sin(x+y)\dfrac{8t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \\ v(x,y,t) = \sin(x+y) + (3h+3h^2+h^3)(\sin(x+y) + (3h^2+2h^3)\sin(x+y)\dfrac{4t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \qquad\qquad +h^3\sin(x+y)\dfrac{8t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \end{cases}.$$

For simplicity, we fix $h=-1$. Then the solution for system (3.2.8) becomes

$$\begin{cases} u(x,y,t) = -\sin(x+y)\left(1 - \dfrac{2t^\alpha}{\Gamma(\alpha+1)} + \dfrac{4t^{2\alpha}}{\Gamma(2\alpha+1)} - \dfrac{8t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = -\sin(x+y)E_\alpha(-2t^\alpha) \\ v(x,y,t) = \sin(x+y)\left(1 - \dfrac{2t^\alpha}{\Gamma(\alpha+1)} + \dfrac{4t^{2\alpha}}{\Gamma(2\alpha+1)} - \dfrac{8t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = \sin(x+y)E_\alpha(-2t^\alpha) \end{cases}, \tag{3.2.12}$$

where $E_\alpha(-2t^\alpha)$ is the Mittag-Leffler function of one parameter α defined by Eq. (1.1.6).

Setting $\alpha = 1$ in (3.2.12), we find

$$\begin{cases} u(x,y,t) = -\sin(x+y)e^{-2t} \\ v(x,y,t) = \sin(x+y)e^{-2t} \end{cases},$$

which is the exact solution of system (3.2.8) under the initial conditions (3.2.9).

Example 3.2.2 *Let us consider the system of nonlinear time-fractional partial differential equations in in three unknowns*

$$\begin{cases} {}^C D_t^\alpha u(x,y,t) - \dfrac{\partial v(x,y,t)}{\partial x}\dfrac{\partial w(x,y,t)}{\partial y} = 1 \\ {}^C D_t^\alpha v(x,y,t) - \dfrac{\partial w(x,y,t)}{\partial x}\dfrac{\partial u(x,y,t)}{\partial y} = 5 \\ {}^C D_t^\alpha w(x,y,t) - \dfrac{\partial u(x,y,t)}{\partial x}\dfrac{\partial v(x,y,t)}{\partial y} = 5 \end{cases}, \tag{3.2.13}$$

under the initial conditions

$$\begin{cases} u(x,y,0) = x+2y \\ v(x,y,0) = x-2y \\ v(x,y,0) = -x+2y \end{cases}, \tag{3.2.14}$$

where ${}^{C}D_t^{\alpha} = \dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ *signify the Caputo time-fractional derivative operator of order* $0 < \alpha \leq 1$.

By following the same steps of the HAA procedure described in paragraph 3.2.

First, we define nonlinear operators $\mathcal{M}_i\left[\phi_i(x,,t,q)\right]$ *for* $i = 1, 2, 3$ *as follows*

$$\begin{cases} \mathcal{M}_1\left[\phi_1(x,y,t,q)\right] = {}^{C} D_t^{\alpha}\phi_1(x,y,t,q) - \dfrac{\partial\phi_2(x,y,t,q)}{\partial x}\dfrac{\partial\phi_3(x,y,t,q)}{\partial y} - 1 \\ \mathcal{M}_2\left[\phi_2(x,y,t,q)\right] = {}^{C} D_t^{\alpha}\phi_2(x,y,t,q) - \dfrac{\partial\phi_3(x,y,t,q)}{\partial x}\dfrac{\partial\phi_1(x,y,t,q)}{\partial y} - 5 \\ \mathcal{M}_3\left[\phi_3(x,y,t,q)\right] = {}^{C} D_t^{\alpha}\phi_3(x,y,t,q) - \dfrac{\partial\phi_1(x,y,t,q)}{\partial x}\dfrac{\partial\phi_2(x,y,t,q)}{\partial y} - 5 \end{cases}.$$

Using the recursive formula (3.2.6), we obtain

$$\begin{cases} u_n(x,y,t) = \chi_n u_{n-1}(x,y,t) + hI_t^{\alpha}\mathcal{R}_{1,n}\left[\overrightarrow{v}_{n-1}(x,y,t), \overrightarrow{w}_{n-1}(x,y,t)\right] \\ v_n(x,y,t) = \chi_n v_{n-1}(x,y,t) + hI_t^{\alpha}\mathcal{R}_{2,n}\left[\overrightarrow{w}_{n-1}(x,y,t), \overrightarrow{u}_{n-1}(x,y,t)\right] \\ w_n(x,y,t) = \chi_n w_{n-1}(x,y,t) + hI_t^{\alpha}\mathcal{R}_{2,n}\left[\overrightarrow{u}_{n-1}(x,y,t), \overrightarrow{v}_{n-1}(x,y,t)\right] \end{cases}, \tag{3.2.15}$$

where

$$\begin{cases} \mathcal{R}_{1,n}\left[\overrightarrow{v}_{n-1}(x,y,t), \overrightarrow{w}_{n-1}(x,y,t)\right] = {}^{C} D_t^{\alpha}u_{n-1}(x,y,t) - \displaystyle\sum_{j=0}^{n-1}\frac{\partial v_j(x,y,t)}{\partial x}\frac{\partial w_{n-1-j}(x,y,t)}{\partial x} - 1 \\ \mathcal{R}_{2,n}\left[\overrightarrow{w}_{n-1}(x,y,,t), \overrightarrow{u}_{n-1}(x,y,,t)\right] = {}^{C} D_t^{\alpha}v_{n-1}(x,y,t) - \displaystyle\sum_{j=0}^{n-1}\frac{\partial w_j(x,y,t)}{\partial x}\frac{\partial u_{n-1-j}(x,y,t)}{\partial x} - 5 \\ \mathcal{R}_{2,n}\left[\overrightarrow{u}_{n-1}(x,y,,t), \overrightarrow{v}_{n-1}(x,y,,t)\right] = {}^{C} D_t^{\alpha}v_{n-1}(x,y,t) - \displaystyle\sum_{j=0}^{n-1}\frac{\partial u_j(x,y,t)}{\partial x}\frac{\partial v_{n-1-j}(x,y,t)}{\partial x} - 5 \end{cases} \tag{3.2.16}$$

We define the the initial approximations

$$\begin{cases} u_0(x,y,t) = x + 2y \\ v_0(x,y,t) = x - 2y \\ w_0(x,y,t) = -x + 2y \end{cases}.$$

By combining the recursive formula (3.2.15) with (3.2.16), we obtain

$$\begin{cases} u_1(x,y,t) = -h\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \\ v_1(x,y,t) = -h\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \\ w_1(x,y,t) = -h\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \end{cases},$$

$$\begin{cases} u_2(x,y,t) = -h(1+h)\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \\ v_2(x,y,t) = -h(1+h)\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \\ w_2(x,y,t) = -h(1+h)\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \end{cases},$$

$$\begin{cases} u_3(x,y,t) = -h(1+h)^2\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \\ v_3(x,y,t) = -h(1+h)^2\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \\ w_3(x,y,t) = -h(1+h)^2\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \end{cases},$$

$$\vdots$$

and so on.

According to Eq. (3.2.7), the expression of the series solutions by HAA can be written as

$$\begin{cases} u(x,y,t) = x+2y-h\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - h(1+h)\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - h(1+h)^2\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - ... \\ v(x,y,t) = x-2y-h\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - h(1+h)\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - h(1+h)^2\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - ... \\ w(x,y,t) = -x+2y-h\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - h(1+h)\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - h(1+h)^2\dfrac{3t^\alpha}{\Gamma(\alpha+1)} - ... \end{cases}.$$

For simplicity, we fix $h=-1$. Then the solution for system (3.2.13) becomes

$$\begin{cases} u(x,y,t) = x+2y+\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \\ v(x,y,t) = x-2y+\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \\ w(x,y,t) = -x+2y+\dfrac{3t^\alpha}{\Gamma(\alpha+1)} \end{cases}, \tag{3.2.17}$$

Setting $\alpha=1$ in (3.2.17), we find

$$\begin{cases} u(x,y,t) = x+2y+3t \\ v(x,y,t) = x-2y++3t \\ w(x,y,t) = -x+2y+3t \end{cases},$$

which is the exact solution of system (3.2.13) under the initial conditions (3.2.14).

3.3 The VIA procedure

To explore the VIA procedure, let us consider the system (3.1.1) under the initial conditions (3.1.2).

Taking the Riemann-Liouville fractional integral operator I_t^α defined by Eq. (1.1.20), to both sides of system (3.1.1), and taking into account relation (1.1.19) as well as the given initial conditions, we obtain

$$u_i(X,t) = \sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!} t^k - I_t^\alpha \left[\mathcal{L}_i \left(u_i(X,t) \right) + \mathcal{N}_i \left(u_i(X,t) \right) - f_i(X,t) \right], \ i = 1..., j. \quad (3.3.1)$$

Practicing $\dfrac{\partial}{dt}$ on both sides of Eq. (3.3.1), we have

$$\frac{\partial_i u(X,t)}{\partial t} = \frac{\partial}{dt} \left(\sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!} t^k - I_t^\alpha \left[\mathcal{L}_i \left(u_i(X,t) \right) + \mathcal{N}_i \left(u_i(X,t) \right) - f_i(X,t) \right] \right), \ i = 1..., j.$$

For $i = 1..., j$, the correction functionals for (3.3.1) using the VIA can be expressed approximately as follows

$$u_{i,n+1}(X,t) = u_{i,n}(X,t) + \int_0^t \lambda_i(X,\tau) \left[\frac{\partial}{d\tau} \left(\sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!} \tau^k - I_\tau^\alpha \left[\begin{array}{c} \mathcal{L}_i \left(u_i(X,\tau) \right) \\ + \mathcal{N}_i \left(u_i(X,\tau) \right) - f_i(X,\tau) \end{array} \right] \right) \right] d\tau. \quad (3.3.2)$$

According to the variational theory, the values of $\lambda_i(X,\tau)$ for (3.3.2) can be obtained as follows

$$1 + \lambda_i(X,\tau) = 0, \ i = 1..., j.$$

Which means that

$$\lambda_i(X,\tau) = -1, \ i = 1..., j.$$

After substituting the values of $\lambda_i(X,\tau)$ into Eq. (3.3.2), we obtain the following iteration formulations, for $i = 1..., j$ and $n = 0, 1, 2, ...$

$$u_{i,n+1}(X,t) = u_{i,n}(X,t) - \int_0^t \left[\frac{\partial}{d\tau} \left(\sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!} \tau^k - I_\tau^\alpha \left[\mathcal{L}_i \left(u_i(X,\tau) \right) + \mathcal{N}_i \left(u_i(X,\tau) \right) - f_i(X,\tau) \right] \right) \right] d\tau.$$

Or otherwise

$$u_{i,n+1}(X,t)=\sum_{k=0}^{m-1}\frac{g_{i,k}(x)}{k!}t^k-I_t^{\alpha}\left[\mathcal{L}_i\left(u_i(X,t)\right)+\mathcal{N}_i\left(u_i(X,t)\right)-f_i(X,t)\right]. \tag{3.3.3}$$

The zero iterations are determined by

$$u_{i,0}(X,t)=\sum_{k=0}^{m-1}\frac{g_{i,k}(x)}{k!}t^k,\ i=1...,j.$$

The successive approximations $u_{i,n}(X,t)$, $i=1...,j$ and $n=0,1,2,...$ converge rapidly to the exact solution of system (3.3.1) as n approaches to infinity,i.e.

$$u_i(X,t)=\lim_{n\longrightarrow\infty}u_{i,n}(X,t),\ i=1...,j.$$

Example 3.3.1 *Let us assume the system of nonlinear time-frational Burgers equations in two unknowns*

$$\begin{cases} {}^{C}D_t^{\alpha}u(x,y,t)=\dfrac{\partial^2u(x,y,t)}{\partial x^2}+\dfrac{\partial^2u(x,y,t)}{\partial y^2}-u(x,y,t)\dfrac{\partial u(x,y,t)}{\partial x}-v(x,y,t)\dfrac{\partial^2u(x,y,t)}{\partial y^2} \\ {}^{C}D_t^{\alpha}v(x,y,t)=\dfrac{\partial^2v(x,y,t)}{\partial x^2}+\dfrac{\partial^2v(x,y,t)}{\partial y^2}-v(x,y,t)\dfrac{\partial v(x,y,t)}{\partial y}-u(x,y,t)\dfrac{\partial^2v(x,y,t)}{\partial x^2} \end{cases}, \tag{3.3.4}$$

under the initial conditions

$$\begin{cases} u(x,y,0)=x+y \\ v(x,y,0)=x-y \end{cases}, \tag{3.3.5}$$

where ${}^{C}D_t^{\alpha}=\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ *signify the Caputo time-fractional derivative operator of order* $0<\alpha\leq 1$.

Utilizing the relation (3.3.3), we obtain the following iteration formula

$$\begin{cases} u_{n+1}(x,y,t)=x+y+I_t^{\alpha}\left[\begin{array}{l}\dfrac{\partial^2u_n(x,y,t)}{\partial x^2}+\dfrac{\partial^2u_n(x,y,t)}{\partial y^2}-u_n(x,y,t)\dfrac{\partial u_n(x,y,t)}{\partial x} \\ \qquad -v_n(x,y,t)\dfrac{\partial^2u_n(x,y,t)}{\partial y^2}\end{array}\right] \\ v_{n+1}(x,y,t)=x-y+I_t^{\alpha}\left[\begin{array}{l}\dfrac{\partial^2v_n(x,y,t)}{\partial x^2}+\dfrac{\partial^2v_n(x,y,t)}{\partial y^2}-v_n(x,y,t)\dfrac{\partial v_n(x,y,t)}{\partial y} \\ \qquad -u_n(x,y,t)\dfrac{\partial^2v_n(x,y,t)}{\partial x^2}\end{array}\right] \end{cases}, \tag{3.3.6}$$

where the zero iterations are determined by

$$\begin{cases} u_0(x,y,t) = x+y \\ v_0(x,y,t) = x-y \end{cases}. \tag{3.3.7}$$

From (3.3.6) and (3.3.7), we obtain

$$\begin{cases} u_0(x,y,t) = x+y \\ v_0(x,y,t) = x-y \end{cases},$$

$$\begin{cases} u_1(x,y,t) = x+y-2x\dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ v_1(x,y,t) = x-y-2y\dfrac{t^\alpha}{\Gamma(\alpha+1)} \end{cases},$$

$$\begin{cases} u_2(x,y,t) = x+y-2x\dfrac{t^\alpha}{\Gamma(\alpha+1)}+4(x+y)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ v_2(x,y,t) = x-y-2y\dfrac{t^\alpha}{\Gamma(\alpha+1)}+4(x-y)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases},$$

$$\begin{cases} u_3(x,y,t) = x+y-2x\dfrac{t^\alpha}{\Gamma(\alpha+1)}+4(x+y)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}-4x\left(4-\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ v_3(x,y,t) = x-y-2y\dfrac{t^\alpha}{\Gamma(\alpha+1)}+4(x-y)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}-4y\left(4-\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{cases},$$

$\vdots$

and so on.

Then, the successive approximations can be expressed as

$$\begin{cases} u_n(x,y,t) = x+y-2x\dfrac{t^\alpha}{\Gamma(\alpha+1)}+4(x+y)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}-4x\left(4+\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}+... \\ v_n(x,y,t) = x-y-2y\dfrac{t^\alpha}{\Gamma(\alpha+1)}+4(x-y)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}-4y\left(4+\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}+... \end{cases}.$$

Putting $\alpha = 1$, we obtain the solution in closed form

$$\begin{cases} u(x,y,t) = \lim\limits_{n\longrightarrow\infty} u_n(x,y,t) \\ v(x,y,t) = \lim\limits_{n\longrightarrow\infty} v_n(x,y,t) \end{cases}$$

$$\Longrightarrow \begin{cases} u(x,y,t) = \lim\limits_{n\longrightarrow\infty} \left((x+y)\left(1+2t^2+4t^4+...\right)-2xt\left(1+2t^2+4t^4+...\right)\right) \\ v(x,y,t) = \lim\limits_{n\longrightarrow\infty} \left((x-y)\left(1+2t^2+4t^4+...\right)-2yt\left(1+2t^2+4t^3+...\right)\right) \end{cases}$$

$$\Longrightarrow \begin{cases} u(x,y,t) = \dfrac{x+y-2xt}{1-2t^2} \\ v(x,y,t) = \dfrac{x-y-2yt}{1-2t^2} \end{cases}.$$

which is the exact solution of system (3.3.4) under the initial conditions (3.3.5).

Example 3.3.2 *Let us assume the system of time-frational nonlinear equations of unsteady flow of a polytropic gas in four unknowns*

$$\begin{cases} {}^{C}D_t^{\alpha}u(x,y,t)+u(x,y,t)\dfrac{\partial u(x,y,t)}{\partial x}+v(x,y,t)\dfrac{\partial u(x,y,t)}{\partial y}+\dfrac{1}{\rho(x,y,t)}\dfrac{\partial w(x,y,t)}{\partial x}=0 \\ {}^{C}D_t^{\alpha}v(x,y,t)+u(x,y,t)\dfrac{\partial v(x,y,t)}{\partial x}+v(x,y,t)\dfrac{\partial v(x,y,t)}{\partial y}+\dfrac{1}{\rho(x,y,t)}\dfrac{\partial w(x,y,t)}{\partial v}=0 \\ {}^{C}D_t^{\alpha}\rho(x,y,t)+u(x,y,t)\dfrac{\partial \rho(x,y,t)}{\partial x}+v(x,y,t)\dfrac{\partial \rho(x,y,t)}{\partial y}+\rho(x,y,t)\left(\dfrac{\partial u(x,y,t)}{\partial x}+\dfrac{\partial v(x,y,t)}{\partial y}\right)=0 \\ {}^{C}D_t^{\alpha}w(x,y,t)+u(x,y,t)\dfrac{\partial w(x,y,t)}{\partial x}+v(x,y,t)\dfrac{\partial w(x,y,t)}{\partial y}+w(x,y,t)\left(\dfrac{\partial u(x,y,t)}{\partial x}+\dfrac{\partial v(x,y,t)}{\partial y}\right)=0 \end{cases}, \tag{3.3.8}$$

under the initial conditions

$$\begin{cases} u(x,y,0)=e^{x+y} \\ v(x,y,0)=-1-e^{x+y} \\ \rho(x,y,0)=e^{x+y} \\ w(x,y,0)=c \end{cases}, \tag{3.3.9}$$

where ${}^{C}D_t^{\alpha}=\dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$ *signify the Caputo time-fractional derivative operator of order* $0<\alpha\leq 1$.

Utilizing relation (3.3.3), we obtain the following iteration formula

$$\begin{cases} u_{n+1}(x,y,t)=e^{x+y}-I_t^{\alpha}\left[\begin{array}{c} u_n(x,y,t)\dfrac{\partial u_n(x,y,t)}{\partial x}+v_n(x,y,t)\dfrac{\partial u_n(x,y,t)}{\partial y} \\ +\dfrac{1}{\rho_n(x,y,t)}\dfrac{\partial w_n(x,y,t)}{\partial x} \end{array}\right] \\ v_{n+1}(x,y,t)=-1-e^{x+y}-I_t^{\alpha}\left[\begin{array}{c} u_n(x,y,t)\dfrac{\partial v_n(x,y,t)}{\partial x}+v_n(x,y,t)\dfrac{\partial v_n(x,y,t)}{\partial y} \\ +\dfrac{1}{\rho_n(x,y,t)}\dfrac{\partial w_n(x,y,t)}{\partial v} \end{array}\right] \\ \rho_{n+1}(x,y,t)=e^{x+y}-I_t^{\alpha}\left[\begin{array}{c} u_n(x,y,t)\dfrac{\partial \rho_n(x,y,t)}{\partial x}+v_n(x,y,t)\dfrac{\partial \rho_n(x,y,t)}{\partial y} \\ +\rho_n(x,y,t)\left(\dfrac{\partial u_n(x,y,t)}{\partial x}+\dfrac{\partial v_n(x,y,t)}{\partial y}\right) \end{array}\right] \\ w_{n+1}(x,y,t)=c-I_t^{\alpha}\left[\begin{array}{c} u_n(x,y,t)\dfrac{\partial w_n(x,y,t)}{\partial x}+v_n(x,y,t)\dfrac{\partial w_n(x,y,t)}{\partial y} \\ +w_n(x,y,t)\left(\dfrac{\partial u_n(x,y,t)}{\partial x}+\dfrac{\partial v_n(x,y,t)}{\partial y}\right) \end{array}\right] \end{cases}, \tag{3.3.10}$$

where the zero iterations are determined by

$$\begin{cases} u_0(x,y,t) = e^{x+y} \\ v_0(x,y,t) = -1 - e^{x+y} \\ \rho_0(x,y,t) = e^{x+y} \\ w_0(x,y,t) = c \end{cases}. \tag{3.3.11}$$

From (3.3.10) and (3.3.11), we obtain

$$\begin{cases} u_0(x,y,t) = e^{x+y} \\ v_0(x,y,t) = -1 - e^{x+y} \\ \rho_0(x,y,t) = e^{x+y} \\ w_0(x,y,t) = c \end{cases},$$

$$\begin{cases} u_1(x,y,t) = e^{x+y} + e^{x+y}\dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ v_1(x,y,t) = -1 - e^{x+y} - e^{x+y}\dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ \rho_1(x,y,t) = e^{x+y} + e^{x+y}\dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ w_1(x,y,t) = c \end{cases},$$

$$\begin{cases} u_2(x,y,t) = e^{x+y} + e^{x+y}\dfrac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ v_2(x,y,t) = -1 - e^{x+y} - e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \rho_2(x,y,t) = e^{x+y} + e^{x+y}\dfrac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ w_2(x,y,t) = c \end{cases},$$

$$\begin{cases} u_3(x,y,t) = e^{x+y} + e^{x+y}\dfrac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{x+y}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ v_3(x,y,t) = -1 - e^{x+y} - e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - e^{x+y}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ \rho_3(x,y,t) = e^{x+y} + e^{x+y}\dfrac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{x+y}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ w_3(x,y,t) = c \end{cases},$$

$$\vdots$$

and so on.

Then, the successive approximations can be expressed as

$$\begin{cases} u_n(x,y,t) = e^{x+y} + e^{x+y}\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{x+y}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... + e^{x+y}\dfrac{t^{n\alpha}}{\Gamma(3n\alpha+1)} \\ v_n(x,y,t) = -1 - e^{x+y} - e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - e^{x+y}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} - ... - e^{x+y}\dfrac{t^{n\alpha}}{\Gamma(3n\alpha+1)} \\ \rho_n(x,y,t) = e^{x+y} + e^{x+y}\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + e^{x+y}\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{x+y}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... + e^{x+y}\dfrac{t^{n\alpha}}{\Gamma(3n\alpha+1)} \\ w_n(x,y,t) = c \end{cases}.$$

Putting $\alpha = 1$, we obtain the solution in closed form

$$\begin{cases} u(x,y,t) = \lim\limits_{n\longrightarrow\infty} u_n(x,y,t) \\ v(x,y,t) = \lim\limits_{n\longrightarrow\infty} v_n(x,y,t) \\ \rho(x,y,t) = \lim\limits_{n\longrightarrow\infty} \rho_n(x,y,t) \\ w(x,y,t) = \lim\limits_{n\longrightarrow\infty} w_n(x,y,t) \end{cases},$$

$$\Longrightarrow \begin{cases} u(x,y,t) = \lim\limits_{n\longrightarrow\infty} e^{x+y}\left(1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + ... + \dfrac{t^n}{n!}\right) \\ v(x,y,t) = \lim\limits_{n\longrightarrow\infty} \left(-1 - e^{x+y}\left(1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + ... + \dfrac{t^n}{n!}\right)\right) \\ \rho(x,y,t) = \lim\limits_{n\longrightarrow\infty} e^{x+y}\left(1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + ... + \dfrac{t^n}{n!}\right) \\ w(x,y,t) = \lim\limits_{n\longrightarrow\infty} c = \end{cases},$$

$$\Longrightarrow \begin{cases} u(x,y,t) = e^{x+y+t} \\ v(x,y,t) = -1 - e^{x+y+t} \\ \rho(x,y,t) = e^{x+y+t} \\ w(x,y,t) = c \end{cases},$$

which is the exact solution of system (3.3.8) under the initial conditions (3.3.9).

3.4 The RDTA procedure

To adapt the RDTA procedure for resolving the system (3.1.1) under the initial conditions (3.1.2), we start by applying the Riemann-Liouville fractional integral operator I_t^{α} defined by Eq. (1.1.20) to both sides of the system (3.1.1), we obtain

$$I_t^{\alpha}\left[{}^{C}D_t^{\alpha}u_i(X,t)\right] + I_t^{\alpha}\left[\mathcal{L}_i\left(u_i(X,t)\right) + \mathcal{N}_i\left(u_i(X,t)\right)\right] = I_t^{\alpha}\left[f_i(X,t)\right],\ i = 1,2,...,j.. \quad (3.4.1)$$

Using relation (1.1.19) and the given initial conditions, Eq. (3.4.1) can be rewritten as

$$u_i(X,t) = h_i(X,t) - I_t^{\alpha}\left[\mathcal{L}_i\left(u_i(X,t)\right) + \mathcal{N}_i\left(u_i(X,t)\right)\right],\ i = 1..., j, \tag{3.4.2}$$

where

$$h_i(X,t) = \sum_{k=0}^{m-1} \frac{g_{i,k}(x)}{k!} t^k + I_t^{\alpha}\left[f_i(X,t)\right],\ i = 1..., j,$$

where $h_i(X,t)$ signify the terms derived from the source terms and the given initial conditions.

Now, by utilizing the fundamental properties of RDTA to Eq. (3.4.2), we obtain

$$U_{i,k+n}(X) = H_{i,k}(X) - I_t^{\alpha}\left[A_{i,k}(X) + B_{i,k}(X)\right], i = 1, 2, ..., j, \tag{3.4.3}$$

where $H_{i,k}(X)$, $A_{i,k}(X)$, and $B_{i,k}(X)$ signify respectively the reduced differential transformed forms of the terms $h_i(X,t)$, $\mathcal{L}_i\left(u_i(X,t)\right)$, and $\mathcal{N}_i\left(u_i(X,t)\right)$ for $i = 1, 2, ..., j$.

From RDTA, the first-order approximations can be expressed as

$$U_{i,0}(X) = H_{i,0}(X),\ i = 1, 2, ..., j. \tag{3.4.4}$$

For $i = 1, 2, ..., j$, we substitute the Eq. (3.4.4) into Eq. (3.4.3), we get

$$\begin{aligned}
U_{i,0}(X) &= H_{i,0}(X), \\
U_{i,1}(X) &= -I_t^{\alpha}\left[A_{i,0}(X) + B_{i,0}(X)\right], \\
U_{i,2}(X) &= -I_t^{\alpha}\left[A_{i,1}(X) + B_{i,1}(X)\right], \\
U_{i,3}(X) &= -I_t^{\alpha}\left[A_{i,2}(X) + B_{i,2}(X)\right], \\
&\vdots
\end{aligned}$$

Finally, the solution of the system (3.1.1) under the initial conditions (3.1.2) can be expressed in the form of an infinite series as follows

$$u_i(X,t) = \sum_{k=0}^{\infty} U_{i,k}(X),\ i = 1, 2, ..., j.$$

Example 3.4.1 *Let's suppose the system of nonlinear time-fractional partial differential equations in two unknowns*

$$\begin{cases} {}^{C}D_t^{\alpha} u(x,t) + \dfrac{\partial^2 u(x,t)}{\partial x^2} + 2u(x,t)\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) = 0 \\ {}^{C}D_t^{\alpha} v(x,t) + \dfrac{\partial^2 v(x,t)}{\partial x^2} + 2v(x,t)\dfrac{\partial v(x,t)}{\partial x} + \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) = 0 \end{cases}, \tag{3.4.5}$$

under the initial conditions

$$\begin{cases} u(x,0) = e^x \\ v(x,0) = -e^x \end{cases}, \tag{3.4.6}$$

where ${}^C D_t^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha}$ *signify the Caputo time-fractional derivative operator of order* $0 < \alpha \leq 1$.

By following the same steps of the RDTA procedure described in paragraph 3.4 and using Theorem 2.4.1, we obtain the following recurrence relation

$$\begin{cases} U_0(x) = e^x \\ V_0(x) = -e^x \end{cases}, \tag{3.4.7}$$

$$\begin{cases} U_k(x) = -\left(\dfrac{\partial^2 U_k(x)}{\partial x^2} + 2\displaystyle\sum_{r=0}^{k} U_r(x)\frac{\partial U_{k-r}(x)}{\partial x} + \frac{\partial}{\partial x}\left(\displaystyle\sum_{r=0}^{k} U_r(x)V_{k-r}(x)\right)\right) \\ V_k(x) = -\left(\dfrac{\partial^2 V_k(x)}{\partial x^2} + 2\displaystyle\sum_{r=0}^{k} V_r(x)\frac{\partial V_{k-r}(x)}{\partial x} + \frac{\partial}{\partial x}\left(\displaystyle\sum_{r=0}^{k} U_r(x)V_{k-r}(x)\right)\right) \end{cases}.$$

By resolving the formula (3.4.7), we have

$$\begin{cases} U_0(x) = e^x \\ V_0(x) = -e^{-x} \end{cases},$$

$$\begin{cases} U_1(x) = -e^x \dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ V_1(x) = e^x \dfrac{t^\alpha}{\Gamma(\alpha+1)} \end{cases},$$

$$\begin{cases} U_2(x) = e^x \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ V_2(x) = -e^x \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases},$$

$$\begin{cases} U_3(x) = -e^x \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ V_3(x) = e^x \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{cases},$$

$$\vdots$$

$$\begin{cases} U_k(x) = (-1)^k e^x \dfrac{t^{k\alpha}}{\Gamma(k\alpha+1)} \\ V_k(x) = (-1)^k \left(-e^x\right) \dfrac{t^{k\alpha}}{\Gamma(k\alpha+1)} \end{cases}.$$

Then, the analytic solution of the system (3.4.5) under the initial conditions (3.4.6) is

$$\begin{cases} u(x,t) = \sum_{k=0}^{\infty} U_k(x) = e^x \left(1 - \dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = e^x E_\alpha(-t^\alpha) \\ v(x,t) = \sum_{k=0}^{\infty} V_k(x) = -e^x \left(1 - \dfrac{t^\alpha}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right) = -e^x E_\alpha(-t^\alpha) \end{cases},$$

where $E_\alpha(-t^\alpha)$ is the Mittag-Leffler function of one parameter α defined by Eq. (1.1.6).

When $\alpha = 1$, the exact solution becomes

$$\begin{cases} u(x,t) = e^{x-t} \\ v(x,t) = -e^{x-t} \end{cases}.$$

Example 3.4.2 *Let's suppose that the system of nonlinear time-frational Navier-Stocks equations in three unknowns*

$$\begin{cases} {}^C D_t^\alpha u + uu_x + vu_y + wu_z = u_{xx} + u_{yy} + u_{zz} \\ {}^C D_t^\alpha v + uv_x + vv_y + wv_z = v_{xx} + v_{yy} + v_{zz} \\ {}^C D_t^\alpha w + uw_x + vw_y + ww_z = u_{xx} + u_{yy} + u_{zz} \end{cases}, \tag{3.4.8}$$

under the initial conditions

$$\begin{cases} u(x,y,z,0) = -0.5x + y + z \\ v(x,y,z,0) = x - 0.5y + z \\ w(x,y,z,0) = x + y - 0.5z \end{cases}, \tag{3.4.9}$$

where ${}^C D_t^\alpha = \dfrac{\partial^\alpha}{\partial t^\alpha}$ signify the Caputo time-fractional derivative operator of order $0 < \alpha \leq 1$ and

$$\begin{cases} u = u(x,y,z,t) \\ v = v(x,y,z,t) \\ w = w(x,y,z,t) \end{cases}. \tag{3.4.9}$$

By following the same steps of the RDTA procedure described in paragraph 3..4 and using Theorem 2.4.1, we obtained the following recurrence relation

$$\begin{cases} U_0(x,y,z) = -0.5x + y + z \\ V_0(x,y,z) = x - 0.5y + z \quad , \\ W_0(x,y,z) = x + y - 0.5z \end{cases} \tag{3.4.10}$$

$$\begin{cases} U_k(x,y,z) = -\left(\sum_{r=0}^{k} U_r \frac{\partial U_{k-r}}{\partial x} + \sum_{r=0}^{k} V_r \frac{\partial U_{k-r}}{\partial y} + \sum_{r=0}^{k} W_r \frac{\partial U_{k-r}}{\partial z}\right) + \frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2} + \frac{\partial^2 U_k}{\partial z^2} \\ V_k(x,y,z) = -\left(\sum_{r=0}^{k} U_r \frac{\partial V_{k-r}}{\partial x} + \sum_{r=0}^{k} V_r \frac{\partial V_{k-r}}{\partial y} + \sum_{r=0}^{k} W_r \frac{\partial V_{k-r}}{\partial z}\right) + \frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2} + \frac{\partial^2 U_k}{\partial z^2} \\ W_k(x,y,z) = -\left(\sum_{r=0}^{k} U_r \frac{\partial W_{k-r}}{\partial x} + \sum_{r=0}^{k} V_r \frac{\partial UW_{k-r}}{\partial y} + \sum_{r=0}^{k} W_r \frac{\partial W_{k-r}}{\partial z}\right) + \frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2} + \frac{\partial^2 U_k}{\partial z^2} \end{cases} .$$

On resolving the formula (3.4.10), we have

$$\begin{cases} U_0(x,y,z) = -0.5x + y + z \\ V_0(x,y,z) = x - 0.5y + z \quad , \\ W_0(x,y,z) = x + y - 0.5z \end{cases}$$

$$\begin{cases} U_1(x,y,z) = -2.25x \dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ V_1(x,y,z) = -2.25y \dfrac{t^\alpha}{\Gamma(\alpha+1)} \quad , \\ W_1(x,y,z) = -2.25z \dfrac{t^\alpha}{\Gamma(\alpha+1)} \end{cases}$$

$$\begin{cases} U_2(x,y,z) = 2(2.25)\left(-0.5x + y + z\right) \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ V_2(x,y,z) = 2(2.25)\left(x - 0.5y + z\right) \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad , \\ W_2(x,y,z) = 2(2.25)\left(x + y - 0.5z\right) \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{cases}$$

$$\begin{cases} U_3(x,z,y) = -(2.25)^2 \left(4 + \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right) x \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ V_3(x,y,z) = -(2.25)^2 \left(4 + \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right) y \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \quad ,, \\ W_3(x,y,z) = -(2.25) \left(4 + \dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right) z \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \end{cases}$$

$$\begin{cases} U_4(x,z,y) = (2.25)^2\left(8+\dfrac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}++\dfrac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right)(-0.5x+y+z)\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ V_4(x,y,z) = (2.25)^2\left(8+\dfrac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}++\dfrac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right)(x-0.5y+z)\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ W_4(x,y,z) = (2.25)^2\left(8+\dfrac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}++\dfrac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right)(x+y-0.5z)\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ \vdots \end{cases},$$

and so on.

Then, the approximate solution of the system (3.4.8) under the initial conditions (3.4.9) is

$$\begin{cases} u(x,y,z,t) = \sum_{k=0}^{\infty} V_k(x,y,z) = -0.5x+y+z-2.25x\dfrac{t^{\alpha}}{\Gamma(\alpha+1)}+2(2.25)(-0.5x+y+z)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \qquad -(2.25)^2\left(4+\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)x\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ \qquad +(2.25)^2\left(8+\dfrac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+\dfrac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right)(-0.5x+y+z)\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ v(x,y,z,t) = \sum_{k=0}^{\infty} V_k(x,y,z) = x-0.5y+z-2.25y\dfrac{t^{\alpha}}{\Gamma(\alpha+1)}+2(2.25)(x-0.5y+z)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \qquad -(2.25)^2\left(4+\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)y\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ \qquad +(2.25)^2\left(8+\dfrac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+\dfrac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right)(x+y-0.5z)\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ w(x,y,z,t) = \sum_{k=0}^{\infty} W_k(x,y,z) = x+y-0.5z-2.25z\dfrac{t^{\alpha}}{\Gamma(\alpha+1)}+2(2.25)(x+y-0.5z)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \qquad -(2.25)\left(4+\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\right)z\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ \qquad +(2.25)^2\left(8+\dfrac{2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}+\dfrac{4\Gamma(3\alpha+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right)(x+y-0.5z)\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \end{cases},$$

For $\alpha = 1$, we obtain the required exact solution as

$$\begin{cases} u(x,y,z,t) = (-0.5x+y+z)\left(1+2.25t^2+(2.25)^2t^4+...\right)-2.25xt\left(1+2.25t^2+...\right) \\ \qquad = \dfrac{-0.5x+y+z-2.25xt}{1-2.25t^2} \\ v(x,y,z,t) = (x-0.5y+z)\left(1+2.25t^2+(2.25)^2t^4+...\right)-2.25yt\left(1+2.25t^2+...\right) \\ \qquad = \dfrac{x-0.5y+z-2.25yt}{1-2.25t^2} \\ w(x,y,z,t) = (x+y-0.5z)\left(1+2.25t^2+(2.25)^2t^4+...\right)-2.25zt\left(1+2.25t^2+...\right) \\ \qquad = \dfrac{x+y-0.5z-2.25zt}{1-2.25t^2} \end{cases}.$$

Chapter 4

A new mixed approach for resolving the system of nonlinear time-fractional Burgers equations

In the current chapter, we suggest a new mixed approach for resolving systems of nonlinear time-fractional partial differential equations, specifically the system of nonlinear time-fractional Burgers equations involving the Caputo fractional derivative. This approach is a mixture of two different approaches: the Khalouta transform approach (KhTA) and the Daftardar-Jafari approach (DJA), it is known as the Khalouta-Daftardar-Jafari approach (KhDJA). We will begin by providing an overview of nonlinear time-fractional Burgers equations, then present the definition and essential characteristics of KhTA, and finally, introduce the procedure of the suggested approach as well as two numerical applications of the system of nonlinear time-fractional Burgers equations.

4.1 History of the nonlinear time-fractional Burgers equation

The nonlinear Burgers equation or nonlinear Bateman–Burgers equation is one of the most important types of diffusion-convection equations. It was initially introduced by Harry Bateman in 1915 [**8**] and later discussed by researcher Johannes Martinus Burgers in [**10**].

It is widely utilized to explain numerous physical problems, such as the unidirectional propagation of weak nonlinear acoustic waves, shock waves in viscous media, flow systems, electromagnetic waves, etc.

The history of the nonlinear time-fractional Burgers equation is a recent development, building on late 19th-and early 20th-century work on the classical nonlinear Burgers equation. The introduction of fractional calculus to model complex physical systems, such as those involving fluid mechanics, heat conduction, and turbulence, led researchers to integrate fractional derivatives with the standard nonlinear Burgers equation, giving rise to the nonlinear time-fractional Burgers equation. This new class of equations allows for a more comprehensive description of phenomena with memory and non-local effects, though developing appropriate fractional derivative definitions and analytical methods for these models remains an active area of research.

4.2 KhT definitions and results

In reference [19], Ali Khalouta proposed a new integral transform in 2023. This new generalized integral transform, which will be abbreviated as KhT in the rest of this chapter, includes a variety of the family of known integral transforms and is particularly useful for solving differential equations, partial differential equations, integral equations, and integro-differential equations.

Definition 4.2.1 *[19] Let $u(x,t)$ is a real-valued function of two variables x and t.*

1) The KhT of $u(x,t)$ respecting the variable t is expressed by

$$\mathbb{KH}\left[u(x,t)\right] = \mathcal{K}(x,s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^{\infty} \exp\left(-\frac{st}{\gamma\eta}\right) u(x,t)dt,$$

where $s,\gamma,\eta > 0$ are the KhT variables.

2) The inverse KhT of $\mathcal{K}(x,s,\gamma,\eta)$ is expressed by

$$\mathbb{KH}^{-1}\left[\mathcal{K}(x,s,\eta,\gamma)\right] = u(x,t), \ \textit{for } t \geq 0,$$

which is equivalent to

$$u(x,t) = \mathbb{KH}^{-1}\left[\mathcal{K}(x,s,\eta,\gamma)\right] = \frac{1}{2\pi i}\int_{\varphi-i\infty}^{\varphi+i\infty} \frac{1}{s}\exp\left(\frac{st}{\gamma\eta}\right)\mathcal{K}(x,s,\eta,\gamma)ds,$$

where φ is a real constant and the integral is taken along $s = \varphi$ in the complex plane $s = u + iv$.

Theorem 4.2.1 *[19] The essential characteristics of KhT are*

(1) Characteristic of the linearity

$$\mathbb{KH}\left[au(x,t) + bv(x,t)\right] = a\mathbb{KH}\left[u(x,t)\right] + b\mathbb{KH}\left[v(x,t)\right], a, b \in \mathbb{R}$$

(2) Characteristic of the derivative

$$\mathbb{KH}\left[u^{(n)}(x,t)\right] = \left(\frac{s}{\gamma\eta}\right)^n \mathbb{KH}\left[u(x,t)\right] - \sum_{k=0}^{n-1} \left(\frac{s}{\gamma\eta}\right)^{n-k} u^{(k)}(x,0), n \geq 1.$$

where $u^{(n)}(x,t)$ is the nth derivative of $u(x,t)$.

(3) Characteristic of the convolution

$$\mathbb{KH}\left[(u * v)(x,t)\right] = \frac{\gamma\eta}{s}\mathbb{KH}\left[u(x,t)\right]\mathbb{KH}\left[v(x,t)\right],$$

where $\mathbb{KH}\left[(u * v)(x,t)\right]$ is the Khalouta convolution of the functions $u(x,t)$ and $v(x,t)$.

(4) KhTs for some common functions

$$\begin{aligned}
\mathbb{KH}\left[1\right] &= 1, \\
\mathbb{KH}\left[t\right] &= \frac{\gamma\eta}{s}, \\
\mathbb{KH}\left[\frac{t^n}{n!}\right] &= \left(\frac{\gamma\eta}{s}\right)^n, n \geq 0, \\
\mathbb{KH}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] &= \left(\frac{\gamma\eta}{s}\right)^\alpha, \alpha > -1,
\end{aligned}$$

Proof. The proof of this theorem is found in detail in the reference [19]. ■

Theorem 4.2.2 *[20] The KhT of the Riemann-Liouville fractional integral of order $\alpha > 0$ is expressed as*

$$\mathbb{KH}\left[I_t^\alpha u(x,t)\right] = \left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}\left[u(x,t)\right].$$

Proof. By performing the KhT on Eq. (1.1.20), we obtain

$$\begin{aligned}
\mathbb{KH}\left[I_t^\alpha u(x,t)\right] &= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} u(x,s)ds\right] \\
&= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * u(x,t)\right].
\end{aligned}$$

By using the characteristic of the convolution (3), we get

$$\begin{aligned}\mathbb{KH}\left[I_t^{\alpha}u(x,t)\right] &= \frac{\gamma\eta}{s}\mathbb{KH}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right]\mathbb{KH}\left[u(x,t)\right]\\ &= \frac{\gamma\eta}{s}\left(\frac{\gamma\eta}{s}\right)^{\alpha-1}\mathbb{KH}\left[u(x,t)\right]\\ &= \left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[u(x,t)\right].\end{aligned}$$

This completes the proof. ■

Theorem 4.2.3 *[20] The KhT of the Caputo fractional derivative of order $n-1<\alpha\leq n$ with $n\geq 1$ is expressed as*

$$\mathbb{KH}\left[{}^{C}D_t^{\alpha}u(x,t)\right]=\left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}\left[u(x,t)\right]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}u^{(k)}(x,0).$$

Proof. We take

$$v(x,t)=u^{(n)}(x,t). \tag{4.2.1}$$

Then, Eq. (1.1.21) transformed into

$$\begin{aligned}{}^{C}D_t^{\alpha}u(x,t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}u^{(n)}(x,s)ds,\\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}v(x,\tau)d\tau\\ &= I_t^{n-\alpha}v(x,t).\end{aligned} \tag{4.2.2}$$

Applying the KhT to both sides of Eq. (5.2.2) and using Theorem 4.2.2 gives

$$\mathbb{KH}\left[{}^{C}D_t^{\alpha}u(x,t)\right]=\mathbb{KH}\left[I_t^{n-\alpha}v(x,t)\right]=\left(\frac{\gamma\eta}{s}\right)^{n-\alpha}\mathcal{V}(x,s,\gamma,\eta), \tag{4.2.3}$$

where $\mathcal{V}(x,s,\gamma,\eta)$ is the KhT of $v(x,t)$.

Implementing the KhT to both sides of Eq. (4.2.1) and using the characteristic of the derivative (2), yields

$$\begin{aligned}\mathbb{KH}\left[v(x,t)\right] &= \mathbb{KH}\left[u^{(n)}(x,t)\right],\\ \mathcal{V}(x,s,\gamma,\eta) &= \left(\frac{s}{\gamma\eta}\right)^{n}\mathbb{KH}\left[u(x,t)\right]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}u^{(k)}(x,0).\end{aligned} \tag{4.2.4}$$

Replacing Eq. (4.2.4) into Eq. (4.2.3), gives

$$\begin{aligned}\mathbb{KH}\left[{}^{C}D_t^{\alpha}u(x,t)\right] &= \left(\frac{\gamma\eta}{s}\right)^{n-\alpha}\left(\left(\frac{s}{\gamma\eta}\right)^{n}\mathbb{KH}\left[u(x,t)\right]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}u^{(k)}(x,0)\right)\\ &= \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}\left[u(x,t)\right]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}u^{(k)}(x,0).\end{aligned}$$

This completes the proof. ■

4.3 Khalouta-Daftardar-Jafari approach (KhDJA)

In this paragraph, we present our main results related to KhDJA

Theorem 4.3.1 *[14] Consider the following general system of nonlinear time-fractional partial differential equations*

$$\begin{cases} {}^{C}D_t^{\alpha}u(x,t)=\mathcal{L}(u(x,t))+\mathcal{N}(u(x,t),v(x,t))+\mathcal{F}(x,t), n-1<\alpha\leq n\\ {}^{C}D_t^{\beta}v(x,t)=\mathcal{P}(v(x,t))+\mathcal{M}(u(x,t),v(x,t))+\mathcal{G}(x,t), n-1<\beta\leq n\end{cases}, n\in\mathbb{N}^{*}, \quad (4.3.1)$$

under the initial conditions

$$\begin{cases} u^{(k)}(x,0)=f_k(x)\\ v^{(k)}(x,0)=g_k(x)\end{cases}, k=0,1,2,...,n-1, \quad (4.3.2)$$

where ${}^{C}D_t^{\alpha}$ *and* ${}^{C}D_t^{\beta}$ *are the Caputo time-fractional derivative operators of the functions* $u(x,t)$ *and* $v(x,t)$ *,* $\mathcal{L},\mathcal{P}$ *are linear operators,* $\mathcal{N},\mathcal{M}$ *are nonlinear operators and* $\mathcal{F}(x,t),\mathcal{G}(x,t)$ *are known continuous functions.*

Using NKhDJA, the solution of the system (4.3.1) under the initial conditions (4.3.2) can be expressed as an infinite series as follows

$$\begin{cases} u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)\\ v(x,t)=\sum_{i=0}^{\infty}v_i(x,t)\end{cases}. \quad (4.3.3)$$

Proof. Employing the KhT on system (4.3.1), we attain the consequent formula

$$\begin{cases} \mathbb{KH}\left[{}^{C}D_t^{\alpha}u(x,t)\right] = \mathbb{KH}\left[\mathcal{L}(u(x,t)) + \mathcal{N}(u(x,t),v(x,t)) + \mathcal{F}(x,t)\right] \\ \mathbb{KH}\left[{}^{C}D_t^{\beta}v(x,t)\right] = \mathbb{KH}\left[\mathcal{P}(v(x,t)) + \mathcal{M}(u(x,t),v(x,t)) + \mathcal{G}(x,t)\right] \end{cases}$$

Utilizing Theorem 4.2.3 and the initial condition (4.3.2), we acquire

$$\begin{cases} \mathbb{KH}\left[u(x,t)\right] = \sum_{k=0}^{n-1}\left(\frac{\gamma\eta}{s}\right)^{k} f_k(x) + \left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\mathcal{F}(x,t)\right] + \left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\begin{array}{c}\mathcal{L}(u(x,t)) \\ +\mathcal{N}(u(x,t),v(x,t))\end{array}\right] \\ \mathbb{KH}\left[v(x,t)\right] = \sum_{k=0}^{n-1}\left(\frac{\gamma\eta}{s}\right)^{k} g_k(x) + \left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\mathcal{G}(x,t)\right] + \left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\begin{array}{c}\mathcal{P}(v(x,t)) \\ +\mathcal{M}(u(x,t),v(x,t))\end{array}\right] \end{cases}, \tag{4.3.4}$$

Implementing the inverse KhT on Eq. (4.3.4), we get

$$\begin{cases} u(x,t) = \mathcal{R}(x,t) + \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\mathcal{L}(u(x,t)) + \mathcal{N}(u(x,t),v(x,t))\right]\right] \\ v(x,t) = \mathcal{T}(x,t) + \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\mathcal{P}(v(x,t)) + +\mathcal{M}(u(x,t),v(x,t))\right]\right] \end{cases}, \tag{4.3.5}$$

where $\mathcal{R}(x,t)$ and $\mathcal{T}(x,t)$ signify the terms due to the initial conditions and the source terms.

Now, we apply the iterative method suggested by Daftardar-Gejji and Jafari [13], the solution to (4.3.1)-(4.3.2) is written as an infinite series

$$\begin{cases} u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \\ v(x,t) = \sum_{i=0}^{\infty} v_i(x,t) \end{cases}. \tag{4.3.6}$$

As $\mathcal{L}$ and $\mathcal{P}$ are linear operators, we have

$$\mathcal{L}\left(\sum_{i=0}^{\infty} u_i(x,t)\right) = \sum_{i=0}^{\infty}\mathcal{L}\left(u_i(x,t)\right), \tag{4.3.7}$$

$$\mathcal{P}\left(\sum_{i=0}^{\infty} v_i(x,t)\right) = \sum_{i=0}^{\infty}\mathcal{P}\left(v_i(x,t)\right). \tag{4.3.8}$$

The nonlinear operators $\mathcal{N}$ and $\mathcal{M}$ can be decomposed as

$$\begin{aligned}\mathcal{N}\left(\sum_{i=0}^{\infty} u_i(x,t), \sum_{i=0}^{\infty} v_i(x,t)\right) &= \mathcal{N}\left(u_0(x,t),v_0(x,t)\right) + \sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} u_j(x,t), \sum_{j=0}^{i} v_j(x,t)\right)\right. \\ &\left. -\mathcal{N}\left(\sum_{j=0}^{i-1} u_j(x,t), \sum_{j=0}^{i-1} v_j(x,t)\right)\right\}, \end{aligned} \tag{4.3.9}$$

$$\mathcal{M}\left(\sum_{i=0}^{\infty}u_i(x,t),\sum_{i=0}^{\infty}v_i(x,t)\right) = \mathcal{M}\left(u_0(x,t),v_0(x,t)\right)+\sum_{i=1}^{\infty}\left\{\mathcal{M}\left(\sum_{j=0}^{i}u_j(x,t),\sum_{j=0}^{i}v_j(x,t)\right)-\mathcal{M}\left(\sum_{j=0}^{i-1}u_j(x,t),\sum_{j=0}^{i-1}v_j(x,t)\right)\right\}. \quad (4.3.10)$$

Exchanging Eqs. (4.3.6)-(4.3.10) into Eq. (4.3.5), we acquire

$$\sum_{i=0}^{\infty}u_i(x,t) = \mathcal{R}(x,t)+\mathbb{KH}^{-1}\left(\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\sum_{i=0}^{\infty}\mathcal{L}\left(u_i(x,t)\right)+\mathcal{N}\left(u_0(x,t),v_0(x,t)\right)+\sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i}u_j(x,t),\sum_{j=0}^{i}v_j(x,t)\right)-\mathcal{N}\left(\sum_{j=0}^{i-1}u_j(x,t),\sum_{j=0}^{i-1}v_j(x,t)\right)\right\}\right]\right), \quad (4.3.11)$$

$$\sum_{i=0}^{\infty}v_i(x,t) = \mathcal{T}(\vartheta,\xi)+\mathbb{KH}^{-1}\left(\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\sum_{i=0}^{\infty}\mathcal{P}\left(u_i(x,t)\right)+\mathcal{M}\left(u_0(x,t),v_0(x,t)\right)+\sum_{i=1}^{\infty}\left\{\mathcal{M}\left(\sum_{j=0}^{i}u_j(x,t),\sum_{j=0}^{i}v_j(x,t)\right)-\mathcal{M}\left(\sum_{j=0}^{i-1}u_j(x,t),\sum_{j=0}^{i-1}v_j(x,t)\right)\right\}\right]\right). \quad (4.3.12)$$

Using Eqs. (4.3.11) and (4.3.12), we derive the following recurrence relation

$$\begin{cases} u_0(x,t)=\mathcal{R}(x,t) \\ v_0(x,t)=\mathcal{T}(x,t) \end{cases},$$

$$\begin{cases} u_1(x,t)=\mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\mathcal{L}\left(u_0(x,t)\right)+\mathcal{N}\left(u_0(x,t),v_0(x,t)\right)\right]\right] \\ v_1(x,t)=\mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\mathcal{L}\left(u_0(x,t)\right)+\mathcal{N}\left(u_0(x,t),v_0(x,t)\right)\right]\right] \end{cases} \quad (4.3.13)$$

$$\begin{cases} u_{m+1}(x,t)=\mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\mathcal{L}\left(u_m(x,t)\right)+\left\{\mathcal{N}\left(\sum_{j=0}^{m}u_j(x,t),\sum_{j=0}^{m}v_j(x,t)\right)-\mathcal{N}\left(\sum_{j=0}^{m-1}u_j(x,t),\sum_{j=0}^{m-1}v_j(x,t)\right)\right\}\right]\right] \\ v_{m+1}(x,t)=\mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\mathcal{P}\left(v_m(x,t)\right)+\left\{\mathcal{M}\left(\sum_{j=0}^{m}u_j(x,t),\sum_{j=0}^{m}v_j(x,t)\right)-\mathcal{M}\left(\sum_{j=0}^{m-1}u_j(x,t),\sum_{j=0}^{m-1}v_j(x,t)\right)\right\}\right]\right] \end{cases}.$$

Consequently, the solution of system (4.3.1) under the initial conditions (4.3.2) is obtained by the subsequent formula as

$$\begin{cases} u(x,t) = \lim\limits_{m\to\infty} \sum\limits_{i=0}^{m} u_i(x,t) = \sum\limits_{i=0}^{\infty} u_i(x,t) \\ v(x,t) = \lim\limits_{m\to\infty} \sum\limits_{i=0}^{m} v_i(x,t) = \sum\limits_{i=0}^{\infty} v_i(x,t) \end{cases}.$$

This completes the proof. ■

4.4 Convergence and absolute error analysis of KhDJA

In this paragraph, we illustrate the convergence and absolute error analysis of KhDJA.

The following theorem explains the convergence conditions of the solution using KhDJA.

Theorem 4.4.1 *Let* $\begin{cases} u(x,t) \\ v(x,t) \end{cases}$ *and* $\begin{cases} u_i(x,t) \\ v_i(x,t) \end{cases}$ *be in Banach space* $\mathcal{B}$. *Then the series solution* $\begin{cases} u(x,t) = \sum\limits_{i=0}^{\infty} u_i(x,t) \\ v(x,t) = \sum\limits_{i=0}^{\infty} v_i(x,t) \end{cases}$ *given by Eq. (4.3.3) is convergent if* $\begin{cases} \exists \varrho \in (0,1) \\ \exists \theta \in (0,1) \end{cases}$ *such as* $\begin{cases} \|u_{i+1}(x,t)\|_{\mathcal{B}} \leq \varrho \|u_i(x,t)\|_{\mathcal{B}} \\ \|v_{i+1}(x,t)\|_{\mathcal{B}} \leq \theta \|v_i(x,t)\|_{\mathcal{B}} \end{cases}$ *and* $\begin{cases} u_0(x,t) \\ v_0(x,t) \end{cases} \in \mathcal{B}$ *is bounded.*

Proof. Considering

$$\begin{cases} \mathcal{S}_0(x,t) = u_0(x,t) \\ \mathcal{S}_0'(x,t) = v_0(x,t) \end{cases},$$

$$\begin{cases} \mathcal{S}_1(x,t) = u_0(x,t) + u_1(x,t) \\ \mathcal{S}_1'(x,t) = v_0(x,t) + v_1(x,t) \end{cases},$$

$$\begin{cases} \mathcal{S}_2(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) \\ \mathcal{S}_2'(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) \end{cases},$$

$$\vdots$$

$$\begin{cases} \mathcal{S}_i(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_i(x,t) \\ \mathcal{S}_i'(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + ... + v_i(x,t) \end{cases}.$$

It is necessary to ensure that the series of partial sums $\{\mathcal{S}_i(x,t)\}_{i=0}^{\infty}$ and $\{\mathcal{S}'_i(x,t)\}_{i=0}^{\infty}$ are Cauchy series in the Banach space $\mathcal{B}$.

For this, we have

$$\begin{cases} \|\mathcal{S}_{i+1}(x,t)-\mathcal{S}_i(x,t)\|_{\mathcal{B}} \leq \|u_{i+1}(x,t)\| \leq \varrho \|u_i(x,t)\|_{\mathcal{B}} \\ \qquad \leq \varrho^2 \|u_{i-1}(x,t)\|_{\mathcal{B}} \leq ... \leq \varrho^{i+1} \|u_0(x,t)\|_{\mathcal{B}} \\ \left\|\mathcal{S}'_{i+1}(x,t)-\mathcal{S}'_i(x,t)\right\|_{\mathcal{B}} \leq \|v_{i+1}(x,t)\|_{\mathcal{B}} \leq \theta \|v_i(x,t)\|_{\mathcal{B}} \\ \qquad \leq \theta^2 \|v_{i-1}(x,t)\|_{\mathcal{B}} \leq ... \leq \theta^{i+1} \|v_0(x,t)\|_{\mathcal{B}} \end{cases}. \tag{4.4.1}$$

For each $i,j \in \mathbb{N}$ with $i \geq j$ and using (4.4.1) and the triangle inequality, we acquire

$$\begin{cases} \|\mathcal{S}_i(x,t)-\mathcal{S}_j(x,t)\|_{\mathcal{B}} = \|\mathcal{S}_i(x,t)-\mathcal{S}_{i-1}(x,t)+\mathcal{S}_{i-1}(x,t)-\mathcal{S}_{i-2}(x,t) \\ \qquad +...+\mathcal{S}_{j+1}(x,t)-\mathcal{S}_j(x,t)\|^{\mathcal{B}} \\ \qquad \leq \|\mathcal{S}_i(x,t)-\mathcal{S}_{i-1}(x,t)\|_{\mathcal{B}} + \|\mathcal{S}_{i-1}(x,t)-\mathcal{S}_{i-2}(x,t)\|_{\mathcal{B}} \\ \qquad +...+\|\mathcal{S}_{j+1}(x,t)-\mathcal{S}_j(x,t)\|_{\mathcal{B}} \\ \qquad \leq \varrho^i \|u_0(x,t)\|_{\mathcal{B}} + \varrho^{i-1}\|u_0(x,t)\|_{\mathcal{B}} + ... + \varrho^{j+1}\|u_0(x,t)\|_{\mathcal{B}} \\ \qquad = \varrho^{j+1}\left(1+\varrho+...+\varrho^{i-j-1}\right)\|u_0(x,t)\|_{\mathcal{B}} \\ \qquad \leq \varrho^{j+1}\left(\dfrac{1-\varrho^{i-j}}{1-\varrho}\right)\|u_0(x,t)\|_{\mathcal{B}} \\ \left\|\mathcal{S}'_i(x,t)-\mathcal{S}'_j(x,t)\right\|_{\mathcal{B}} = \left\|\mathcal{S}'_i(x,t)-\mathcal{S}'_{i-1}(x,t)+\mathcal{S}'_{i-1}(x,t)-\mathcal{S}'_{i-2}(x,t)\right. \\ \qquad \left.+...+\mathcal{S}'_{j+1}(x,t)-\mathcal{S}'_j(x,t)\right\|_{\mathcal{B}} \\ \qquad \leq \left\|\mathcal{S}'_i(x,t)-\mathcal{S}'_{i-1}(x,t)\right\|_{\mathcal{B}} + \left\|\mathcal{S}'_{i-1}(x,t)-\mathcal{S}'_{i-2}(x,t)\right\|_{\mathcal{B}} \\ \qquad +...+\left\|\mathcal{S}'_{j+1}(x,t)-\mathcal{S}'_j(x,t)\right\| \\ \qquad \leq \theta^i \|v_0(x,t)\|_{\mathcal{B}} + \theta^{i-1}\|v_0(x,t)\|_{\mathcal{B}} + ... + \theta^{j+1}\|v_0(x,t)\|_{\mathcal{B}} \\ \qquad = \theta^{j+1}\left(1+\theta+...+\theta^{i-j-1}\right)\|v_0(x,t)\|_{\mathcal{B}} \\ \qquad \leq \theta^{j+1}\left(\dfrac{1-\theta^{i-j}}{1-\theta}\right)\|v_0(x,t)\|_{\mathcal{B}} \end{cases}. \tag{4.4.2}$$

Since $\begin{cases} 0<\varrho<1 \\ 0<\theta<1 \end{cases}$, then $\begin{cases} 1-\varrho^{i-j}<1 \\ 1-\theta^{i-j}<1 \end{cases}$.

Therefore, from inequality (4.4.2), we have

$$\begin{cases} \|\mathcal{S}_i(x,t)-\mathcal{S}_j(x,t)\|_{\mathcal{B}} \leq \dfrac{\varrho^{j+1}}{1-\varrho}\|u_0(x,t)\|_{\mathcal{B}} \\ \left\|\mathcal{S}'_i(x,t)-\mathcal{S}'_j(x,t)\right\|_{\mathcal{B}} \leq \dfrac{\theta^{j+1}}{1-\theta}\|v_0(x,t)\|_{\mathcal{B}} \end{cases}.$$

Because $\begin{cases} \|u_0(x,t)\|_{\mathcal{B}} < \infty \\ \|v_0(x,t)\|_{\mathcal{B}} < \infty \end{cases}$, hence we have

$$\begin{cases} \lim\limits_{i,j\to\infty} \|\mathcal{S}_i(x,t) - \mathcal{S}_j(x,t)\|_{\mathcal{B}} = 0 \\ \lim\limits_{i,j\to\infty} \left\|\mathcal{S}'_i(x,t) - \mathcal{S}'_j(x,t)\right\|_{\mathcal{B}} = 0 \end{cases}.$$

In consequence, $\{\mathcal{S}_i(x,t)\}_{i=0}^{\infty}$ and $\{\mathcal{S}'_i(x,t)\}_{i=0}^{\infty}$ are Cauchy series in Banach space $\mathcal{B}$. So, the solution of system (4.3.1) is convergent.

This completes the proof. ■

Now, we propose an absolute error study of the KhDJA in the following theorem

Theorem 4.4.2 *Let* $\begin{cases} u(x,t) \\ v(x,t) \end{cases}$ *be the approximate solution of the truncated finite series* $\begin{cases} u(x,t) = \sum\limits_{i=0}^{N} u_i(x,t) \\ v(x,t) = \sum\limits_{i=0}^{N} v_i(x,t) \end{cases}$*, then the maximum absolute error is*

$$\begin{cases} \left\| u(x,t) - \sum\limits_{i=0}^{N} u_i(x,t) \right\|_{\mathcal{B}} \leq \dfrac{\varrho^{N+1}}{1-\varrho} \|u_0(x,t)\|_{\mathcal{B}} \\ \left\| v(x,t) - \sum\limits_{i=0}^{N} v_i(x,t) \right\|_{\mathcal{B}} \leq \dfrac{\theta^{N+1}}{1-\theta} \|v_0(x,t)\|_{\mathcal{B}} \end{cases}.$$

Proof. Let the series $\begin{cases} \sum\limits_{i=0}^{N} u_i(x,t) \\ \sum\limits_{i=0}^{N} v_i(x,t) \end{cases}$ be finite. Then

$$\begin{cases} \left\| u(x,t) - \sum_{i=0}^{N} u_i(x,t) \right\|_{\mathcal{B}} \leq \left\| \sum_{i=N+1}^{\infty} u_i(x,t) \right\|_{\mathcal{B}} \\ \leq \sum_{i=N+1}^{\infty} \|u_i(x,t)\|_{\mathcal{B}} \\ \leq \sum_{i=N+1}^{\infty} \varrho^i \|u_0(x,t)\|_{\mathcal{B}} \\ \leq \varrho^{N+1} (1 + \varrho + \varrho^2 + ...) \|u_0(x,t)\|_{\mathcal{B}} \\ \leq \dfrac{\varrho^{N+1}}{1-\varrho} \|u_0(x,t)\|_{\mathcal{B}} \\ \left\| v(x,t) - \sum_{i=0}^{N} v_i(x,t) \right\|_{\mathcal{B}} \leq \left\| \sum_{i=N+1}^{\infty} v_i(x,t) \right\|_{\mathcal{B}} \\ \leq \sum_{i=N+1}^{\infty} \|v_i(x,t)\|_{\mathcal{B}} \\ \leq \sum_{i=N+1}^{\infty} \theta^i \|v_0(x,t)\|_{\mathcal{B}} \\ \leq \theta^{N+1} \left(1 + \theta + \theta^2 + ...\right) \|v_0(x,t)\|_{\mathcal{B}} \\ \leq \dfrac{\theta^{N+1}}{1-\theta} \|v_0(x,t)\|_{\mathcal{B}} \end{cases}.$$

This completes the proof. ■

4.5 Numerical applications

In this paragraph, we provide two systems of nonlinear time-fractional Burgers equations to illustrate the performance and relevance of the KhDJA.

Example 4.5.1 *Let us examine the system of nonlinear time-fractional Burgers equations*

$$\begin{cases} {}^{C}D_t^{\alpha} u(x,t) = u_{xx}(x,t) + 2u(x,t)u_x(x,t) - (u(x,t)v(x,t))_x \\ {}^{C}D_t^{\beta} v(x,t) = v_{xx}(x,t) + 2v(x,t)v_x(x,t) - (u(x,t)v(x,t))_x \end{cases} \tag{4.5.1}$$

under the initial conditions

$$\begin{cases} u(x,0) = \sin(x) \\ v(x,0) = \sin(x) \end{cases}, \tag{4.5.2}$$

where ${}^{C}D_t^{\alpha} u(x,t)$ *and* ${}^{C}D_t^{\beta} v(x,t)$ *are the Caputo time-fractional derivatives of order* α *and* β *with* $0 < \alpha \leq 1$ *and* $0 < \beta \leq 1$*, respectively.*

Here we have

$$\mathcal{L}(u(x,t)) = u_{xx}(x,t),\ \mathcal{N}(u(x,t),v(x,t) = 2u(x,t)u_x(x,t) - (u(x,t)v(x,t))_x,\ \textit{and}\ \mathcal{F}(x,t) = 0,$$
$$\mathcal{P}(v(x,t)) = v_{xx}(x,t),\ \mathcal{M}(u(x,t),v(x,t) = 2v(x,t)v_x(x,t) - (u(x,t)v(x,t))_x,\ \textit{and}\ \mathcal{G}(x,t) = 0.$$

Now, operating the KhT on (4.5.1) and using Theorem 4.2.3 we obtain the consequent formula

$$\begin{cases} \mathbb{KH}\left[u(x,t)\right] = \sin(x) + \left(\dfrac{\gamma\eta}{s}\right)^{\alpha} \mathbb{KH}\left[u_{xx}(x,t) + 2u(x,t)u_x(x,t) - (u(x,t)v(x,t))_x\right] \\ \mathbb{KH}\left[v(x,t)\right] = \sin(x) + \left(\dfrac{\gamma\eta}{s}\right)^{\beta} \mathbb{KH}\left[v_{xx}(x,t) + 2v(x,t)v_x(x,t) - (u(x,t)v(x,t))_x\right] \end{cases}. \tag{4.5.3}$$

Next, employing the inverse KhT on (4.5.3), we acquire the subsequent formula

$$\begin{cases} u(x,t) = \sin(x) + \mathbb{KH}^{-1}\left[\left(\dfrac{\gamma\eta}{s}\right)^{\alpha} \mathbb{KH}\left[u_{xx}(x,t) + 2u(x,t)u_x(x,t) - (u(x,t)v(x,t))_x\right]\right] \\ v(x,t) = \sin(x) + \mathbb{KH}^{-1}\left[\left(\dfrac{\gamma\eta}{s}\right)^{\beta} \mathbb{KH}\left[v_{xx}(x,t) + 2v(x,t)v_x(x,t) - (u(x,t)v(x,t))_x\right]\right] \end{cases}.$$

According to KhDJA, we attain the iterative terms of $\begin{cases} u(x,t) \\ v(x,t) \end{cases}$ *as follows*

$$\begin{cases} u_0(x,t) = \sin(x) \\ v_0(x,t) = \sin(x) \end{cases},$$

$$\begin{cases} u_1(x,t) = -\sin(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ v_1(x,t) = -\sin(x)\frac{t^{\beta}}{\Gamma(\beta+1)} \end{cases},$$

$$\begin{cases} u_2(x,t) = \sin(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2\sin(x)\cos(x)\left(\begin{array}{c} -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ +\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)}\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \end{array}\right) \\ v_2(x,t) = \sin(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)} + 2\sin(x)\cos(x)\left(\begin{array}{c} -\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+1)}\frac{t^{3\beta}}{\Gamma(3\beta+1)} \\ +\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{t^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)} \end{array}\right) \end{cases},$$

$\vdots$

and so on.

Thus, the approximate solution of the system (4.5.1) under the initial conditions (4.5.2) is written as

$$\begin{cases} u(x,t) = \sin(x)\left(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + ...\right) + 2\sin(x)\cos(x\left(\begin{array}{c} -\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ +\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)}\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \\ +... \end{array}\right) \\ v(x,t) = \sin(x)\left(1 - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + ...\right) + 2\sin(x)\cos(x)\left(\begin{array}{c} -\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+1)}\frac{t^{3\beta}}{\Gamma(3\beta+1)} \\ +\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{t^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)} \\ +... \end{array}\right) \end{cases}. \tag{4.5.4}$$

If $\alpha = \beta$ in (4.5.4), then the approximate solution of the system (4.5.1) under the initial conditions (4.5.2) reduces to

$$\begin{cases} u(x,t) = \sin(x)\left(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + ...\right) = \sin(x)E_{\alpha}(-t^{\alpha}) \\ v(x,t) = \sin(x)\left(1 - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + ...\right) = \sin(x)E_{\beta}(-t^{\beta}) \end{cases}, \tag{4.5.6}$$

where $E_{\alpha}(-t^{\alpha})$ and $E_{\beta}(-t^{\beta})$ are the Mittag-Leffler functions defined by Eq. (1.1.6).

When $\alpha = \beta = 1$ in (4.5.6), we arrived at

$$\begin{cases} u(x,t) = e^{-t}\sin(x) \\ v(x,t) = e^{-t}\sin(x) \end{cases},$$

which is the exact solution of the system of nonlinear Burgers equations in the classical case.

Example 4.5.2 *Let us examine the system of nonlinear time-fractional Burgers equations*

$$\begin{cases} {}^{C}D_t^{\alpha}u(x,t) = -u_{xx}(x,t) - 2u(x,t)u_x(x,t) - (u(x,t)v(x,t))_x \\ {}^{C}D_t^{\beta}v(x,t) = -v_{xx}(x,t) - 2v(x,t)v_x(x,t) - (u(x,t)v(x,t))_x \end{cases} \tag{4.5.7}$$

under the initial conditions

$$\begin{cases} u(x,0) = e^{x} \\ v(x,0) = -e^{x} \end{cases}, \tag{4.5.8}$$

where ${}^{C}D_t^{\alpha}u(x,t)$ and ${}^{C}D_t^{\beta}v(x,t)$ are the Caputo time-fractional derivatives of order α and β with $0 < \alpha \leq 1$ and $0 < \beta \leq 1$, respectively.

Here we have

$$\mathcal{L}(u(x,t)) = -u_{xx}(x,t),\ \mathcal{N}(u(x,t),v(x,t) = -2u(x,t)u_x(x,t) - (u(x,t)v(x,t))_x,\ \text{and } \mathcal{F}(x,t) = 0,$$
$$\mathcal{P}(v(x,t)) = -v_{xx}(x,t),\ \mathcal{M}(u(x,t),v(x,t) = -2v(x,t)v_x(x,t) - (u(x,t)v(x,t))_x,\ \text{and } \mathcal{G}(x,t) = 0.$$

Now, operating the KhT on (4.5.7) and using Theorem 4.2.3 we obtain the consequent formula

$$\begin{cases} \mathbb{KH}\left[u(x,t)\right] = e^x - \left(\frac{\gamma\eta}{s}\right)^{\alpha} \mathbb{KH}\left[u_{xx}(x,t) + 2u(x,t)u_x(x,t) + (u(x,t)v(x,t))_x\right] \\ \mathbb{KH}\left[v(x,t)\right] = -e^x - \left(\frac{\gamma\eta}{s}\right)^{\beta} \mathbb{KH}\left[v_{xx}(x,t) + 2v(x,t)v_x(x,t) + (u(x,t)v(x,t))_x\right] \end{cases}. \tag{4.5.9}$$

Next, employing the inverse KhT on (4.5.9), we acquire the subsequent formula

$$\begin{cases} u(x,t) = e^x - \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha} \mathbb{KH}\left[u_{xx}(x,t) + 2u(x,t)u_x(x,t) + (u(x,t)v(x,t))_x\right]\right] \\ v(x,t) = -e^x - \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\beta} \mathbb{KH}\left[v_{xx}(x,t) + 2v(x,t)v_x(x,t) + (u(x,t)v(x,t))_x\right]\right] \end{cases}.$$

According to KhDJA, we attain the iterative terms of $\begin{cases} u(x,t) \\ v(x,t) \end{cases}$ *as follows*

$$\begin{cases} u_0(x,t) = e^x \\ v_0(x,t) = -e^x \end{cases},$$

$$\begin{cases} u_1(x,t) = -e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ v_1(x,t) = e^x \frac{t^{\beta}}{\Gamma(\beta+1)} \end{cases},$$

$$\begin{cases} u_2(x,t) = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 2e^{2x} \begin{pmatrix} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ -\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)} \frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)} \end{pmatrix} \\ v_2(x,t) = -e^x \frac{t^{2\beta}}{\Gamma(2\beta+1)} + 2e^{2x} \begin{pmatrix} \frac{t^{2\beta}}{\Gamma(2\beta+1)} - \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+1)} \frac{t^{3\beta}}{\Gamma(3\beta+1)} \\ -\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{t^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)} \end{pmatrix} \end{cases},$$

$$\vdots$$

and so on.

Thus, the approximate solution of the system (4.5.1) under the initial conditions (4.5.2) is written as

$$\begin{cases} u(x,t) = e^x\left(1-\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+...\right)+2e^{2x}\begin{pmatrix}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\ -\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)}\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\\ +...\end{pmatrix} \\ v(x,t) = -e^x\left(1-\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{2\beta}}{\Gamma(2\beta+1)}+...\right)+2e^{2x}\begin{pmatrix}\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+1)}\frac{t^{3\beta}}{\Gamma(3\beta+1)}\\ -\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{t^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)}\\ +...\end{pmatrix} \end{cases}. \tag{4.5.10}$$

If $\alpha=\beta$ in (4.5.10), then the approximate solution of the system (4.5.7) under the initial conditions (4.5.8) reduces to

$$\begin{cases} u(x,t) = e^x\left(1-\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+...\right)=e^xE_\alpha(-t^\alpha) \\ v(x,t) = -e^x\left(1-\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{2\beta}}{\Gamma(2\beta+1)}+...\right)=-e^xE_\beta(-t^\beta) \end{cases}, \tag{4.5.11}$$

where $E_\alpha(-t^\alpha)$ and $E_\beta(-t^\beta)$ are the Mittag-Leffler functions defined by Eq. (1.1.6).

When $\alpha=\beta=1$ in (4.5.6), we arrived at

$$\begin{cases} u(x,t) = e^{x-t} \\ v(x,t) = -e^{x-t} \end{cases},$$

which is the exact solution of the system of nonlinear Burgers equations in the classical case.

4.6 Numerical simulations and discussion

In this paragraph, we discuss the numerical results obtained using MATLAB programming to evaluate the performance of the KhDJA. Numerical simulations provide insights into the practical implications of the theoretical results.

Figures 1 and 2 describe the two-dimensional graphs of the approximate and exact solutions obtained by KhDJA for systems (4.5.1) and (4.5.7) for different values of the fractional parameters α and β. These figures show a strong agreement between the obtained solution and exact solution at $\alpha=\beta=1$. Tables 1, 2 and 3 present the numerical values of the KhDJA solutions of systems (4.5.1) and (4.5.7) for different values of α and β, compared to the exact solutions. The results in the table seem to coincide with the values of the exact solutions, which means that KhDJA is an accurate approach.

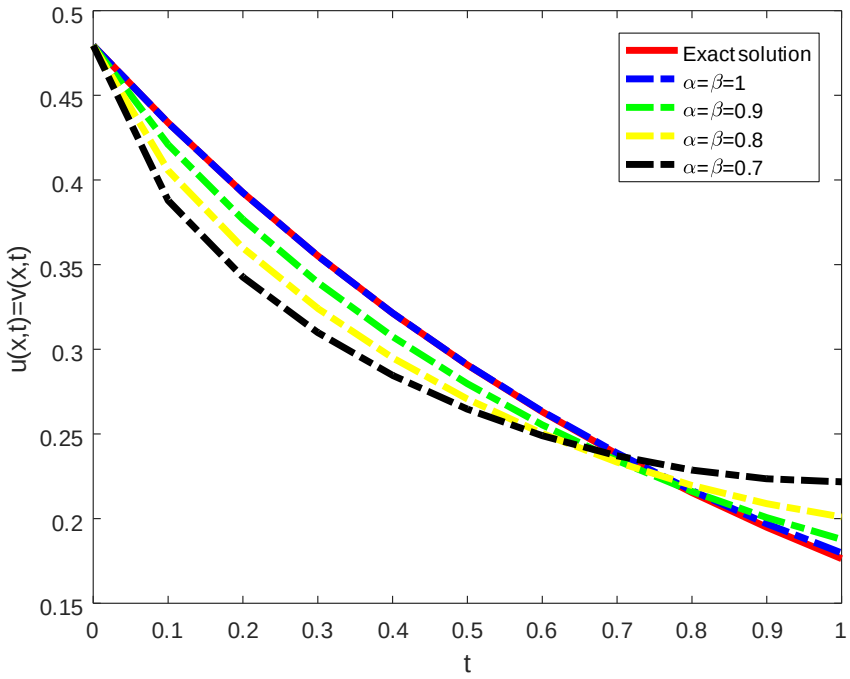

Figure 1: 2D graphs of the KhDJA solutions and exact solutions $u(x,t) = v(x,t)$ for system (4.5.1) at different values of $\alpha = \beta$ and for $x = 1$.

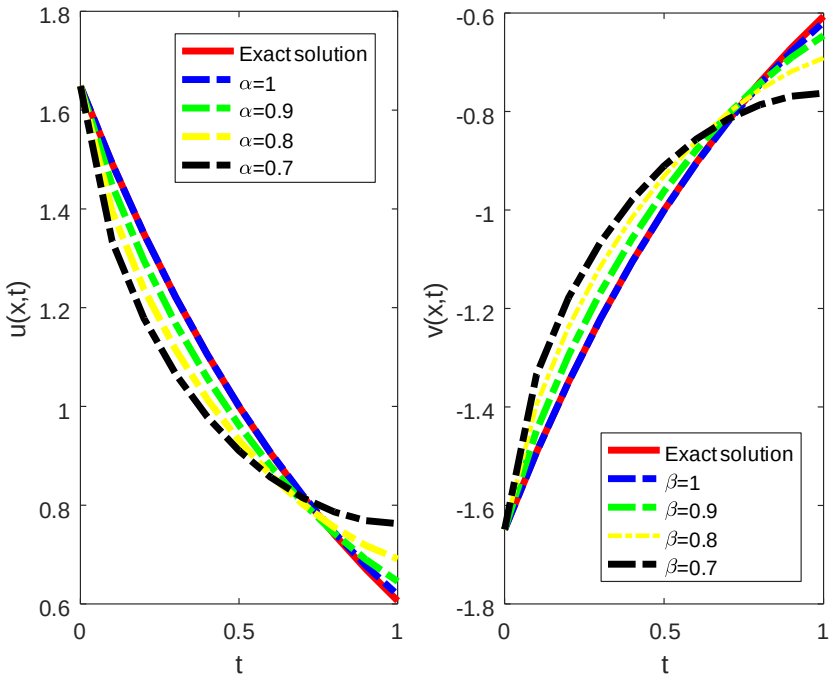

Figure 2: 2D graphs of the KhDJA solutions and exact solutions $u(x,t)$ and $v(x,t)$ for system (4.5.7) at different values of α and β and for $x = 1$.

t	$\alpha=\beta=0.7$	$\alpha=\beta=0.8$	$\alpha=\beta=0.9$	$\alpha=\beta=1$	*Exact solution*	$\lvert u_{Exact}-u_{KhDJA}\rvert$
0.01	0.45902	0.46670	0.47160	0.45604	0.45604	3.9886×10^{-13}
0.03	0.43682	0.44948	0.45870	0.46526	0.46526	9.6600×10^{-11}
0.05	0.42006	0.43523	0.44706	0.45604	0.45604	1.2382×10^{-9}
0.07	0.40598	0.42260	0.43620	0.44701	0.44701	6.6372×10^{-9}
0.09	0.39362	0.41109	0.42593	0.43816	0.43816	2.3242×10^{-8}

Table 1: Comparison of numerical values of the KhDJA solutions and exact solutions $u(x,t)=v(x,t)$ of system (4.5.1) for different values of $\alpha=\beta$.

t	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$	*Exact solution*	$\lvert u_{Exact}-u_{KhDJA}\rvert$
0.01	1.5785	1.6050	1.6218	1.6323	1.6323	1.3716×10^{-12}
0.03	1.5022	1.5457	1.5774	1.6000	1.6000	3.3220×10^{-10}
0.05	1.4446	1.4967	1.5374	1.5683	1.5683	4.2580×10^{-9}
0.07	1.3961	1.4533	1.5001	1.5373	1.5373	2.2825×10^{-8}
0.09	1.3537	1.4137	1.4648	1.5068	1.5068	7.9928×10^{-8}

Table 2: Comparison of numerical values of the KhDJA solutions and exact solutions $u(x,t)$ of system (4.5.7) for different values of α.

t	$\beta=0.7$	$\beta=0.8$	$\beta=0.9$	$\beta=1$	*Exact solution*	$\lvert v_{Exact}-v_{KhDJA}\rvert$
0.01	-1.5785	-1.6050	-1.6218	-1.6323	-1.6323	1.3716×10^{-12}
0.03	-1.5022	-1.5457	-1.5774	-1.6000	-1.6000	3.3220×10^{-10}
0.05	-1.4446	-1.4967	-1.5374	-1.5683	-1.5683	4.2580×10^{-9}
0.07	-1.3961	-1.4533	-1.5001	-1.5373	-1.5373	2.2825×10^{-8}
0.09	-1.3537	-1.4137	-1.4648	-1.5068	-1.5068	7.9928×10^{-8}

Table 3: Comparison of numerical values of the KhDJA solutions and exact solutions $v(x,t)$ of system (4.5.7) for different values of β.

Conclusions and future research directions

This thesis focuses on obtaining new solutions for systems of nonlinear time-fractional partial differential equations using the Caputo fractional derivative and a new mixed approach called the Khalouta-Daftardar-Jafari approach (KhDJA). By combining the Khalouta transform with the Daftardar-Jafari approach and carefully selecting the definition of the fractional derivative, we obtained accurate and efficient solutions to the system of nonlinear time-fractional Burgers equations, as demonstrated by the results presented in the tables and 2D graphs. In conclusion, the KhDJA, applied with the Caputo fractional derivative, exhibits excellent ability to resolve nonlinear fractional problems, making it a promising approach for addressing even more complex nonlinear fractional problems in various fields.

The findings contribute significantly to different fields of fractional calculus, offering new insights and practical tools that can be summarized in the following points:

- In the near future, we will attempt to use the Caputo fractional derivative to model real-world phenomena, including reaction, diffusion, and convection.

- Study the stability and relative error analysis of KhDJA.

- Extend the applicability of KhDJA to resolve systems of nonlinear time-space fractional partial differential equations.

- Extend the Khalouta integral transform to other fractional derivatives such as conformable, Caputo-Fabrizio, and Atangana-Baleanu-Caputo fractional derivative orders.

Bibliography

[1] **G. Adomian**, *A review of the decomposition method in applied mathematics.* Journal of mathematical analysis and applications, **135**(2) (1988), pp 501-544.

[2] **O P. Agrawal**, *Formulation of Euler-Lagrange equations for fractional variational problems*, Journal of Mathematical Analysis and Applications, **272** (2002), pp 368-379.

[3] **O.P. Agarwal, M. Jleli, and B. Samet,** *Fixed point theory in metric spaces*, Recent Advances and Applications, **10** (2018), pp. 978-981.

[4] **R. Almeida, S. Pooseh, and D.F.M. Torres**, *Fractional variational problems depending on indefinite integrals*, Nonlinear Analysis, **75** (2012), pp 1009-1025.

[5] **M. Arshad, S.F. Aldosary, S Batool, I. Hussain, and N. Hussain,** *Exploring fractional-order new coupled Korteweg-de Vries system via improved Adomian decomposition method.* PLoS ONE, **19**(5) (2024), e0303426. https://doi.org/10.1371/journal.pone.0303426

[6] **R.L. Bagley and P.J. Torvik**, *A theoretical basis for the application of fractional calculus in viscoelasticity*, Journal of Rheology, **27** (1983), pp 201-210.

[7] **R.L. Bagley and P.J. Torvik**, *On the fractional calculus model of viscoelasticity behavior*, Journal of Rheology, **30** (1986), pp 133-155.

[8] **H. Bateman**, *Some recent researches on the motion of fluids*, Monthly Weather Review, **43 (**1915), pp 163-170.

[9] **H. Brézis.** *Analyse fonctionnelle, Théorie et applications.* Dunod Paris, ISBN 2 10 004314 5, 1999.

[10] **J.M. Burgers,** *A mathematical model illustrating the theory of turbulence*, In Advances in Applied Mechanics; Elsevier: Amsterdam, The Netherlands, **1** (1948), pp 171-199.

[11] **M. Caputo**, *Linear models of dissipation whose Q is almost frequency independent*, Geophysical Journal of the Royal Astronomical Society, **13** (1967), pp 529-539.

[12] **M. Caputo and F.Mainardi**, *Linear models of dissipation in an elastic solids*, Rivista del Nuovo Cimento, **1** (1971), pp 161-198.

[13] **V. Daftardar-Gejji and H. Jafari**, *An Iterative Method for Solving Nonlinear Functional Equations*. Journal of Mathematical Analysis and Application, **316** (2006), pp 753-763.

[14] **N. Driai and A. Khalouta**, *A New Analytical Solutions for Systems of Nonlinear Time-Fractional Partial Differential Equations using the Khalouta-Daftardar-Jafari Method,* Annals of the University of Craiova, Mathematics and Computer Science Series, **51**(2) (2024), pp 526-539.

[15] **J.H. He,** *Variational iteration method a kind of non linear analytical technique : some exemples,* International Journal of Non-Linear Mechanics, **34**(4) (1999), pp 699-708.

[16] **T. Hélie and D. Matignon**, *Diffusive representations for the analysis and simulation of flared acoustic pipes with visco-thermal losses*, Mathematical Models and Methods in Applied Sciences, **16**(04) (2006), pp 503-536.

[17] **H.M. Jaradat, S. Al-Shar'a, Q.J.A. Khan, M. Alquran, and K. AlKhaled,** *Analytical solution of time-fractional drinfeld-sokolov-wilson system using residual power series method,* IAENG International Journal of Applied Mathematics, **46**(1) (2016), pp 64–70.

[18] **Y. Keskin and G. Oturanc,** *Reduced differential transform method for partial differffential equations*, International Journal of Nonlinear Sciences and Numerical Simulation, **10**(6) (2009), pp 741-749.

[19] **A. Khalouta**, *A new exponential type kernel integral transform: Khalouta transform and its applications*, Mathematica Montisnigri, **57** (2023), pp 5-23.

[20] **A. Khalouta**, A. Khalouta transform via different fractional derivative operators, Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki. **28**(3) (2024), pp 407-425.

[21] **A. Khalouta**, *New approaches for solving Caputo time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas,* International Journal of Analysis and Applications, **14**(3) (2023), pp 33-46.

[22] **A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo**, *Theory and applications of fractional differential equations*, North-HollandMathematical studies 204, Ed VanMill , Amsterdam, 2006.

[23] **S.J. Liao**, *The proposed homotopy analysis technique for the solution of nonlinear problems*, PhD thesis, Shanghai Jiao Tong University, Shanghai, 1992.

[24] **T. Pfitzenreiter**, *A physical basis for fractional derivatives in constitutive equations*, Zeitschrift für Angewandte Mathematik und Mechanik, **84**(4) (2004), pp 284-287.

[25] **I. Podlubny**, *Fractional differential equations*, Academic Press, San Diego, 1999.

[26] **Podlubny**, *Fractional-order system and fractional-order controllers*, Academy of Sciences, Slovakia, 1994.

[27] **R. Hilfer**, *Applications of Fractional Calculus in Physics*,World Scientific, 2000.

[28] **F. Riewe**, *Mechanics with fractional derivatives*, Physical Review E, **55** (1997), 3581-3592.

[29] **F. Riewe**, *Nonconservative Lagrangian and Hamiltonian mechanics*, Physical Review E, **53** (1996), pp 1890-1899.

[30] **A. Oustaloup**, *La dérivation non entière: théorie, synthèse et applications*, Hermès, Paris, 1995.

[31] **B.K. Singh and P. Kumar,** *FRDTM for numerical simulation of multi-dimensional, time-fractional model of Navier–Stokes equation,* Ain Shams Engineering Journal, **9** (2018), pp 827-834.

[32] **J. Sabatier, O.P. Agrawal, and J.A. Tenreiro Machado**, *Advances in fractional calculus*, Springer, 2007.

ملخص:

الهدف الرئيسي من هذه الرسالة هو اقتراح نهج فعال، يُسمى نهج خلوطة-دفتردار-جعفري، وهو مزيج من تحويل خلوطة، بمعنى مشتقة كابوتو الكسرية، والنهج التكراري الذي استخدمه دفتردار-جيجي وجعفري، لإيجاد حلول تقريبية ومغلقة لأنظمة المعادلات التفاضلية الجزئية غير الخطية ذات الكسري الزمني، وخاصةً نظام معادلات برغر غير الخطية ذات الكسري الزمني .تكمن ميزة هذا النهج الجديد، الذي خططنا له، في بساطته الحسابية ودقته الفائقة، مما يسمح لنا بتقريب النتيجة النهائية.

كلمات مفتاحية: نظام معادلات برغر الكسرية، المشتقة الكسرية لكابوتو، تحويل خلوطة

Résumé:

L'objectif principal de cette thèse est de proposer une approche efficace, appelée l'approche de Khalouta-Daftardar-Jafari, qui combine la transformée de Khalouta au sens de la dérivée fractionnaire de Caputo et l'approche itérative utilisée par Daftardar-Gejji et Jafari pour trouver des solutions approchées et sous forme fermée aux systèmes d'équations aux dérivées partielles non-linéaires fractionnaires en temps, en particulier le système d'équations de Burgers non-linéaires fractionnaires en temps. L'avantage de cette nouvelle approche, que nous avons envisagée, réside dans sa simplicité de calcul et sa précision supérieure, ce qui nous permet d'approximer le résultat final.

Mots clés: Système d'équations fractionnaires de Burgers, Dérivée fractionnaire de Caputo, Transformation de Khalouta.

Abstract:

The main goal of this thesis is to propose an effectual approach, called Khalouta-Daftardar-Jafari approach, which is the collaboration of the Khalouta transform in the sense of the Caputo fractional derivative and the iterative approach used by Daftardar-Gejji and Jafari for finding approximate and closed-form solutions to systems of nonlinear time-fractional partial differential equations, especially the system of nonlinear time-fractional Burgers equations. The advantage of this new approach, which we have planned, lies in its computational simplicity and superior accuracy, which allows us to approximate the final result.

Key words: System of fractional Burgers equations, Caputo fractional derivative, Khalouta transform.