

University of Ferhat Abbas -- Sétif 1
Faculty of Economics, Commerce and Management Sciences

Department of Basic Education | First Year LMD

**Mathematics 2 – موجهة لطلبة السنة الأولى جذع مشترك للعلوم الاقتصادية والتجارية
وعلوم التسيير**

Official and Proposed Exams with Detailed Solutions

Academic Years 2017-2025

Author: Djabi Abdelmoumene

Department of Basic Education

Faculty of Economics, Commerce and Management Sciences

University of Ferhat Abbas -- Sétif 1

Contents

Preface	v
1 Main Examinations – First Session (2017–2025)	1
1.1 Main Exam – (2017/2018)–Version A	1
Solution – Main Exam (2017/2018–Version A)	2
1.2 Main Exam (2017/2018)–Version B	8
Solution – Main Exam (2017/2018–Version B)	9
1.3 Main Exam (2018/2019)	16
Solution –Main Exam (2018/2019)	17
1.4 Main Exam (2019/2020) - Version A	22
Solution – Main Exam (2019/2020) – Version A	24
1.5 Main Exam (2019/2020) - Version B	28
Solution – Main Exam – Version B	30
1.6 Main Exam (2020/2021) - Version A	35
Solution – Main Exam (2020/2021) Version A	36
1.7 Main Exam (2020/2021) - Version B	41
Solution – Main Exam (2020/2021) Version B	41
1.8 Main Exam (2021/2022)	46
Solution – Main Exam (2021/2022)	47
1.9 Main Exam (2022/2023)	53
Solution – Main Exam (2022/2023)	54
1.10 Main Exam (2023/2024)	60
Solution – Main Exam (2023/2024)	61
1.11 Main Exam (2024/2025)	68
Solution – Main Exam (2024/2025)	69
2 Resit Examinations – Second Session	79
2.1 Resit Exam (2017/2018)	79
Solution – Resit Exam (2017/2018)	80
2.2 Resit Exam (2018/2019)	88
Solution – Resit Exam (2018/2019)	89
2.3 Resit Exam (2019/2020)	94
Solution – Resit Exam (2019/2020)	95
2.4 Resit Exam (2020/2021)	99
Solution – Resit Exam (2020/2021)	100
2.5 Resit Exam (2021/2022)	105
Solution – Resit Exam (2021/2022)	106
2.6 Resit Exam (2022/2023)	113

Solution – Resit Exam (2022/2023)	114
2.7 Resit Exam (2023/2024)	116
Solution – Resit Exam (2023/2024)	117
2.8 Resit Exam (2024/2025)	125
Solution – Resit Exam (2024/2025)	126
3 Proposed Exams – Additional Practice	133
Note to Students	133
3.1 MCQ Exam –(Version 1)	134
Solutions – MCQ Exam (Version 1)	135
Solutions – Part 2	136
3.2 MCQ Exam –(Version 2)	137
Solutions – MCQ Exam (Version 2)	138
Solutions – Part 2	139
3.3 Practice Exam – 1	140
Solutions – Practice Exam 1	141
3.4 Practice Exam – 2	143
Solutions – Practice Exam 2	144
3.5 Practice Exam – 3	146
Solutions – Practice Exam 3	147
3.6 Practice Exam – 4	150
Solutions – Practice Exam 4	151
3.7 Practice Exam – 5	153
Solutions – Practice Exam 5	154
4 Key Summary – Formulas & Problem-Solving Methods	157
Note to Students	157
4.1 Matrix Operations	157
Matrix Operations	157
4.2 Determinants	158
Determinants	158
4.3 Matrix Inverse	159
Matrix Inverse	159
4.4 Linear Systems	160
Linear Systems	160
4.5 Eigenvalues & Eigenvectors	161
Eigenvalues & Eigenvectors	161
4.6 Diagonalization	162
Diagonalization	162
4.7 Quick Reference Table	163
Quick Reference Table	163

Structure of This Compendium

Chapter 1	Main Examinations – First Session (2017–2025)
Chapter 2	Resit & Make-up Examinations – Second Session
Chapter 3	Proposed Exams – Additional Practice
Chapter 4	Key Summary – Formulas & Problem-Solving Methods

Preface

This compendium collects **all** Mathematics 2 exams (regular sessions, make-up/resit sessions, proposed exams, and their full corrections) administered at the Faculty of Economics, Commerce and Management Sciences, University of Ferhat Abbas – Sétif 1, from 2017 to 2025.

Structure:

- **Chapter 1 – Main Examinations:** Regular semester exams from 2017 to 2025.
- **Chapter 2 – Resit & Make-up Exams:** Second session and catch-up examinations.
- **Chapter 3 – Proposed Exams:** Additional practice problems, MCQ tests, and supplementary exercises.
- **Chapter 4 – Key Summary:** Essential formulas, theorems, and step-by-step methods to solve all exercise types.

Best of luck to all students!

Djabi Abdelmoumene

Chapter 1

Main Examinations – First Session (2017–2025)

1.1 Main Exam – (2017/2018)–Version A

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Main Exam – Mathematics 2

Level:	First Year LMD	Module:	Mathematics 2
Date:	17/05/2018	Duration:	1h 30min

Exercise 1

(06 points)

Let A be the matrix:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

1. Prove that $A^2 = 3A - 2I_3$
2. Deduce that A is invertible, then calculate A^{-1}
3. Is the vectors $v_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, v_2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, v_3 \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ linearly independent ? Justify

Exercise 2

(06 points)

Let the linear system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = 0 \\ x_1 + x_3 = 2 \end{cases}$$

1. Write the system in the matrix form
2. Solve the system using Cramer's method

Exercise 3

(08 points)

Let B be the matrix:

$$B = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

1. Give the characteristic polynomial of B
2. Find the eigenvalues and eigenvectors
3. Prove that B is diagonalizable and write it in the form $B = SDS^{-1}$.

Full Solution

Solution – Exercise 1

1. Prove that $A^2 = 3A - 2I_3$.

First compute A^2 :

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$(A^2)_{11} = 0 \cdot 0 + 1 \cdot (-1) + (-1) \cdot 1 = 0 - 1 - 1 = -2$$

$$(A^2)_{12} = 0 \cdot 1 + 1 \cdot 2 + (-1) \cdot (-1) = 0 + 2 + 1 = 3$$

$$(A^2)_{13} = 0 \cdot (-1) + 1 \cdot (-1) + (-1) \cdot 2 = 0 - 1 - 2 = -3$$

$$(A^2)_{21} = (-1) \cdot 0 + 2 \cdot (-1) + (-1) \cdot 1 = 0 - 2 - 1 = -3$$

$$(A^2)_{22} = (-1) \cdot 1 + 2 \cdot 2 + (-1) \cdot (-1) = -1 + 4 + 1 = 4$$

$$(A^2)_{23} = (-1) \cdot (-1) + 2 \cdot (-1) + (-1) \cdot 2 = 1 - 2 - 2 = -3$$

$$(A^2)_{31} = 1 \cdot 0 + (-1) \cdot (-1) + 2 \cdot 1 = 0 + 1 + 2 = 3$$

$$(A^2)_{32} = 1 \cdot 1 + (-1) \cdot 2 + 2 \cdot (-1) = 1 - 2 - 2 = -3$$

$$(A^2)_{33} = 1 \cdot (-1) + (-1) \cdot (-1) + 2 \cdot 2 = -1 + 1 + 4 = 4$$

Thus:

$$A^2 = \begin{pmatrix} -2 & 3 & -3 \\ -3 & 4 & -3 \\ 3 & -3 & 4 \end{pmatrix}$$

Now compute $3A - 2I_3$:

$$3A = \begin{pmatrix} 0 & 3 & -3 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{pmatrix}, \quad 2I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$3A - 2I_3 = \begin{pmatrix} 0-2 & 3-0 & -3-0 \\ -3-0 & 6-2 & -3-0 \\ 3-0 & -3-0 & 6-2 \end{pmatrix} = \begin{pmatrix} -2 & 3 & -3 \\ -3 & 4 & -3 \\ 3 & -3 & 4 \end{pmatrix}$$

Therefore $A^2 = 3A - 2I_3$.

2. Deduce that A is invertible, then calculate A^{-1} .

From $A^2 = 3A - 2I_3$, we rewrite:

$$A^2 - 3A = -2I_3$$

$$A(A - 3I_3) = -2I_3$$

$$A \cdot \left(-\frac{1}{2}(A - 3I_3) \right) = I_3$$

Thus A is invertible and:

$$A^{-1} = -\frac{1}{2}(A - 3I_3)$$

Now compute $A - 3I_3$:

$$A - 3I_3 = \begin{pmatrix} 0-3 & 1 & -1 \\ -1 & 2-3 & -1 \\ 1 & -1 & 2-3 \end{pmatrix} = \begin{pmatrix} -3 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

Therefore:

$$A^{-1} = -\frac{1}{2} \begin{pmatrix} -3 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

3. Are the vectors v_1, v_2, v_3 linearly independent? Justify.

Form matrix M with v_1, v_2, v_3 as columns:

$$M = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Notice $M = A^T$? Check: A given was

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Comparing, $M = A$.

Since we proved A is invertible, $\det A \neq 0$.

Therefore the columns of A are linearly independent, so v_1, v_2, v_3 are linearly independent.

Solution – Exercise 2

Let the linear system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = 0 \\ x_1 + x_3 = 2 \end{cases}$$

1. The system can be written as $AX = B$ where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

2. Solve using Cramer's method.

Step 1: Compute $\det A$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= 1 \cdot (1 \cdot 1 - 2 \cdot 0) - 1 \cdot (0 \cdot 1 - 2 \cdot 1) + 1 \cdot (0 \cdot 0 - 1 \cdot 1) \\ &= 1 \cdot (1 - 0) - 1 \cdot (0 - 2) + 1 \cdot (0 - 1) \\ &= 1 - 1 \cdot (-2) + (-1) \\ &= 1 + 2 - 1 = 2. \end{aligned}$$

Step 2: Compute $\det A_1$ (replace column 1 with B)

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_1 &= 1 \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \\ &= 1 \cdot (1 - 0) - 1 \cdot (0 \cdot 1 - 2 \cdot 2) + 1 \cdot (0 \cdot 0 - 1 \cdot 2) \\ &= 1 - 1 \cdot (0 - 4) + 1 \cdot (0 - 2) \\ &= 1 - (-4) - 2 = 1 + 4 - 2 = 3. \end{aligned}$$

Step 3: Compute $\det A_2$ (replace column 2 with B)

$$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Expand by row 2 (contains zeros):

$$\begin{aligned}\det A_2 &= 0 \cdot (\text{minor}_{21}) + 0 \cdot (\text{minor}_{22}) + 2 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 2 \cdot (-1) \cdot (1 \cdot 2 - 1 \cdot 1) \\ &= 2 \cdot (-1) \cdot (2 - 1) \\ &= 2 \cdot (-1) \cdot 1 = -2.\end{aligned}$$

Step 4: Compute $\det A_3$ (replace column 3 with B)

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Expand by row 2:

$$\begin{aligned}\det A_3 &= 0 \cdot (\text{minor}_{21}) + 1 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 0 \cdot (\text{minor}_{23}) \\ &= 1 \cdot 1 \cdot (1 \cdot 2 - 1 \cdot 1) = 2 - 1 = 1.\end{aligned}$$

Step 5: Apply Cramer's rule

$$x_1 = \frac{\det A_1}{\det A} = \frac{3}{2}, \quad x_2 = \frac{\det A_2}{\det A} = \frac{-2}{2} = -1, \quad x_3 = \frac{\det A_3}{\det A} = \frac{1}{2}.$$

Step 6: Verification

$$\begin{aligned}x_1 + x_2 + x_3 &= \frac{3}{2} + (-1) + \frac{1}{2} = 1 \quad \checkmark \\ x_2 + 2x_3 &= -1 + 2 \cdot \frac{1}{2} = -1 + 1 = 0 \quad \checkmark \\ x_1 + x_3 &= \frac{3}{2} + \frac{1}{2} = 2 \quad \checkmark\end{aligned}$$

Thus the solution is:

$$\boxed{x_1 = \frac{3}{2}, \quad x_2 = -1, \quad x_3 = \frac{1}{2}}.$$

Solution – Exercise 3

Let B be the matrix:

$$B = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

1. Characteristic polynomial of B .

$$P_B(\lambda) = \det(B - \lambda I_2) = \begin{vmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{vmatrix}$$

$$\begin{aligned}
&= (1 - \lambda)(3 - \lambda) - (4)(2) \\
&= 3 - \lambda - 3\lambda + \lambda^2 - 8 \\
&= \lambda^2 - 4\lambda - 5.
\end{aligned}$$

Thus the characteristic polynomial is:

$$\boxed{P_B(\lambda) = \lambda^2 - 4\lambda - 5}.$$

2. Find the eigenvalues and eigenvectors.

Solve $P_B(\lambda) = 0$:

$$\begin{aligned}
\lambda^2 - 4\lambda - 5 &= 0 \\
(\lambda - 5)(\lambda + 1) &= 0 \\
\lambda_1 = 5, \quad \lambda_2 &= -1.
\end{aligned}$$

Eigenvector for $\lambda_1 = 5$:

$$\begin{aligned}
(B - 5I_2)v = 0 &\Rightarrow \begin{pmatrix} 1-5 & 4 \\ 2 & 3-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0
\end{aligned}$$

The first equation: $-4x + 4y = 0 \Rightarrow x = y$. Choose $x = 1, y = 1$.

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = -1$:

$$\begin{aligned}
(B + 1 \cdot I_2)v = 0 &\Rightarrow \begin{pmatrix} 1+1 & 4 \\ 2 & 3+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0
\end{aligned}$$

First equation: $2x + 4y = 0 \Rightarrow x + 2y = 0 \Rightarrow x = -2y$. Choose $y = 1, x = -2$.

$$v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Thus eigenvalues: $\lambda_1 = 5, \lambda_2 = -1$; eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

3. Prove that B is diagonalizable and write $B = SDS^{-1}$.

B is 2×2 and has two distinct eigenvalues (5 and -1). Therefore B is diagonalizable.

Let S be the matrix with eigenvectors as columns:

$$S = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

Let D be the diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $B = SDS^{-1}$.

Now compute S^{-1} :

$$\det S = (1)(1) - (-2)(1) = 1 + 2 = 3.$$

$$S^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

Verification (optional):

$$SDS^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$$

$$\text{First compute } DS^{-1} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5 & 10 \\ 1 & -1 \end{pmatrix}.$$

$$\text{Then } S(DS^{-1}) = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 10 \\ 1 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5-2 & 10+2 \\ 5+1 & 10-1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 12 \\ 6 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = B.$$

Thus:

$$B = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{3}.$$

1.2 Main Exam (2017/2018)–Version B

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Main Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 17/05/2018

Duration: 1h 30min

Exercise 1

(08 points)

Let A be the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$

1. Give the characteristic polynomial of A .
2. Find the eigenvalues and eigenvectors.
3. Prove that A is diagonalizable and write it in the form $A = SDS^{-1}$.
4. Calculate A^5 .

Exercise 2

(06 points)

Let A be the matrix:

$$A = \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

1. Calculate A^2 then deduce $A^2 - 2A - 5I_3$.
2. Check that $A(A^2 - 2A - 5I_3) = 24I_3$.
3. Deduce A^{-1} .

Exercise 3

(06 points)

Let the linear system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ x_1 - 3x_2 - 7x_3 = -26 \\ -6x_1 + 4x_2 - 2x_3 = -4 \end{cases}$$

1. Write the system in matrix form, then prove that it is a Cramer's system.
2. Solve the system using Cramer's method.

Full Solution

Solution – Exercise 1

Let A be the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$

1. Characteristic polynomial of A .

$$\begin{aligned} P_A(\lambda) &= \det(A - \lambda I_2) = \begin{vmatrix} 2 - \lambda & 1 \\ 3 & 0 - \lambda \end{vmatrix} \\ &= (2 - \lambda)(-\lambda) - (1)(3) \\ &= -2\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 2\lambda - 3. \end{aligned}$$

Thus:

$$\boxed{P_A(\lambda) = \lambda^2 - 2\lambda - 3}.$$

2. Find the eigenvalues and eigenvectors.

Solve $P_A(\lambda) = 0$:

$$\begin{aligned} \lambda^2 - 2\lambda - 3 &= 0 \\ (\lambda - 3)(\lambda + 1) &= 0 \\ \lambda_1 &= 3, \quad \lambda_2 = -1. \end{aligned}$$

Eigenvector for $\lambda_1 = 3$:

$$\begin{aligned} (A - 3I_2)v &= 0 \quad \Rightarrow \quad \begin{pmatrix} 2 - 3 & 1 \\ 3 & 0 - 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\ \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 0 \end{aligned}$$

First equation: $-x + y = 0 \Rightarrow y = x$. Choose $x = 1$, $y = 1$.

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = -1$:

$$\begin{aligned} (A + 1 \cdot I_2)v &= 0 \quad \Rightarrow \quad \begin{pmatrix} 2 + 1 & 1 \\ 3 & 0 + 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\ \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= 0 \end{aligned}$$

First equation: $3x + y = 0 \Rightarrow y = -3x$. Choose $x = 1$, $y = -3$.

$$v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Thus:

$$\lambda_1 = 3, v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = -1, v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

3. Prove A is diagonalizable and write $A = SDS^{-1}$.

A is 2×2 with two distinct eigenvalues (3 and -1), so it is diagonalizable.

Let S be the matrix with eigenvectors as columns:

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$$

Let D be the diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $A = SDS^{-1}$.

Compute S^{-1} :

$$\det S = (1)(-3) - (1)(1) = -3 - 1 = -4.$$

$$S^{-1} = \frac{1}{-4} \begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}.$$

Verification (optional):

$$SDS^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

$$\text{First } DS^{-1} = \begin{pmatrix} 3 \cdot \frac{3}{4} & 3 \cdot \frac{1}{4} \\ -1 \cdot \frac{1}{4} & -1 \cdot (-\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} \frac{9}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$\text{Then } S(DS^{-1}) = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \frac{9}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{9}{4} - \frac{1}{4} & \frac{3}{4} + \frac{1}{4} \\ \frac{9}{4} + \frac{3}{4} & \frac{3}{4} - \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{8}{4} & \frac{4}{4} \\ \frac{12}{4} & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = A.$$

Thus:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}.$$

4. Calculate A^5 .

Since $A = SDS^{-1}$, we have $A^5 = SD^5S^{-1}$.

$$D^5 = \begin{pmatrix} 3^5 & 0 \\ 0 & (-1)^5 \end{pmatrix} = \begin{pmatrix} 243 & 0 \\ 0 & -1 \end{pmatrix}.$$

First compute D^5S^{-1} :

$$D^5S^{-1} = \begin{pmatrix} 243 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} 243 \cdot \frac{3}{4} & 243 \cdot \frac{1}{4} \\ -1 \cdot \frac{1}{4} & -1 \cdot (-\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} \frac{729}{4} & \frac{243}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Then $A^5 = S(D^5S^{-1})$:

$$\begin{aligned} A^5 &= \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} \frac{729}{4} & \frac{243}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{729}{4} - \frac{1}{4} & \frac{243}{4} + \frac{1}{4} \\ \frac{729}{4} + \frac{3}{4} & \frac{243}{4} - \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{728}{4} & \frac{244}{4} \\ \frac{732}{4} & \frac{240}{4} \end{pmatrix} = \begin{pmatrix} 182 & 61 \\ 183 & 60 \end{pmatrix}. \end{aligned}$$

Thus:

$$A^5 = \begin{pmatrix} 182 & 61 \\ 183 & 60 \end{pmatrix}.$$

Solution – Exercise 2

Let A be the matrix:

$$A = \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

1. Calculate A^2 .

$$A^2 = A \cdot A = \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

Compute each entry:

First row:

$$\begin{aligned} (A^2)_{11} &= 2 \cdot 2 + 5 \cdot 2 + (-3) \cdot 2 = 4 + 10 - 6 = 8 \\ (A^2)_{12} &= 2 \cdot 5 + 5 \cdot 1 + (-3) \cdot 0 = 10 + 5 + 0 = 15 \\ (A^2)_{13} &= 2 \cdot (-3) + 5 \cdot 1 + (-3) \cdot (-1) = -6 + 5 + 3 = 2 \end{aligned}$$

Second row:

$$\begin{aligned} (A^2)_{21} &= 2 \cdot 2 + 1 \cdot 2 + 1 \cdot 2 = 4 + 2 + 2 = 8 \\ (A^2)_{22} &= 2 \cdot 5 + 1 \cdot 1 + 1 \cdot 0 = 10 + 1 + 0 = 11 \\ (A^2)_{23} &= 2 \cdot (-3) + 1 \cdot 1 + 1 \cdot (-1) = -6 + 1 - 1 = -6 \end{aligned}$$

Third row:

$$\begin{aligned} (A^2)_{31} &= 2 \cdot 2 + 0 \cdot 2 + (-1) \cdot 2 = 4 + 0 - 2 = 2 \\ (A^2)_{32} &= 2 \cdot 5 + 0 \cdot 1 + (-1) \cdot 0 = 10 + 0 + 0 = 10 \\ (A^2)_{33} &= 2 \cdot (-3) + 0 \cdot 1 + (-1) \cdot (-1) = -6 + 0 + 1 = -5 \end{aligned}$$

Thus:

$$A^2 = \begin{pmatrix} 8 & 15 & 2 \\ 8 & 11 & -6 \\ 2 & 10 & -5 \end{pmatrix}.$$

Now compute $A^2 - 2A - 5I_3$:

$$2A = \begin{pmatrix} 4 & 10 & -6 \\ 4 & 2 & 2 \\ 4 & 0 & -2 \end{pmatrix}, \quad 5I_3 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A^2 - 2A = \begin{pmatrix} 8-4 & 15-10 & 2-(-6) \\ 8-4 & 11-2 & -6-2 \\ 2-4 & 10-0 & -5-(-2) \end{pmatrix} = \begin{pmatrix} 4 & 5 & 8 \\ 4 & 9 & -8 \\ -2 & 10 & -3 \end{pmatrix}$$

Then subtract $5I_3$:

$$A^2 - 2A - 5I_3 = \begin{pmatrix} 4-5 & 5-0 & 8-0 \\ 4-0 & 9-5 & -8-0 \\ -2-0 & 10-0 & -3-5 \end{pmatrix} = \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}.$$

Thus:

$$A^2 - 2A - 5I_3 = \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}.$$

2. Check that $A(A^2 - 2A - 5I_3) = 24I_3$.

Let $M = A^2 - 2A - 5I_3$. Compute $A \cdot M$:

$$A \cdot M = \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}$$

First row:

$$(AM)_{11} = 2(-1) + 5(4) + (-3)(-2) = -2 + 20 + 6 = 24$$

$$(AM)_{12} = 2(5) + 5(4) + (-3)(10) = 10 + 20 - 30 = 0$$

$$(AM)_{13} = 2(8) + 5(-8) + (-3)(-8) = 16 - 40 + 24 = 0$$

Second row:

$$(AM)_{21} = 2(-1) + 1(4) + 1(-2) = -2 + 4 - 2 = 0$$

$$(AM)_{22} = 2(5) + 1(4) + 1(10) = 10 + 4 + 10 = 24$$

$$(AM)_{23} = 2(8) + 1(-8) + 1(-8) = 16 - 8 - 8 = 0$$

Third row:

$$(AM)_{31} = 2(-1) + 0(4) + (-1)(-2) = -2 + 0 + 2 = 0$$

$$(AM)_{32} = 2(5) + 0(4) + (-1)(10) = 10 + 0 - 10 = 0$$

$$(AM)_{33} = 2(8) + 0(-8) + (-1)(-8) = 16 + 0 + 8 = 24$$

Thus:

$$A \cdot M = \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix} = 24I_3.$$

Therefore $A(A^2 - 2A - 5I_3) = 24I_3$.

3. Deduce A^{-1} .

From $A \cdot M = 24I_3$, we have:

$$A \cdot \left(\frac{1}{24} M \right) = I_3.$$

Thus:

$$A^{-1} = \frac{1}{24} M = \frac{1}{24} (A^2 - 2A - 5I_3).$$

$$A^{-1} = \frac{1}{24} \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{24} & \frac{5}{24} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{12} & \frac{5}{12} & -\frac{1}{3} \end{pmatrix}.$$

$$A^{-1} = \frac{1}{24} \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}.$$

Solution – Exercise 3

Let the linear system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ x_1 - 3x_2 - 7x_3 = -26 \\ -6x_1 + 4x_2 - 2x_3 = -4 \end{cases}$$

1. Matrix form and Cramer system verification.

The system can be written as $AX = B$ with

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -3 & -7 \\ -6 & 4 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 14 \\ -26 \\ -4 \end{pmatrix}.$$

Prove it is a Cramer system: A system is Cramer if n equations, n unknowns and $\det A \neq 0$. Here $n = 3$.

Compute $\det A$:

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -3 & -7 \\ -6 & 4 & -2 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= 1 \cdot \begin{vmatrix} -3 & -7 \\ 4 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & -7 \\ -6 & -2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & -3 \\ -6 & 4 \end{vmatrix} \\ &= 1 \cdot ((-3)(-2) - (-7)(4)) - 2 \cdot ((1)(-2) - (-7)(-6)) + 3 \cdot ((1)(4) - (-3)(-6)) \\ &= 1 \cdot (6 + 28) - 2 \cdot (-2 - 42) + 3 \cdot (4 - 18) \\ &= 1 \cdot 34 - 2 \cdot (-44) + 3 \cdot (-14) \\ &= 34 + 88 - 42 = 80. \end{aligned}$$

Since $\det A = 80 \neq 0$, the system is a Cramer system.

2. Solve using Cramer's method.

$$x_1 = \frac{\det A_1}{\det A}, \quad x_2 = \frac{\det A_2}{\det A}, \quad x_3 = \frac{\det A_3}{\det A},$$

where A_i is A with column i replaced by B .

Compute $\det A_1$:

$$A_1 = \begin{pmatrix} 14 & 2 & 3 \\ -26 & -3 & -7 \\ -4 & 4 & -2 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_1 &= 14 \cdot \begin{vmatrix} -3 & -7 \\ 4 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} -26 & -7 \\ -4 & -2 \end{vmatrix} + 3 \cdot \begin{vmatrix} -26 & -3 \\ -4 & 4 \end{vmatrix} \\ &= 14 \cdot (6 + 28) - 2 \cdot (52 - 28) + 3 \cdot (-104 - 12) \\ &= 14 \cdot 34 - 2 \cdot 24 + 3 \cdot (-116) \\ &= 476 - 48 - 348 = 80. \end{aligned}$$

Compute $\det A_2$:

$$A_2 = \begin{pmatrix} 1 & 14 & 3 \\ 1 & -26 & -7 \\ -6 & -4 & -2 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_2 &= 1 \cdot \begin{vmatrix} -26 & -7 \\ -4 & -2 \end{vmatrix} - 14 \cdot \begin{vmatrix} 1 & -7 \\ -6 & -2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & -26 \\ -6 & -4 \end{vmatrix} \\ &= 1 \cdot (52 - 28) - 14 \cdot (-2 - 42) + 3 \cdot (-4 - 156) \\ &= 1 \cdot 24 - 14 \cdot (-44) + 3 \cdot (-160) \\ &= 24 + 616 - 480 = 160. \end{aligned}$$

Compute $\det A_3$:

$$A_3 = \begin{pmatrix} 1 & 2 & 14 \\ 1 & -3 & -26 \\ -6 & 4 & -4 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_3 &= 1 \cdot \begin{vmatrix} -3 & -26 \\ 4 & -4 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & -26 \\ -6 & -4 \end{vmatrix} + 14 \cdot \begin{vmatrix} 1 & -3 \\ -6 & 4 \end{vmatrix} \\ &= 1 \cdot (12 + 104) - 2 \cdot (-4 - 156) + 14 \cdot (4 - 18) \\ &= 1 \cdot 116 - 2 \cdot (-160) + 14 \cdot (-14) \\ &= 116 + 320 - 196 = 240. \end{aligned}$$

Now apply Cramer's rule:

$$x_1 = \frac{\det A_1}{\det A} = \frac{80}{80} = 1, \quad x_2 = \frac{\det A_2}{\det A} = \frac{160}{80} = 2, \quad x_3 = \frac{\det A_3}{\det A} = \frac{240}{80} = 3.$$

Thus the solution is:

$$\boxed{x_1 = 1, \quad x_2 = 2, \quad x_3 = 3}.$$

Verification:

$$1 + 2(2) + 3(3) = 1 + 4 + 9 = 14 \quad \checkmark$$

$$1 - 3(2) - 7(3) = 1 - 6 - 21 = -26 \quad \checkmark$$

$$-6(1) + 4(2) - 2(3) = -6 + 8 - 6 = -4 \quad \checkmark$$

1.3 Main Exam (2018/2019)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Main Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 16/09/2019 **Duration:** 1h 30min

Exercise 1

(08 points)

Consider the following system of linear equations:

$$(S) : \begin{cases} x_2 - x_3 = -1 \\ -3x_1 + 4x_2 - 3x_3 = -6 \\ -x_1 + x_2 = -1 \end{cases}$$

1. Write the system (S) in the matrix form $AX = B$.
2. Show that the matrix A satisfies the equation: $A^2 - 3A + 2I_3 = 0_M$. Deduce that A is invertible and determine A^{-1} .
3. Solve the system (S) using two methods: (a) Cramer's rule, and (b) the matrix inverse method.

Exercise 2

(06 points)

Answer by **True** or **False**, correcting the false statements:

1. The product of two matrices $A \cdot B$ is possible if the number of rows of A equals the number of columns of B .
2. Every homogeneous system of linear equations is a Cramer system and admits just infinitely many solutions.
3. Every non-square matrix has a determinant equal to zero and is invertible.
4. If the number of variables equals the number of equations and the determinant is non-zero, then it is a Cramer system.

Exercise 3

(06 points)

Let the following two matrices be:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & -3 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix}$$

1. Calculate $|A|$, $|B|$, $|A + B|$, $A \times B$, and $|A \times B|$.
2. Is $|A + B| = |A| + |B|$?
3. Is $|A \times B| = |A| \times |B|$?

Full Solution

Solution – Exercise 1

Consider the following system of linear equations:

$$(S) : \begin{cases} x_2 - x_3 = -1 \\ -3x_1 + 4x_2 - 3x_3 = -6 \\ -x_1 + x_2 = -1 \end{cases}$$

1. Matrix form $AX = B$.

The system can be written as:

$$\begin{cases} 0x_1 + 1x_2 - 1x_3 = -1 \\ -3x_1 + 4x_2 - 3x_3 = -6 \\ -1x_1 + 1x_2 + 0x_3 = -1 \end{cases}$$

Thus:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ -6 \\ -1 \end{pmatrix}.$$

2. Show that A satisfies $A^2 - 3A + 2I_3 = 0_M$. Deduce A^{-1} .

First compute A^2 :

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}$$

Compute each entry:

Row 1:

$$(A^2)_{11} = 0 \cdot 0 + 1 \cdot (-3) + (-1) \cdot (-1) = 0 - 3 + 1 = -2$$

$$(A^2)_{12} = 0 \cdot 1 + 1 \cdot 4 + (-1) \cdot 1 = 0 + 4 - 1 = 3$$

$$(A^2)_{13} = 0 \cdot (-1) + 1 \cdot (-3) + (-1) \cdot 0 = 0 - 3 + 0 = -3$$

Row 2:

$$(A^2)_{21} = (-3) \cdot 0 + 4 \cdot (-3) + (-3) \cdot (-1) = 0 - 12 + 3 = -9$$

$$(A^2)_{22} = (-3) \cdot 1 + 4 \cdot 4 + (-3) \cdot 1 = -3 + 16 - 3 = 10$$

$$(A^2)_{23} = (-3) \cdot (-1) + 4 \cdot (-3) + (-3) \cdot 0 = 3 - 12 + 0 = -9$$

Row 3:

$$(A^2)_{31} = (-1) \cdot 0 + 1 \cdot (-3) + 0 \cdot (-1) = 0 - 3 + 0 = -3$$

$$(A^2)_{32} = (-1) \cdot 1 + 1 \cdot 4 + 0 \cdot 1 = -1 + 4 + 0 = 3$$

$$(A^2)_{33} = (-1) \cdot (-1) + 1 \cdot (-3) + 0 \cdot 0 = 1 - 3 + 0 = -2$$

Thus:

$$A^2 = \begin{pmatrix} -2 & 3 & -3 \\ -9 & 10 & -9 \\ -3 & 3 & -2 \end{pmatrix}.$$

Now compute $A^2 - 3A + 2I_3$:

$$3A = \begin{pmatrix} 0 & 3 & -3 \\ -9 & 12 & -9 \\ -3 & 3 & 0 \end{pmatrix}, \quad 2I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A^2 - 3A = \begin{pmatrix} -2 - 0 & 3 - 3 & -3 - (-3) \\ -9 - (-9) & 10 - 12 & -9 - (-9) \\ -3 - (-3) & 3 - 3 & -2 - 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Then add $2I_3$:

$$A^2 - 3A + 2I_3 = \begin{pmatrix} -2 + 2 & 0 & 0 \\ 0 & -2 + 2 & 0 \\ 0 & 0 & -2 + 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_M.$$

Thus $A^2 - 3A + 2I_3 = 0$, as required.

Deduce A^{-1} : From $A^2 - 3A + 2I_3 = 0$, we have:

$$A^2 - 3A = -2I_3$$

$$A(A - 3I_3) = -2I_3$$

$$A \cdot \left(-\frac{1}{2}(A - 3I_3) \right) = I_3.$$

Thus A is invertible and:

$$A^{-1} = -\frac{1}{2}(A - 3I_3).$$

Compute $A - 3I_3$:

$$A - 3I_3 = \begin{pmatrix} 0-3 & 1 & -1 \\ -3 & 4-3 & -3 \\ -1 & 1 & 0-3 \end{pmatrix} = \begin{pmatrix} -3 & 1 & -1 \\ -3 & 1 & -3 \\ -1 & 1 & -3 \end{pmatrix}.$$

Therefore:

$$A^{-1} = -\frac{1}{2} \begin{pmatrix} -3 & 1 & -1 \\ -3 & 1 & -3 \\ -1 & 1 & -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 & 1 \\ 3 & -1 & 3 \\ 1 & -1 & 3 \end{pmatrix}.$$

3. Solve the system (S) using two methods.

(a) Cramer's rule:

First compute $\det A$ (from earlier, A^{-1} exists so $\det A \neq 0$). Using $A^{-1} = -\frac{1}{2}(A - 3I_3)$, but easier: $\det A$ can be computed directly.

$$\det A = \begin{vmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= 0 \cdot \begin{vmatrix} 4 & -3 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -3 & -3 \\ -1 & 0 \end{vmatrix} + (-1) \cdot \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} \\ &= 0 - 1 \cdot ((-3)(0) - (-3)(-1)) - 1 \cdot ((-3)(1) - (4)(-1)) \\ &= -1 \cdot (0 - 3) - 1 \cdot (-3 + 4) \\ &= -1 \cdot (-3) - 1 \cdot (1) \\ &= 3 - 1 = 2. \end{aligned}$$

Now compute $\det A_1$ (replace column 1 with B):

$$A_1 = \begin{pmatrix} -1 & 1 & -1 \\ -6 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \det A_1 &= (-1) \cdot \begin{vmatrix} 4 & -3 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -6 & -3 \\ -1 & 0 \end{vmatrix} + (-1) \cdot \begin{vmatrix} -6 & 4 \\ -1 & 1 \end{vmatrix} \\ &= (-1) \cdot (0 + 3) - 1 \cdot (0 - 3) - 1 \cdot (-6 + 4) \\ &= -3 - 1 \cdot (-3) - 1 \cdot (-2) \\ &= -3 + 3 + 2 = 2. \end{aligned}$$

Compute $\det A_2$ (replace column 2 with B):

$$A_2 = \begin{pmatrix} 0 & -1 & -1 \\ -3 & -6 & -3 \\ -1 & -1 & 0 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_2 &= 0 \cdot \begin{vmatrix} -6 & -3 \\ -1 & 0 \end{vmatrix} - (-1) \cdot \begin{vmatrix} -3 & -3 \\ -1 & 0 \end{vmatrix} + (-1) \cdot \begin{vmatrix} -3 & -6 \\ -1 & -1 \end{vmatrix} \\ &= 0 + 1 \cdot ((-3)(0) - (-3)(-1)) - 1 \cdot ((-3)(-1) - (-6)(-1)) \\ &= 1 \cdot (0 - 3) - 1 \cdot (3 - 6) \\ &= -3 - 1 \cdot (-3) = -3 + 3 = 0. \end{aligned}$$

Compute $\det A_3$ (replace column 3 with B):

$$A_3 = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -6 \\ -1 & 1 & -1 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_3 &= 0 \cdot \begin{vmatrix} 4 & -6 \\ 1 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} -3 & -6 \\ -1 & -1 \end{vmatrix} + (-1) \cdot \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} \\ &= 0 - 1 \cdot ((-3)(-1) - (-6)(-1)) - 1 \cdot ((-3)(1) - (4)(-1)) \\ &= -1 \cdot (3 - 6) - 1 \cdot (-3 + 4) \\ &= -1 \cdot (-3) - 1 \cdot (1) \\ &= 3 - 1 = 2. \end{aligned}$$

Thus by Cramer's rule:

$$x_1 = \frac{\det A_1}{\det A} = \frac{2}{2} = 1, \quad x_2 = \frac{\det A_2}{\det A} = \frac{0}{2} = 0, \quad x_3 = \frac{\det A_3}{\det A} = \frac{2}{2} = 1.$$

$$\boxed{x_1 = 1, x_2 = 0, x_3 = 1}.$$

(b) Matrix inverse method:

We have $X = A^{-1}B$.

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 & 1 \\ 3 & -1 & 3 \\ 1 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ -6 \\ -1 \end{pmatrix}$$

$$X = \frac{1}{2} \begin{pmatrix} 3 & -1 & 1 \\ 3 & -1 & 3 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ -6 \\ -1 \end{pmatrix}$$

Compute:

First component:

$$3(-1) + (-1)(-6) + 1(-1) = -3 + 6 - 1 = 2$$

$$x_1 = \frac{1}{2} \cdot 2 = 1.$$

Second component:

$$3(-1) + (-1)(-6) + 3(-1) = -3 + 6 - 3 = 0$$

$$x_2 = \frac{1}{2} \cdot 0 = 0.$$

Third component:

$$1(-1) + (-1)(-6) + 3(-1) = -1 + 6 - 3 = 2$$

$$x_3 = \frac{1}{2} \cdot 2 = 1.$$

Thus:

$$\boxed{x_1 = 1, \quad x_2 = 0, \quad x_3 = 1}.$$

Both methods give the same solution.

Solution – Exercise 2

1. **False.** The product $A \cdot B$ is possible if the number of **columns** of A equals the number of **rows** of B .
2. **False.** A homogeneous system is a Cramer system only if $|A| \neq 0$, in which case it admits **only the trivial solution** $(0, 0, 0)$.
3. **False.** Determinants are only defined for **square matrices**, and only square matrices with non-zero determinants are invertible.
4. **True.**

Solution – Exercise 3

$$1. \text{ **Calculations: } |A| = 15, |B| = 3. A + B = \begin{pmatrix} -2 & 0 & 0 \\ -2 & -2 & -1 \\ 1 & 0 & 2 \end{pmatrix} \implies |A + B| = 8.**$$

$$A \times B = \begin{pmatrix} -5 & 2 & 2 \\ 7 & -4 & 2 \\ 2 & -1 & -2 \end{pmatrix}. |A \times B| = 15 \times 3 = 45.$$

2. **No.** $|A + B| \neq |A| + |B|$ as $8 \neq 15 + 3$.
3. **Yes.** $|A \times B| = |A| \times |B|$ as $45 = 15 \times 3$.

1.4 Main Exam (2019/2020) - Version A

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Main Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 03/10/2020 **Duration:** 1h 00min

Exercise 1

(10 points)

Let A , B , and C be the following matrices:

$$A = \begin{pmatrix} -1 & 0 & 3 \\ 2 & 6 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 5 & 6 \end{pmatrix}$$

Choose the correct answer for each case:

- The sum $A + B$:
 (a) Is a square matrix (b) Cannot be calculated (c) Is a non-square matrix.
- The product $A \times C$:
 (a) Is a non-square matrix (b) Result is $\begin{pmatrix} 14 & 18 \\ 34 & 42 \end{pmatrix}$ (c) Cannot be calculated.
- The product $A \times B \times C$:
 (a) Is a matrix $D_{(3 \times 3)}$ (b) Is a matrix $D_{(2 \times 2)}$ (c) Cannot be calculated.
- The trace of matrix B :
 (a) $tr(B) = 2$ (b) $tr(B) = -2$ (c) Cannot be calculated.
- The transpose C^T :
 (a) $C^T = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 3 & 6 \end{pmatrix}$ (b) $C^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 5 & 6 \end{pmatrix}$ (c) Cannot be calculated.
- The determinant $|A|$:
 (a) $|A| = 5$ (b) $|A| = -5$ (c) Cannot be calculated.
- The determinant $|A \times B \times C|$:
 (a) $|A| \times |B| \times |C|$ (b) Is not equal to $|A| \times |B| \times |C|$ (c) Cannot be calculated.
- The determinant $|B|$:
 (a) $|B| = 5$ (b) $|B| = -5$ (c) $|B| = 15$.

9. The inverse B^{-1} satisfies:
 (a) $B^{-1} \times B = I_3$ (b) $B^{-1} \times B = I_4$ (c) Cannot be calculated.
10. The inverse C^{-1} :
 (a) Is a matrix $C_{(3 \times 2)}^{-1}$ (b) Is a matrix $C_{(2 \times 3)}^{-1}$ (c) Cannot be calculated.

Exercise 2**(10 points)**

Let A and B be the matrices defined as follows:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = -\frac{1}{2}A + \frac{3}{2}I_3 = \begin{pmatrix} 3/2 & -1/2 & 1/2 \\ 3/2 & -1/2 & 3/2 \\ 1/2 & -1/2 & 3/2 \end{pmatrix}$$

1. Compute $|A|$ and $|B|$:

$$|A| = \begin{vmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{vmatrix} =$$

$$|B| = \begin{vmatrix} 3/2 & -1/2 & 1/2 \\ 3/2 & -1/2 & 3/2 \\ 1/2 & -1/2 & 3/2 \end{vmatrix} =$$

2. Are the matrices A and B invertible or not? Justify your answer.

3. Compute A^2 :

$$A^2 = \begin{pmatrix} _ & _ & _ \\ _ & _ & _ \\ _ & _ & _ \end{pmatrix} \times \begin{pmatrix} _ & _ & _ \\ _ & _ & _ \\ _ & _ & _ \end{pmatrix} =$$

4. Verify that: $A^2 = 3A - 2I_3$

5. Deduce the relationship between matrices A and B .

Full Solution – Version A

Solution – Exercise 1

1. **(b) Cannot be calculated.** Matrix A is 2×3 and B is 3×3 . Addition is only defined for matrices of the same dimensions.
2. **(b) Result is** $\begin{pmatrix} 14 & 18 \\ 34 & 42 \end{pmatrix}$.
Calculation: $\begin{pmatrix} (-1)(1) + (0)(2) + (3)(5) & (-1)(0) + (0)(3) + (3)(6) \\ (2)(1) + (6)(2) + (4)(5) & (2)(0) + (6)(3) + (4)(6) \end{pmatrix} = \begin{pmatrix} 14 & 18 \\ 34 & 42 \end{pmatrix}$.
3. **(b) Is a matrix** $D_{(2 \times 2)}$. Dimensions: $(2 \times 3) \times (3 \times 3) \times (3 \times 2) = 2 \times 2$.
4. **(b) $\text{tr}(B) = -2$.** The trace is the sum of the main diagonal: $-2 + 1 + (-1) = -2$.
5. **(a) $C^T = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 3 & 6 \end{pmatrix}$.** Rows become columns.
6. **(c) Cannot be calculated.** Determinants are only defined for square matrices; A is 2×3 .
7. **(b) Is not equal to $|A| \times |B| \times |C|$.** Because $|A|$ and $|C|$ do not exist individually.
8. **(a) $|B| = 5$.** $|B| = -2(-1 - 4) - 4(0 - 2) + 3(0 - 1) = 10 + 8 - 3 = \mathbf{15}$. (Note: Corrected calculation from $10 - 8 - 3 = -1$).
9. **(a) $B^{-1} \times B = I_3$.** By definition of the inverse of a 3×3 matrix.
10. **(c) Cannot be calculated.** Only square matrices can have an inverse.

Solution – Exercise 2

Exam Problem:

Let A and B be the matrices defined as follows:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = -\frac{1}{2}A + \frac{3}{2}I_3 = \begin{pmatrix} 3/2 & -1/2 & 1/2 \\ 3/2 & -1/2 & 3/2 \\ 1/2 & -1/2 & 3/2 \end{pmatrix}$$

1. Compute $|A|$ and $|B|$.

$$|A| = \begin{vmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned}
 |A| &= 0 \cdot \begin{vmatrix} 4 & -3 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -3 & -3 \\ -1 & 0 \end{vmatrix} + (-1) \cdot \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} \\
 &= 0 - 1 \cdot ((-3)(0) - (-3)(-1)) - 1 \cdot ((-3)(1) - (4)(-1)) \\
 &= -1 \cdot (0 - 3) - 1 \cdot (-3 + 4) \\
 &= -1 \cdot (-3) - 1 \cdot (1) = 3 - 1 = 2.
 \end{aligned}$$

Thus:

$$\boxed{|A| = 2}.$$

Now compute $|B|$. Since $B = -\frac{1}{2}A + \frac{3}{2}I_3$, but we can also compute directly:

$$|B| = \begin{vmatrix} \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{vmatrix}$$

Factor $\frac{1}{2}$ from each row: Row 1 factor $\frac{1}{2}$, Row 2 factor $\frac{1}{2}$, Row 3 factor $\frac{1}{2}$:

$$|B| = \left(\frac{1}{2}\right)^3 \begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 3 \\ 1 & -1 & 3 \end{vmatrix} = \frac{1}{8} \begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 3 \\ 1 & -1 & 3 \end{vmatrix}$$

Compute the 3×3 determinant:

$$\begin{aligned}
 \begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 3 \\ 1 & -1 & 3 \end{vmatrix} &= 3 \cdot \begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} \\
 &= 3 \cdot ((-1)(3) - (3)(-1)) + 1 \cdot ((3)(3) - (3)(1)) + 1 \cdot ((3)(-1) - (-1)(1)) \\
 &= 3 \cdot (-3 + 3) + 1 \cdot (9 - 3) + 1 \cdot (-3 + 1) \\
 &= 3 \cdot 0 + 1 \cdot 6 + 1 \cdot (-2) = 6 - 2 = 4.
 \end{aligned}$$

Thus:

$$|B| = \frac{1}{8} \cdot 4 = \frac{1}{2}.$$

$$\boxed{|B| = \frac{1}{2}}.$$

2. Are the matrices A and B invertible or not? Justify your answer.

A square matrix is invertible if and only if its determinant is non-zero.

$|A| = 2 \neq 0$, so A is invertible.

$|B| = \frac{1}{2} \neq 0$, so B is invertible.

Both A and B are invertible because their determinants are non-zero.

3. Compute A^2 .

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}$$

Compute each entry:

Row 1:

$$(A^2)_{11} = 0 \cdot 0 + 1 \cdot (-3) + (-1) \cdot (-1) = 0 - 3 + 1 = -2$$

$$(A^2)_{12} = 0 \cdot 1 + 1 \cdot 4 + (-1) \cdot 1 = 0 + 4 - 1 = 3$$

$$(A^2)_{13} = 0 \cdot (-1) + 1 \cdot (-3) + (-1) \cdot 0 = 0 - 3 + 0 = -3$$

Row 2:

$$(A^2)_{21} = (-3) \cdot 0 + 4 \cdot (-3) + (-3) \cdot (-1) = 0 - 12 + 3 = -9$$

$$(A^2)_{22} = (-3) \cdot 1 + 4 \cdot 4 + (-3) \cdot 1 = -3 + 16 - 3 = 10$$

$$(A^2)_{23} = (-3) \cdot (-1) + 4 \cdot (-3) + (-3) \cdot 0 = 3 - 12 + 0 = -9$$

Row 3:

$$(A^2)_{31} = (-1) \cdot 0 + 1 \cdot (-3) + 0 \cdot (-1) = 0 - 3 + 0 = -3$$

$$(A^2)_{32} = (-1) \cdot 1 + 1 \cdot 4 + 0 \cdot 1 = -1 + 4 + 0 = 3$$

$$(A^2)_{33} = (-1) \cdot (-1) + 1 \cdot (-3) + 0 \cdot 0 = 1 - 3 + 0 = -2$$

Thus:

$$A^2 = \begin{pmatrix} -2 & 3 & -3 \\ -9 & 10 & -9 \\ -3 & 3 & -2 \end{pmatrix}.$$

$$A^2 = \begin{pmatrix} -2 & 3 & -3 \\ -9 & 10 & -9 \\ -3 & 3 & -2 \end{pmatrix}.$$

4. Verify that $A^2 = 3A - 2I_3$.

First compute $3A$:

$$3A = \begin{pmatrix} 0 & 3 & -3 \\ -9 & 12 & -9 \\ -3 & 3 & 0 \end{pmatrix}$$

Then $3A - 2I_3$:

$$3A - 2I_3 = \begin{pmatrix} 0-2 & 3-0 & -3-0 \\ -9-0 & 12-2 & -9-0 \\ -3-0 & 3-0 & 0-2 \end{pmatrix} = \begin{pmatrix} -2 & 3 & -3 \\ -9 & 10 & -9 \\ -3 & 3 & -2 \end{pmatrix}$$

This matches A^2 computed above. Thus $A^2 = 3A - 2I_3$.

$$\text{Verified: } A^2 = 3A - 2I_3.$$

5. Deduce the relationship between matrices A and B .

We have $B = -\frac{1}{2}A + \frac{3}{2}I_3$.

From $A^2 - 3A + 2I_3 = 0$, divide by -2 :

$$-\frac{1}{2}A^2 + \frac{3}{2}A - I_3 = 0$$

But also from $B = -\frac{1}{2}A + \frac{3}{2}I_3$, we have:

Notice $B^2 = \left(-\frac{1}{2}A + \frac{3}{2}I_3\right)^2 = \frac{1}{4}A^2 - \frac{3}{2}A + \frac{9}{4}I_3$.

Alternatively, using $A^2 = 3A - 2I_3$, substitute into B : From $B = -\frac{1}{2}A + \frac{3}{2}I_3$, multiply by A :

$$\begin{aligned} BA &= -\frac{1}{2}A^2 + \frac{3}{2}A = -\frac{1}{2}(3A - 2I_3) + \frac{3}{2}A \\ &= -\frac{3}{2}A + I_3 + \frac{3}{2}A = I_3. \end{aligned}$$

Thus $BA = I_3$, so $B = A^{-1}$.

The relationship is $B = A^{-1}$.

1.5 Main Exam (2019/2020) - Version B

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Main Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 03/10/2020 **Duration:** 1h 00min

Exercise 1

(10 points)

Let A , B , and C be the matrices defined as follows:

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 4 & 2 \\ 3 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 3 & 3 \\ 0 & 1 & 5 \\ 1 & 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 \\ 5 & 4 \\ 1 & 7 \end{pmatrix}$$

Choose the correct answer from the proposed options for each case:

- The sum $B + A$:
 (a) is a square matrix (b) is a non-square matrix (c) cannot be computed
- The product $B \times C$:
 (a) $B \times C = \begin{pmatrix} 12 & 33 \\ 10 & 39 \\ 12 & 1 \end{pmatrix}$ (b) $B \times C = \begin{pmatrix} 17 & 37 & 13 \\ 60 & 12 & 1 \end{pmatrix}$ (c) cannot be computed
- The product $A^2 \times B \times C$:
 (a) is a $D_{(3 \times 3)}$ matrix (b) is a $D_{(3 \times 2)}$ matrix (c) cannot be computed
- The trace of matrix A :
 (a) $\text{tr}(A) = 5$ (b) $\text{tr}(A) = -8$ (c) cannot be computed
- The square of matrix C :
 (a) $C^2 = \begin{pmatrix} 9 & 0 \\ 25 & 16 \\ 1 & 49 \end{pmatrix}$ (b) $C^2 = \begin{pmatrix} 9 & 0 & 25 \\ 16 & 1 & 49 \end{pmatrix}$ (c) cannot be computed
- The determinant of matrix B :
 (a) $|B| = 34$ (b) $|B| = 0$ (c) cannot be computed
- The inverse of matrix B :
 (a) is a $B_{(3 \times 3)}^{-1}$ matrix (b) is a real number $B^{-1} = \frac{1}{34}$ (c) cannot be computed
- The inverse of matrix C :
 (a) is a $C_{(2 \times 3)}^{-1}$ matrix (b) is a $C_{(3 \times 2)}^{-1}$ matrix (c) cannot be computed

9. The transpose of matrix C :

(a) $C_{3 \times 3}^T$

(b) $C_{3 \times 2}^T$

(c) $C_{2 \times 3}^T$

10. If $\text{Com}(A)$ is the adjugate (comatrix) of A , then:

(a) $A^{-1} = \frac{1}{|A|} (\text{Com}(A))^T$ (b) $A^{-1} = \frac{1}{|A|} \text{Com}(A)$ (c) $A^{-1} = \frac{1}{A} (\text{Com}(A))^T$

Exercise 2

(10 points)

Let A and B be the matrices defined by:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 6 & 8 \\ 0 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ -6 & \frac{3}{2} & -4 \\ 4 & -1 & 3 \end{pmatrix}$$

1. Compute $|A|$ and $|B|$:

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 6 & 8 \\ 0 & 2 & 3 \end{vmatrix} = \dots\dots\dots$$

$$|B| = \begin{vmatrix} 1 & 0 & 0 \\ -6 & \frac{3}{2} & -4 \\ 4 & -1 & 3 \end{vmatrix} = \dots\dots\dots$$

2. Are the matrices A and B invertible or not? Justify your answer.

.....

3. Compute the adjugate (comatrix) of A : $(\text{Com}(A))$

$$\text{Com}(A) = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

4. Find the transpose of the adjugate matrix of A : $((\text{Com}(A))^T)$

$$(\text{Com}(A))^T = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

5. Compute A^{-1} then deduce B^{-1}

$$A^{-1} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Full Solution – Version B

Solution – Exercise 1

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 4 & 2 \\ 3 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 3 & 3 \\ 0 & 1 & 5 \\ 1 & 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 \\ 5 & 4 \\ 1 & 7 \end{pmatrix}$$

Choose the correct answer from the proposed options for each case:

1. The sum $B + A$:

(a) is a square matrix (b) is a non-square matrix (c) cannot be computed

Solution: A and B are both 3×3 matrices, so $B + A$ is defined and is 3×3 , hence a square matrix. (a) is a square matrix.

2. The product $B \times C$:

(a) $B \times C = \begin{pmatrix} 12 & 33 \\ 10 & 39 \\ 12 & 1 \end{pmatrix}$ (b) $B \times C = \begin{pmatrix} 17 & 37 & 13 \\ 60 & 12 & 1 \end{pmatrix}$ (c) cannot be computed

Solution: B is 3×3 , C is 3×2 , so BC is 3×2 . Compute:

$$BC = \begin{pmatrix} -2 & 3 & 3 \\ 0 & 1 & 5 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 5 & 4 \\ 1 & 7 \end{pmatrix}$$

First column: $(-2)(3) + 3(5) + 3(1) = -6 + 15 + 3 = 12$; $0(3) + 1(5) + 5(1) = 0 + 5 + 5 = 10$; $1(3) + 2(5) + (-1)(1) = 3 + 10 - 1 = 12$

Second column: $(-2)(0) + 3(4) + 3(7) = 0 + 12 + 21 = 33$; $0(0) + 1(4) + 5(7) = 0 + 4 + 35 = 39$; $1(0) + 2(4) + (-1)(7) = 0 + 8 - 7 = 1$

$$BC = \begin{pmatrix} 12 & 33 \\ 10 & 39 \\ 12 & 1 \end{pmatrix}$$

Matches option (a). (a).

3. The product $A^2 \times B \times C$:

(a) is a $D_{(3 \times 3)}$ matrix (b) is a $D_{(3 \times 2)}$ matrix (c) cannot be computed

Solution: A is 3×3 , so A^2 is 3×3 . $A^2 B$ is $(3 \times 3)(3 \times 3) = 3 \times 3$. Then $(A^2 B)C$ is $(3 \times 3)(3 \times 2) = 3 \times 2$. Thus $A^2 BC$ is 3×2 . (b).

4. The trace of matrix A :

(a) $\text{tr}(A) = 5$ (b) $\text{tr}(A) = -8$ (c) cannot be computed

Solution: $\text{tr}(A) = (-1) + 4 + 2 = 5$. (a).

5. The square of matrix C :

(a) $C^2 = \begin{pmatrix} 9 & 0 \\ 25 & 16 \\ 1 & 49 \end{pmatrix}$ (b) $C^2 = \begin{pmatrix} 9 & 0 & 25 \\ 16 & 1 & 49 \end{pmatrix}$ (c) cannot be computed

Solution: C is 3×2 . For $C^2 = C \cdot C$ to be defined, the number of columns of first C must equal rows of second C : $2 \neq 3$, so C^2 is not defined. (c).

6. The determinant of matrix B :

- (a) $|B| = 34$ (b) $|B| = 0$ (c) cannot be computed

Solution: Compute $\det B$:

$$\det B = \begin{vmatrix} -2 & 3 & 3 \\ 0 & 1 & 5 \\ 1 & 2 & -1 \end{vmatrix}$$

$$\begin{aligned} &\text{Expand by row 2: } 0 \cdot (\dots) + 1 \cdot (-1)^{2+2} \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} + 5 \cdot (-1)^{2+3} \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 1 \cdot ((-2)(-1) - 3(1)) + 5 \cdot (-1) \cdot ((-2)(2) - 3(1)) \\ &= (2 - 3) - 5 \cdot (-4 - 3) = (-1) - 5 \cdot (-7) = -1 + 35 = 34. \det B = 34 \neq 0. \end{aligned}$$
(a).

7. The inverse of matrix B :

- (a) is a $B_{(3 \times 3)}^{-1}$ matrix (b) is a real number $B^{-1} = \frac{1}{34}$ (c) cannot be computed

Solution: Since $\det B \neq 0$, B is invertible, and B^{-1} is also 3×3 (not a scalar). (a).

8. The inverse of matrix C :

- (a) is a $C_{(2 \times 3)}^{-1}$ matrix (b) is a $C_{(3 \times 2)}^{-1}$ matrix (c) cannot be computed

Solution: C is 3×2 , not square, so it has no inverse. (c).

9. The transpose of matrix C :

- (a) $C_{3 \times 3}^T$ (b) $C_{3 \times 2}^T$ (c) $C_{2 \times 3}^T$

Solution: C is 3×2 , so C^T is 2×3 , denoted $C_{2 \times 3}^T$. (c).

10. If $\text{Com}(A)$ is the adjugate (comatrix) of A , then:

- (a) $A^{-1} = \frac{1}{|A|} (\text{Com}(A))^T$ (b) $A^{-1} = \frac{1}{|A|} \text{Com}(A)$ (c) $A^{-1} = \frac{1}{A} (\text{Com}(A))^T$

Solution: The correct formula is $A^{-1} = \frac{1}{\det A} (\text{Com}(A))^T$, where $\text{Com}(A)$ is the cofactor matrix. Option (a) matches this if $\text{Com}(A)$ is the cofactor matrix. (a).

Solution – Exercise 2

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 6 & 8 \\ 0 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ -6 & \frac{3}{2} & -4 \\ 4 & -1 & 3 \end{pmatrix}$$

1. Compute $|A|$ and $|B|$.

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 6 & 8 \\ 0 & 2 & 3 \end{vmatrix}$$

Expand by row 1:

$$|A| = 1 \cdot \begin{vmatrix} 6 & 8 \\ 2 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 4 & 8 \\ 0 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 4 & 6 \\ 0 & 2 \end{vmatrix}$$

$$|A| = 1 \cdot (6 \cdot 3 - 8 \cdot 2) = 18 - 16 = 2.$$

$$|B| = \begin{vmatrix} 1 & 0 & 0 \\ -6 & \frac{3}{2} & -4 \\ 4 & -1 & 3 \end{vmatrix}$$

Expand by row 1:

$$|B| = 1 \cdot \begin{vmatrix} \frac{3}{2} & -4 \\ -1 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} -6 & -4 \\ 4 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} -6 & \frac{3}{2} \\ 4 & -1 \end{vmatrix}$$

$$|B| = 1 \cdot \left(\frac{3}{2} \cdot 3 - (-4)(-1) \right) = \frac{9}{2} - 4 = \frac{9}{2} - \frac{8}{2} = \frac{1}{2}.$$

Thus:

$$\boxed{|A| = 2}, \quad \boxed{|B| = \frac{1}{2}}.$$

2. Are the matrices A and B invertible or not? Justify your answer.

A square matrix is invertible if and only if its determinant is non-zero.

Since $|A| = 2 \neq 0$, A is invertible.

Since $|B| = \frac{1}{2} \neq 0$, B is invertible.

Both A and B are invertible because their determinants are non-zero.

3. Compute the adjugate (comatrix) of A : $\text{Com}(A)$.

First compute all cofactors C_{ij} of A .

$$\begin{aligned}
C_{11} &= (-1)^{1+1} \begin{vmatrix} 6 & 8 \\ 2 & 3 \end{vmatrix} = 1 \cdot (18 - 16) = 2 \\
C_{12} &= (-1)^{1+2} \begin{vmatrix} 4 & 8 \\ 0 & 3 \end{vmatrix} = -1 \cdot (12 - 0) = -12 \\
C_{13} &= (-1)^{1+3} \begin{vmatrix} 4 & 6 \\ 0 & 2 \end{vmatrix} = 1 \cdot (8 - 0) = 8 \\
C_{21} &= (-1)^{2+1} \begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = -1 \cdot (0 - 0) = 0 \\
C_{22} &= (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 1 \cdot (3 - 0) = 3 \\
C_{23} &= (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -1 \cdot (2 - 0) = -2 \\
C_{31} &= (-1)^{3+1} \begin{vmatrix} 0 & 0 \\ 6 & 8 \end{vmatrix} = 1 \cdot (0 - 0) = 0 \\
C_{32} &= (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 4 & 8 \end{vmatrix} = -1 \cdot (8 - 0) = -8 \\
C_{33} &= (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 4 & 6 \end{vmatrix} = 1 \cdot (6 - 0) = 6
\end{aligned}$$

Thus the cofactor matrix (adjugate/comatrix) is:

$$\text{Com}(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} 2 & -12 & 8 \\ 0 & 3 & -2 \\ 0 & -8 & 6 \end{pmatrix}.$$

$$\boxed{\text{Com}(A) = \begin{pmatrix} 2 & -12 & 8 \\ 0 & 3 & -2 \\ 0 & -8 & 6 \end{pmatrix}}.$$

4. Find the transpose of the adjugate matrix of A : $(\text{Com}(A))^T$.

$$(\text{Com}(A))^T = \begin{pmatrix} 2 & 0 & 0 \\ -12 & 3 & -8 \\ 8 & -2 & 6 \end{pmatrix}.$$

$$\boxed{(\text{Com}(A))^T = \begin{pmatrix} 2 & 0 & 0 \\ -12 & 3 & -8 \\ 8 & -2 & 6 \end{pmatrix}}.$$

5. Compute A^{-1} then deduce B^{-1} .

We have $A^{-1} = \frac{1}{\det A} (\text{Com}(A))^T$.

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ -12 & 3 & -8 \\ 8 & -2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -6 & \frac{3}{2} & -4 \\ 4 & -1 & 3 \end{pmatrix}.$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -6 & \frac{3}{2} & -4 \\ 4 & -1 & 3 \end{pmatrix}.$$

Notice that A^{-1} is exactly the matrix B given in the problem.

Thus $B = A^{-1}$, so $B^{-1} = (A^{-1})^{-1} = A$.

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 6 & 8 \\ 0 & 2 & 3 \end{pmatrix}.$$

1.6 Main Exam (2020/2021) - Version A

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Main Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 17/06/2021 **Duration:** 1h 30min

Exercise 1

(06 points)

Answer by **True** or **False**, correcting the false statements:

1. The trace of square matrix is the product its main diagonal elements.
2. The product of two matrices is possible if the two matrices are for the same size.
3. The determinant of matrix lower or upper triangular or diagonal matrix is the sum of it's diagonal elements.
4. The invertible matrix is the matrix which has zero as determinant.

Exercise 2

(06 points)

Let A and B two matrices given by

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$$

1. Calculate A^T and $tr(B)$.
2. Find the matrix C , while $3A - B - C = 0_M$
3. Calculate AB and BA , Then deduce.

Exercise 3

(08 points)

Let the linear system (S)

$$(S) \begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 + 2x_2 - x_3 = 2 \\ x_1 - 4x_3 = -3 \end{cases}$$

1. Write the system in the matrix form, the prove that is Cramer's system.
2. Solve the system using Cramer's and inverse methods.

Full Solution – Version A

Solution – Exercise 1

1. **False.**

Correction: The trace of a square matrix is the **sum** of its main diagonal elements, not the product.

2. **False.**

Correction: The product of two matrices A (size $m \times n$) and B (size $p \times q$) is possible if and only if $n = p$ (the number of columns of A equals the number of rows of B), not necessarily if they are the same size.

3. **False.**

Correction: The determinant of a lower triangular, upper triangular, or diagonal matrix is the **product** of its diagonal elements, not the sum.

4. **False.**

Correction: An invertible matrix is a matrix whose determinant is **non-zero**. A matrix with zero determinant is singular (non-invertible).

Solution – Exercise 2

Let A and B be two matrices given by:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$$

1. Calculate A^T and $\text{tr}(B)$.

$$A^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}.$$

$$\text{tr}(B) = (-1) + 4 = 3.$$

Thus:

$$A^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \quad \text{tr}(B) = 3.$$

2. Find the matrix C such that $3A - B - C = 0_M$.

The equation $3A - B - C = 0_M$ implies:

$$C = 3A - B.$$

First compute $3A$:

$$3A = \begin{pmatrix} 3 & 6 \\ 0 & 9 \end{pmatrix}.$$

Then:

$$C = \begin{pmatrix} 3 & 6 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 - (-1) & 6 - 2 \\ 0 - 3 & 9 - 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ -3 & 5 \end{pmatrix}.$$

$$\boxed{C = \begin{pmatrix} 4 & 4 \\ -3 & 5 \end{pmatrix}}.$$

3. Calculate AB and BA . Deducing?

$$AB = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot (-1) + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 4 \\ 0 \cdot (-1) + 3 \cdot 3 & 0 \cdot 2 + 3 \cdot 4 \end{pmatrix} = \begin{pmatrix} -1 + 6 & 2 + 8 \\ 0 + 9 & 0 + 12 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 9 & 12 \end{pmatrix}.$$

$$BA = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 \cdot 1 + 2 \cdot 0 & -1 \cdot 2 + 2 \cdot 3 \\ 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 2 + 4 \cdot 3 \end{pmatrix} = \begin{pmatrix} -1 + 0 & -2 + 6 \\ 3 + 0 & 6 + 12 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 3 & 18 \end{pmatrix}.$$

Thus:

$$AB = \begin{pmatrix} 5 & 10 \\ 9 & 12 \end{pmatrix}, \quad BA = \begin{pmatrix} -1 & 4 \\ 3 & 18 \end{pmatrix}.$$

Deducing: $AB \neq BA$, so matrix multiplication is not commutative in general.

Solution – Exercise 3

Let the linear system (S) :

$$(S) \begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 + 2x_2 - x_3 = 2 \\ x_1 - 4x_3 = -3 \end{cases}$$

1. Write the system in matrix form $AX = B$:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & -4 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.$$

Prove it is a Cramer system: A system is Cramer if it has n equations, n unknowns, and $\det A \neq 0$. Here $n = 3$.

Compute $\det A$:

$$\det A = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & -4 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= 1 \cdot \begin{vmatrix} 2 & -1 \\ 0 & -4 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & -1 \\ 1 & -4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \\ &= 1 \cdot ((2)(-4) - (-1)(0)) + 1 \cdot ((1)(-4) - (-1)(1)) + 1 \cdot ((1)(0) - (2)(1)) \\ &= 1 \cdot (-8 - 0) + 1 \cdot (-4 + 1) + 1 \cdot (0 - 2) \\ &= -8 + (-3) + (-2) = -13. \end{aligned}$$

Since $\det A = -13 \neq 0$, the system is a Cramer system.

2. Solve the system using two methods.

(a) Cramer's rule:

We have $\det A = -13$.

Compute $\det A_1$ (replace column 1 with B):

$$A_1 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & -1 \\ -3 & 0 & -4 \end{pmatrix}$$

Expand by row 3 (contains a zero):

$$\begin{aligned} \det A_1 &= (-3) \cdot (-1)^{3+1} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + 0 \cdot (\dots) + (-4) \cdot (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} \\ &= (-3) \cdot 1 \cdot ((-1)(-1) - (1)(2)) + (-4) \cdot 1 \cdot ((1)(2) - (-1)(2)) \\ &= -3 \cdot (1 - 2) - 4 \cdot (2 + 2) \\ &= -3 \cdot (-1) - 4 \cdot 4 = 3 - 16 = -13. \end{aligned}$$

Compute $\det A_2$ (replace column 2 with B):

$$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -3 & -4 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_2 &= 1 \cdot \begin{vmatrix} 2 & -1 \\ -3 & -4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & -1 \\ 1 & -4 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} \\ &= 1 \cdot ((2)(-4) - (-1)(-3)) - 1 \cdot ((1)(-4) - (-1)(1)) + 1 \cdot ((1)(-3) - (2)(1)) \\ &= 1 \cdot (-8 - 3) - 1 \cdot (-4 + 1) + 1 \cdot (-3 - 2) \\ &= -11 - 1 \cdot (-3) + (-5) = -11 + 3 - 5 = -13. \end{aligned}$$

Compute $\det A_3$ (replace column 3 with B):

$$A_3 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & -3 \end{pmatrix}$$

Expand by row 3:

$$\begin{aligned} \det A_3 &= 1 \cdot (-1)^{3+1} \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} + 0 \cdot (\dots) + (-3) \cdot (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} \\ &= 1 \cdot 1 \cdot ((-1)(2) - (1)(2)) + (-3) \cdot 1 \cdot ((1)(2) - (-1)(1)) \\ &= (-2 - 2) - 3 \cdot (2 + 1) = -4 - 9 = -13. \end{aligned}$$

Now by Cramer's rule:

$$x_1 = \frac{\det A_1}{\det A} = \frac{-13}{-13} = 1, \quad x_2 = \frac{\det A_2}{\det A} = \frac{-13}{-13} = 1, \quad x_3 = \frac{\det A_3}{\det A} = \frac{-13}{-13} = 1.$$

Thus:

$$\boxed{x_1 = 1, \quad x_2 = 1, \quad x_3 = 1}.$$

Verification:

$$1 - 1 + 1 = 1 \quad \checkmark$$

$$1 + 2 - 1 = 2 \quad \checkmark$$

$$1 - 4 = -3 \quad \checkmark$$

(b) Inverse method:

We have $X = A^{-1}B$.

First compute A^{-1} using the adjugate method.

Compute cofactor matrix of A :

$$C_{11} = (-1)^2 \begin{vmatrix} 2 & -1 \\ 0 & -4 \end{vmatrix} = 1 \cdot (-8 - 0) = -8$$

$$C_{12} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 1 & -4 \end{vmatrix} = -1 \cdot (-4 + 1) = 3$$

$$C_{13} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 2) = -2$$

$$C_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 0 & -4 \end{vmatrix} = -1 \cdot (4 - 0) = -4$$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} = 1 \cdot (-4 - 1) = -5$$

$$C_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -1 \cdot (0 + 1) = -1$$

$$C_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = 1 \cdot (1 - 2) = -1$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 \cdot (-1 - 1) = 2$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 1 \cdot (2 + 1) = 3$$

Cofactor matrix:

$$C = \begin{pmatrix} -8 & 3 & -2 \\ -4 & -5 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

Adjugate matrix = C^T :

$$\text{adj}(A) = \begin{pmatrix} -8 & -4 & -1 \\ 3 & -5 & 2 \\ -2 & -1 & 3 \end{pmatrix}$$

Then:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A) = \frac{1}{-13} \begin{pmatrix} -8 & -4 & -1 \\ 3 & -5 & 2 \\ -2 & -1 & 3 \end{pmatrix}$$

Now compute $X = A^{-1}B$:

$$X = \frac{1}{-13} \begin{pmatrix} -8 & -4 & -1 \\ 3 & -5 & 2 \\ -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

First component:

$$(-8)(1) + (-4)(2) + (-1)(-3) = -8 - 8 + 3 = -13$$

$$x_1 = \frac{-13}{-13} = 1$$

Second component:

$$3(1) + (-5)(2) + 2(-3) = 3 - 10 - 6 = -13$$

$$x_2 = \frac{-13}{-13} = 1$$

Third component:

$$(-2)(1) + (-1)(2) + 3(-3) = -2 - 2 - 9 = -13$$

$$x_3 = \frac{-13}{-13} = 1$$

Thus the solution is $x_1 = x_2 = x_3 = 1$, as before.

1.7 Main Exam (2020/2021) - Version B

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Main Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 2021

Duration: 1h 30min

Exercise 1

(06 points)

Choose the correct answer for each of the following statements: (*Refer to Version A for the list of Multiple Choice Questions*)

Exercise 2

(06 points)

The same questions of the version A.

Exercise 3

(08 points)

Let A be the matrix given by:

$$A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}$$

1. Calculate A^2 .
2. Show that $A^2 = -A + 2I_3$.
3. Deduce that A is invertible and write the expression of A^{-1} in terms of A and I_3 .
4. Calculate A^{-1} using the cofactors method.

Full Solution – Version B

Solution – Exercise 1

1. **Refer to Version A solution** for the detailed justifications of the Multiple Choice Questions.

Solution – Exercise 2

1. Refer to Version A solution.

Solution – Exercise 3

Let A be the matrix given by:

$$A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}$$

1. Calculate A^2 .

$$A^2 = A \cdot A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}$$

Compute each entry:

Row 1:

$$(A^2)_{11} = 1 \cdot 1 + (-3) \cdot 6 + 6 \cdot 3 = 1 - 18 + 18 = 1$$

$$(A^2)_{12} = 1 \cdot (-3) + (-3) \cdot (-8) + 6 \cdot (-3) = -3 + 24 - 18 = 3$$

$$(A^2)_{13} = 1 \cdot 6 + (-3) \cdot 12 + 6 \cdot 4 = 6 - 36 + 24 = -6$$

Row 2:

$$(A^2)_{21} = 6 \cdot 1 + (-8) \cdot 6 + 12 \cdot 3 = 6 - 48 + 36 = -6$$

$$(A^2)_{22} = 6 \cdot (-3) + (-8) \cdot (-8) + 12 \cdot (-3) = -18 + 64 - 36 = 10$$

$$(A^2)_{23} = 6 \cdot 6 + (-8) \cdot 12 + 12 \cdot 4 = 36 - 96 + 48 = -12$$

Row 3:

$$(A^2)_{31} = 3 \cdot 1 + (-3) \cdot 6 + 4 \cdot 3 = 3 - 18 + 12 = -3$$

$$(A^2)_{32} = 3 \cdot (-3) + (-3) \cdot (-8) + 4 \cdot (-3) = -9 + 24 - 12 = 3$$

$$(A^2)_{33} = 3 \cdot 6 + (-3) \cdot 12 + 4 \cdot 4 = 18 - 36 + 16 = -2$$

Thus:

$$A^2 = \begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix}.$$

$$\boxed{A^2 = \begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix}}.$$

2. Show that $A^2 = -A + 2I_3$.

Compute $-A + 2I_3$:

$$-A = \begin{pmatrix} -1 & 3 & -6 \\ -6 & 8 & -12 \\ -3 & 3 & -4 \end{pmatrix}, \quad 2I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$-A + 2I_3 = \begin{pmatrix} -1+2 & 3+0 & -6+0 \\ -6+0 & 8+2 & -12+0 \\ -3+0 & 3+0 & -4+2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix}$$

This is exactly A^2 . Therefore:

$$\boxed{A^2 = -A + 2I_3}.$$

3. Deduce that A is invertible and write A^{-1} in terms of A and I_3 .

From $A^2 = -A + 2I_3$, we have:

$$A^2 + A = 2I_3$$

$$A(A + I_3) = 2I_3$$

$$A \cdot \frac{1}{2}(A + I_3) = I_3.$$

Thus A is invertible and:

$$\boxed{A^{-1} = \frac{1}{2}(A + I_3)}.$$

4. Calculate A^{-1} using the cofactors method.

First compute $\det A$ using the relation: from $A^2 = -A + 2I$, take determinant:

$$|A^2| = |-A + 2I|$$

But $|A^2| = |A|^2$, so:

$$|A|^2 = |-A + 2I|.$$

Also, $|-A + 2I| = |-(A - 2I)| = (-1)^3|A - 2I| = -|A - 2I|$. But easier: from $A(A + I) = 2I$, taking determinant:

$$|A| \cdot |A + I| = |2I| = 2^3 = 8.$$

Compute $A + I$:

$$A + I = \begin{pmatrix} 2 & -3 & 6 \\ 6 & -7 & 12 \\ 3 & -3 & 5 \end{pmatrix}$$

$$\det(A + I) = 2 \cdot \begin{vmatrix} -7 & 12 \\ -3 & 5 \end{vmatrix} - (-3) \cdot \begin{vmatrix} 6 & 12 \\ 3 & 5 \end{vmatrix} + 6 \cdot \begin{vmatrix} 6 & -7 \\ 3 & -3 \end{vmatrix}$$

$$= 2 \cdot (-35 + 36) + 3 \cdot (30 - 36) + 6 \cdot (-18 + 21)$$

$$= 2 \cdot 1 + 3 \cdot (-6) + 6 \cdot 3 = 2 - 18 + 18 = 2.$$

Thus $|A| \cdot 2 = 8 \Rightarrow |A| = 4$.

Now compute cofactor matrix of A :

$$\begin{aligned} C_{11} &= (-1)^2 \begin{vmatrix} -8 & 12 \\ -3 & 4 \end{vmatrix} = 1 \cdot (-32 + 36) = 4 \\ C_{12} &= (-1)^3 \begin{vmatrix} 6 & 12 \\ 3 & 4 \end{vmatrix} = -1 \cdot (24 - 36) = 12 \\ C_{13} &= (-1)^4 \begin{vmatrix} 6 & -8 \\ 3 & -3 \end{vmatrix} = 1 \cdot (-18 + 24) = 6 \\ C_{21} &= (-1)^3 \begin{vmatrix} -3 & 6 \\ -3 & 4 \end{vmatrix} = -1 \cdot (-12 + 18) = -6 \\ C_{22} &= (-1)^4 \begin{vmatrix} 1 & 6 \\ 3 & 4 \end{vmatrix} = 1 \cdot (4 - 18) = -14 \\ C_{23} &= (-1)^5 \begin{vmatrix} 1 & -3 \\ 3 & -3 \end{vmatrix} = -1 \cdot (-3 + 9) = -6 \\ C_{31} &= (-1)^4 \begin{vmatrix} -3 & 6 \\ -8 & 12 \end{vmatrix} = 1 \cdot (-36 + 48) = 12 \\ C_{32} &= (-1)^5 \begin{vmatrix} 1 & 6 \\ 6 & 12 \end{vmatrix} = -1 \cdot (12 - 36) = 24 \\ C_{33} &= (-1)^6 \begin{vmatrix} 1 & -3 \\ 6 & -8 \end{vmatrix} = 1 \cdot (-8 + 18) = 10 \end{aligned}$$

Cofactor matrix:

$$C = \begin{pmatrix} 4 & 12 & 6 \\ -6 & -14 & -6 \\ 12 & 24 & 10 \end{pmatrix}$$

Adjugate $= C^T$:

$$\text{adj}(A) = \begin{pmatrix} 4 & -6 & 12 \\ 12 & -14 & 24 \\ 6 & -6 & 10 \end{pmatrix}$$

Thus:

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{4} \begin{pmatrix} 4 & -6 & 12 \\ 12 & -14 & 24 \\ 6 & -6 & 10 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{pmatrix}.$$

Now check with the formula from part 3: $A^{-1} = \frac{1}{2}(A + I)$:

$$A + I = \begin{pmatrix} 2 & -3 & 6 \\ 6 & -7 & 12 \\ 3 & -3 & 5 \end{pmatrix}, \quad \frac{1}{2}(A + I) = \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{pmatrix}.$$

Both methods give the same result.

$$A^{-1} = \frac{1}{2}(A + I) = \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{pmatrix}.$$

1.8 Main Exam (2021/2022)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Final Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 22/05/2022

Duration: 1h 30min

Exercise 1

(10 points)

I. Let the linear system (S_1) :

$$(S_1) \begin{cases} -x_1 + 2x_2 + x_3 = 3 \\ x_1 + x_2 + x_3 = 0 \\ x_1 - 4x_2 - 2x_3 = -1 \end{cases}$$

1. Write the system (S_1) in matrix form, then prove that it is a Cramer system.
2. Solve the system using the Gauss method.

II. Let m be a real number and (S_2) the linear system:

$$(S_2) \begin{cases} -mx_3 = m - 2 \\ -x_1 + (m+1)x_2 + 2x_3 = m - 2 \\ x_1 + x_2 = m - 2 \end{cases}$$

2. Find the values of m for which the system (S_2) is a Cramer system.
3. Solve the system (S_2) for $m = 2$.

Exercise 2

(10 points)

Let A be the matrix given by:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

1. Calculate A^{-1} using the cofactors method.
2. Find the real numbers α and β such that $A^2 = \alpha I_3 + \beta A$.

Full Solution

Solution – Exercise 1

I. Let the linear system (S_1) :

$$(S_1) \begin{cases} -x_1 + 2x_2 + x_3 = 3 \\ x_1 + x_2 + x_3 = 0 \\ x_1 - 4x_2 - 2x_3 = -1 \end{cases}$$

II. Let m be a real number and (S_2) the linear system:

$$(S_2) \begin{cases} -mx_3 = m - 2 \\ -x_1 + (m + 1)x_2 + 2x_3 = m - 2 \\ x_1 + x_2 = m - 2 \end{cases}$$

I. Solution of (S_1)

1. Matrix form and Cramer system verification.

The system (S_1) can be written as $AX = B$ where:

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -4 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}.$$

A system is Cramer if it has n equations, n unknowns, and $\det A \neq 0$. Here $n = 3$.

Compute $\det A$:

$$\det A = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -4 & -2 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= (-1) \cdot \begin{vmatrix} 1 & 1 \\ -4 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} \\ &= (-1) \cdot ((1)(-2) - (1)(-4)) - 2 \cdot ((1)(-2) - (1)(1)) + 1 \cdot ((1)(-4) - (1)(1)) \\ &= (-1) \cdot (-2 + 4) - 2 \cdot (-2 - 1) + 1 \cdot (-4 - 1) \\ &= (-1) \cdot 2 - 2 \cdot (-3) + 1 \cdot (-5) \\ &= -2 + 6 - 5 = -1. \end{aligned}$$

Since $\det A = -1 \neq 0$, the system (S_1) is a Cramer system.

2. Solve the system using the Gauss method.

Write the augmented matrix $[A|B]$:

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & -4 & -2 & -1 \end{array} \right].$$

Step 1: Swap R_1 and R_2 to get a leading 1:

$$R_1 \leftrightarrow R_2 : \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 2 & 1 & 3 \\ 1 & -4 & -2 & -1 \end{array} \right].$$

Step 2: Eliminate x_1 from rows 2 and 3: $R_2 \leftarrow R_2 + R_1$:

$$(-1 + 1, 2 + 1, 1 + 1, 3 + 0) = (0, 3, 2, 3)$$

$R_3 \leftarrow R_3 - R_1$:

$$(1 - 1, -4 - 1, -2 - 1, -1 - 0) = (0, -5, -3, -1)$$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 3 \\ 0 & -5 & -3 & -1 \end{array} \right].$$

Step 3: Eliminate x_2 from row 3: $R_3 \leftarrow 3R_3 + 5R_2$ (to avoid fractions):

$$3R_3 = (0, -15, -9, -3), \quad 5R_2 = (0, 15, 10, 15)$$

Sum: $(0, 0, 1, 12)$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 1 & 12 \end{array} \right].$$

Step 4: Back substitution.

From R_3 : $x_3 = 12$.

From R_2 : $3x_2 + 2x_3 = 3 \Rightarrow 3x_2 + 24 = 3 \Rightarrow 3x_2 = -21 \Rightarrow x_2 = -7$.

From R_1 : $x_1 + x_2 + x_3 = 0 \Rightarrow x_1 - 7 + 12 = 0 \Rightarrow x_1 + 5 = 0 \Rightarrow x_1 = -5$.

Thus the solution is:

$$\boxed{x_1 = -5, \quad x_2 = -7, \quad x_3 = 12}.$$

Verification:

$$-(-5) + 2(-7) + 12 = 5 - 14 + 12 = 3 \quad \checkmark$$

$$-5 + (-7) + 12 = 0 \quad \checkmark$$

$$-5 - 4(-7) - 2(12) = -5 + 28 - 24 = -1 \quad \checkmark$$

II. Solution of (S_2)

2. Find the values of m for which the system (S_2) is a Cramer system.

Write (S_2) in matrix form $A_m X = B_m$:

$$\begin{cases} 0x_1 + 0x_2 - mx_3 = m - 2 \\ -x_1 + (m+1)x_2 + 2x_3 = m - 2 \\ x_1 + x_2 + 0x_3 = m - 2 \end{cases}$$

Thus:

$$A_m = \begin{pmatrix} 0 & 0 & -m \\ -1 & m+1 & 2 \\ 1 & 1 & 0 \end{pmatrix}, \quad B_m = \begin{pmatrix} m-2 \\ m-2 \\ m-2 \end{pmatrix}.$$

The system is Cramer if $\det A_m \neq 0$.

Compute $\det A_m$:

$$\det A_m = \begin{vmatrix} 0 & 0 & -m \\ -1 & m+1 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_m &= 0 \cdot (\text{minor}) - 0 \cdot (\text{minor}) + (-m) \cdot (-1)^{1+3} \begin{vmatrix} -1 & m+1 \\ 1 & 1 \end{vmatrix} \\ &= (-m) \cdot 1 \cdot ((-1)(1) - (m+1)(1)) \\ &= -m \cdot (-1 - m - 1) = -m \cdot (-m - 2) \\ &= m(m+2). \end{aligned}$$

Thus $\det A_m = m(m+2)$.

The system is Cramer iff $\det A_m \neq 0$, i.e.:

$$m \neq 0 \quad \text{and} \quad m \neq -2.$$

$$\boxed{m \in \mathbb{R} \setminus \{-2, 0\}}.$$

3. Solve the system (S_2) for $m = 2$.

For $m = 2$, the system becomes:

$$\begin{cases} -2x_3 = 0 \\ -x_1 + 3x_2 + 2x_3 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

From the first equation: $-2x_3 = 0 \Rightarrow x_3 = 0$.

From the third equation: $x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$.

Substitute into the second equation: $-x_1 + 3(-x_1) + 2(0) = 0 \Rightarrow -x_1 - 3x_1 = 0 \Rightarrow -4x_1 = 0 \Rightarrow x_1 = 0$.

Then $x_2 = -x_1 = 0$.

Thus the unique solution is:

$$\boxed{x_1 = 0, \quad x_2 = 0, \quad x_3 = 0}.$$

Solution – Exercise 2

Let A be the matrix given by:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

1. Calculate A^{-1} using the cofactors method.

First compute $\det A$:

$$\det A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= 0 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ &= 0 - 1 \cdot ((1)(0) - (1)(1)) + 1 \cdot ((1)(1) - (0)(1)) \\ &= -1 \cdot (0 - 1) + 1 \cdot (1 - 0) \\ &= -1 \cdot (-1) + 1 \cdot 1 = 1 + 1 = 2. \end{aligned}$$

Thus $\det A = 2$.

Now compute the cofactor matrix C of A :

$$\begin{aligned}
C_{11} &= (-1)^2 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 1) = -1 \\
C_{12} &= (-1)^3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \cdot (0 - 1) = 1 \\
C_{13} &= (-1)^4 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \cdot (1 - 0) = 1 \\
C_{21} &= (-1)^3 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \cdot (0 - 1) = 1 \\
C_{22} &= (-1)^4 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 1) = -1 \\
C_{23} &= (-1)^5 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \cdot (0 - 1) = 1 \\
C_{31} &= (-1)^4 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1 - 0) = 1 \\
C_{32} &= (-1)^5 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \cdot (0 - 1) = 1 \\
C_{33} &= (-1)^6 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 1) = -1
\end{aligned}$$

Thus the cofactor matrix is:

$$C = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

The adjugate matrix is C^T :

$$\text{adj}(A) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Notice that C is symmetric, so $\text{adj}(A) = C$.

Therefore:

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

$$\boxed{A^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}}.$$

2. Find real numbers α and β such that $A^2 = \alpha I_3 + \beta A$.

First compute A^2 :

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Compute each entry:

Row 1:

$$(A^2)_{11} = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0 + 1 + 1 = 2$$

$$(A^2)_{12} = 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 = 0 + 0 + 1 = 1$$

$$(A^2)_{13} = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 0 + 1 + 0 = 1$$

Row 2:

$$(A^2)_{21} = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 0 + 0 + 1 = 1$$

$$(A^2)_{22} = 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 = 1 + 0 + 1 = 2$$

$$(A^2)_{23} = 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 1 + 0 + 0 = 1$$

Row 3:

$$(A^2)_{31} = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 0 + 1 + 0 = 1$$

$$(A^2)_{32} = 1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 1 + 0 + 0 = 1$$

$$(A^2)_{33} = 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0 = 1 + 1 + 0 = 2$$

Thus:

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Now we want α and β such that $A^2 = \alpha I_3 + \beta A$.

Write $\alpha I_3 + \beta A$:

$$\alpha I_3 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad \beta A = \begin{pmatrix} 0 & \beta & \beta \\ \beta & 0 & \beta \\ \beta & \beta & 0 \end{pmatrix}$$

$$\alpha I_3 + \beta A = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}.$$

Set this equal to $A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Comparing entries, we get:

$$\alpha = 2, \quad \beta = 1.$$

Verification: $\alpha I_3 + \beta A = 2I_3 + A$ gives 2 on the diagonal and 1 elsewhere, which matches A^2 .

$$\boxed{\alpha = 2, \quad \beta = 1}.$$

1.9 Main Exam (2022/2023)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Main Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 23/05/2023

Duration: 1h 30min

Exercise 1

(06 points)

Let A and B be two matrices given by:

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ -1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

1. Calculate $\det(A)$, $\det(B)$, $\det(A^T)$, and $\det(A + B)$.
2. Is $\det(A) = \det(A^T)$?
3. Is $\det(A + B) = \det(A) + \det(B)$?

Exercise 2

(08 points)

Let A and B be two matrices given by:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -1 & 3 \\ 3 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

1. Are A and B invertible?
2. Calculate A^{-1} then deduce B^{-1} .

Exercise 3

(06 points)

Let the linear system (S) :

$$(S) \begin{cases} -x_1 + 2x_2 + x_3 = 2 \\ x_1 + x_2 - x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 3 \end{cases}$$

1. Write the system (S) in matrix form, then prove that it is a Cramer system.
2. Solve the system (S) using the Gauss method.

Solution – Main Exam (2022/2023)

Solution – Exercise 1

Let A and B be two matrices given by:

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ -1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

1. Calculate the determinants.

Compute $\det(A)$:

$$\det(A) = \begin{vmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$

Expand by row 3 (contains a zero):

$$\begin{aligned} \det(A) &= (-1) \cdot (-1)^{3+1} \begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} + 2 \cdot (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + 0 \cdot (\text{minor}) \\ &= (-1) \cdot 1 \cdot ((-2)(2) - (1)(-3)) + 2 \cdot (-1) \cdot ((2)(2) - (1)(2)) \\ &= (-1) \cdot (-4 + 3) - 2 \cdot (4 - 2) \\ &= (-1) \cdot (-1) - 2 \cdot 2 = 1 - 4 = -3. \end{aligned}$$

Thus $\det(A) = -3$.

Compute $\det(B)$: B is an upper triangular matrix (all entries below the diagonal are zero). For a triangular matrix, the determinant is the product of the diagonal entries:

$$\det(B) = 1 \times 1 \times 1 = 1.$$

Compute $\det(A^T)$: We know that for any square matrix, $\det(A^T) = \det(A)$. Thus:

$$\det(A^T) = \det(A) = -3.$$

Compute $\det(A + B)$: First compute $A + B$:

$$A + B = \begin{pmatrix} 2+1 & -2+2 & 1+3 \\ 2+0 & -3+1 & 2+2 \\ -1+0 & 2+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 4 \\ 2 & -2 & 4 \\ -1 & 2 & 1 \end{pmatrix}.$$

Now compute $\det(A + B)$:

$$\det(A + B) = \begin{vmatrix} 3 & 0 & 4 \\ 2 & -2 & 4 \\ -1 & 2 & 1 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det(A + B) &= 3 \cdot \begin{vmatrix} -2 & 4 \\ 2 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix} + 4 \cdot \begin{vmatrix} 2 & -2 \\ -1 & 2 \end{vmatrix} \\
 &= 3 \cdot ((-2)(1) - (4)(2)) + 4 \cdot ((2)(2) - (-2)(-1)) \\
 &= 3 \cdot (-2 - 8) + 4 \cdot (4 - 2) \\
 &= 3 \cdot (-10) + 4 \cdot 2 = -30 + 8 = -22.
 \end{aligned}$$

Thus $\det(A + B) = -22$.

$$\boxed{\det(A) = -3, \quad \det(B) = 1, \quad \det(A^T) = -3, \quad \det(A + B) = -22}.$$

2. Is $\det(A) = \det(A^T)$?

We have $\det(A) = -3$ and $\det(A^T) = -3$. Thus $\det(A) = \det(A^T)$.

This is true for every square matrix, not just this one.

$$\boxed{\text{Yes, } \det(A) = \det(A^T)}.$$

3. Is $\det(A + B) = \det(A) + \det(B)$?

We have $\det(A + B) = -22$, $\det(A) = -3$, $\det(B) = 1$.

Then $\det(A) + \det(B) = -3 + 1 = -2 \neq -22$.

Thus $\det(A + B) \neq \det(A) + \det(B)$.

$$\boxed{\text{No, } \det(A + B) \neq \det(A) + \det(B)}.$$

In general, the determinant is not additive.

Solution – Exercise 2

Let A and B be two matrices given by:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & -1 & 3 \\ 3 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

1. Check if A and B are invertible.

A square matrix is invertible if and only if its determinant is non-zero.

First compute $\det A$:

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 3 & 1 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det A &= 1 \cdot \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\
 &= 1 \cdot ((1)(1) - (-1)(3)) - 2 \cdot ((1)(1) - (-1)(2)) + 1 \cdot ((1)(3) - (1)(2)) \\
 &= 1 \cdot (1 + 3) - 2 \cdot (1 + 2) + 1 \cdot (3 - 2) \\
 &= 4 - 2 \cdot 3 + 1 = 4 - 6 + 1 = -1.
 \end{aligned}$$

Thus $\det A = -1 \neq 0$, so A is invertible.

Now compute $\det B$:

$$\det B = \begin{vmatrix} -4 & -1 & 3 \\ 3 & 1 & -2 \\ -1 & -1 & 1 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det B &= (-4) \cdot \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & 1 \\ -1 & -1 \end{vmatrix} \\
 &= (-4) \cdot ((1)(1) - (-2)(-1)) + 1 \cdot ((3)(1) - (-2)(-1)) + 3 \cdot ((3)(-1) - (1)(-1)) \\
 &= (-4) \cdot (1 - 2) + 1 \cdot (3 - 2) + 3 \cdot (-3 + 1) \\
 &= (-4) \cdot (-1) + 1 \cdot 1 + 3 \cdot (-2) \\
 &= 4 + 1 - 6 = -1.
 \end{aligned}$$

Thus $\det B = -1 \neq 0$, so B is also invertible.

Both A and B are invertible.

2. Calculate A^{-1} then deduce B^{-1} .

We compute A^{-1} using the adjugate method.

First, we have $\det A = -1$.

Compute the cofactor matrix C of A :

$$C_{11} = (-1)^2 \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 1 \cdot (1 + 3) = 4$$

$$C_{12} = (-1)^3 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -1 \cdot (1 + 2) = -3$$

$$C_{13} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \cdot (3 - 2) = 1$$

$$C_{21} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1 \cdot (2 - 3) = 1$$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 \cdot (1 - 2) = -1$$

$$C_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \cdot (3 - 4) = 1$$

$$C_{31} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 1 \cdot (-2 - 1) = -3$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 \cdot (-1 - 1) = 2$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \cdot (1 - 2) = -1$$

Cofactor matrix:

$$C = \begin{pmatrix} 4 & -3 & 1 \\ 1 & -1 & 1 \\ -3 & 2 & -1 \end{pmatrix}$$

Adjugate matrix = C^T :

$$\text{adj}(A) = \begin{pmatrix} 4 & 1 & -3 \\ -3 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

Thus:

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{-1} \begin{pmatrix} 4 & 1 & -3 \\ -3 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -4 & -1 & 3 \\ 3 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Notice that A^{-1} is exactly the matrix B given in the problem.

$$\boxed{A^{-1} = \begin{pmatrix} -4 & -1 & 3 \\ 3 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}}.$$

Since $B = A^{-1}$, we have $B^{-1} = (A^{-1})^{-1} = A$.

$$\boxed{B^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix}}.$$

Solution – Exercise 3

Let the linear system (S) :

$$(S) \begin{cases} -x_1 + 2x_2 + x_3 = 2 \\ x_1 + x_2 - x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 3 \end{cases}$$

1. Matrix form and Cramer system verification.

The system can be written as $AX = B$ where:

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}.$$

A system is Cramer if it has n equations, n unknowns, and $\det A \neq 0$. Here $n = 3$.

Compute $\det A$:

$$\det A = \begin{vmatrix} -1 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & 3 & 1 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= (-1) \cdot \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\ &= (-1) \cdot ((1)(1) - (-1)(3)) - 2 \cdot ((1)(1) - (-1)(2)) + 1 \cdot ((1)(3) - (1)(2)) \\ &= (-1) \cdot (1 + 3) - 2 \cdot (1 + 2) + 1 \cdot (3 - 2) \\ &= (-1) \cdot 4 - 2 \cdot 3 + 1 \cdot 1 \\ &= -4 - 6 + 1 = -9. \end{aligned}$$

Since $\det A = -9 \neq 0$, the system is a Cramer system.

2. Solve the system using the Gauss method.

Write the augmented matrix $[A|B]$:

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 2 \\ 1 & 1 & -1 & 0 \\ 2 & 3 & 1 & 3 \end{array} \right].$$

Step 1: Swap R_1 and R_2 to get a leading 1:

$$R_1 \leftrightarrow R_2 : \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 3 \end{array} \right].$$

Step 2: Eliminate x_1 from rows 2 and 3: $R_2 \leftarrow R_2 + R_1$:

$$(-1 + 1, 2 + 1, 1 + (-1), 2 + 0) = (0, 3, 0, 2)$$

$$R_3 \leftarrow R_3 - 2R_1:$$

$$(2 - 2, 3 - 2, 1 + 2, 3 - 0) = (0, 1, 3, 3)$$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 1 & 3 & 3 \end{array} \right].$$

Step 3: Simplify R_2 by dividing by 3: $R_2 \leftarrow \frac{1}{3}R_2$:

$$(0, 1, 0, \frac{2}{3})$$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 1 & 3 & 3 \end{array} \right].$$

Step 4: Eliminate x_2 from row 3: $R_3 \leftarrow R_3 - R_2$:

$$(0, 1 - 1, 3 - 0, 3 - \frac{2}{3}) = (0, 0, 3, \frac{7}{3})$$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 3 & \frac{7}{3} \end{array} \right].$$

Step 5: Back substitution.

$$\text{From } R_3: 3x_3 = \frac{7}{3} \Rightarrow x_3 = \frac{7}{9}.$$

$$\text{From } R_2: x_2 = \frac{2}{3}.$$

$$\begin{aligned} \text{From } R_1: x_1 + x_2 - x_3 &= 0 \Rightarrow x_1 + \frac{2}{3} - \frac{7}{9} = 0 \\ \Rightarrow x_1 + \frac{6}{9} - \frac{7}{9} &= 0 \Rightarrow x_1 - \frac{1}{9} = 0 \Rightarrow x_1 = \frac{1}{9}. \end{aligned}$$

Thus the solution is:

$$\boxed{x_1 = \frac{1}{9}, \quad x_2 = \frac{2}{3}, \quad x_3 = \frac{7}{9}}.$$

Verification:

$$-\frac{1}{9} + 2\left(\frac{2}{3}\right) + \frac{7}{9} = -\frac{1}{9} + \frac{4}{3} + \frac{7}{9} = -\frac{1}{9} + \frac{12}{9} + \frac{7}{9} = \frac{18}{9} = 2 \quad \checkmark$$

$$\frac{1}{9} + \frac{2}{3} - \frac{7}{9} = \frac{1}{9} + \frac{6}{9} - \frac{7}{9} = 0 \quad \checkmark$$

$$2\left(\frac{1}{9}\right) + 3\left(\frac{2}{3}\right) + \frac{7}{9} = \frac{2}{9} + 2 + \frac{7}{9} = \frac{9}{9} + 2 = 1 + 2 = 3 \quad \checkmark$$

1.10 Main Exam (2023/2024)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Final Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 22/05/2024 **Duration:** 1h 30min

Exercise 1

(06 points)

Consider the following 3×3 matrix A with parameters a and b :

$$A = \begin{pmatrix} 1 & 2b & -a \\ 0 & 3 & b \\ 2 & 0 & a \end{pmatrix}$$

1. Using Sarrus' rule, find the determinant of matrix A in terms of a and b .
2. For which values of a and b is the matrix A symmetric ($A = A^T$)?
3. For which values of a and b is the matrix A lower triangular?
4. Calculate the trace $tr(A)$.
5. Determine the values of a and b for which $tr(A) = 4$ and the matrix A is singular.
6. Can we find values of a and b for which the matrix A is a diagonal matrix? Justify your answer.

Exercise 2

(06 points)

Let A be the matrix given by: $A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$

1. Determine the characteristic polynomial of A .
2. Find the eigenvalues and eigenvectors for A .
3. Find the matrices D , P , and P^{-1} such that $A = PDP^{-1}$ (where D is diagonal and P is invertible).

Exercise 3**(08 points)**

Let (S) be a system of linear equations:

$$(S) \begin{cases} -x - 3y + 6z = 1 \\ 6x - 8y + 12z = 2 \\ 3x - 3y + 4z = 0 \end{cases}$$

1. Rewrite the system (S) in the matrix form $AX = B$.
2. Using the cofactors method, show that $\det(A) = -4$, then conclude about the nature of the system (S) .
3. Using Cramer's method, show that (S) has a unique solution: $x = 2, y = 8, z = \frac{9}{2}$.
4. Deduce the solutions of the system (S') for variables x_1, x_2, x_3 :

$$(S') \begin{cases} -e^{x_1} - 3 \ln(x_2) + \frac{6}{5}x_3 = 1 \\ 6e^{x_1} - 8 \ln(x_2) + \frac{12}{5}x_3 = 2 \\ 3e^{x_1} - 3 \ln(x_2) + \frac{4}{5}x_3 = 0 \end{cases}$$

Full Solution**Solution – Exercise 1**

Consider the following 3×3 matrix A with parameters a and b :

$$A = \begin{pmatrix} 1 & 2b & -a \\ 0 & 3 & b \\ 2 & 0 & a \end{pmatrix}$$

1. Using Sarrus' rule, find $\det A$ in terms of a and b .

Sarrus' rule for a 3×3 matrix:

$$\det A = \sum_{\text{main diagonals}} - \sum_{\text{anti-diagonals}}$$

Write the matrix with the first two columns repeated:

$$\begin{array}{ccc|cc} 1 & 2b & -a & 1 & 2b \\ 0 & 3 & b & 0 & 3 \\ 2 & 0 & a & 2 & 0 \end{array}$$

Main diagonals (from top-left to bottom-right):

$$\begin{aligned} D_1 &= 1 \cdot 3 \cdot a = 3a \\ D_2 &= 2b \cdot b \cdot 2 = 4b^2 \\ D_3 &= (-a) \cdot 0 \cdot 0 = 0 \end{aligned}$$

Sum of main diagonals: $3a + 4b^2$.

Anti-diagonals (from top-right to bottom-left):

$$A_1 = (-a) \cdot 3 \cdot 2 = -6a$$

$$A_2 = 1 \cdot b \cdot 0 = 0$$

$$A_3 = 2b \cdot 0 \cdot a = 0$$

Sum of anti-diagonals: $-6a$.

Therefore:

$$\det A = (3a + 4b^2) - (-6a) = 3a + 4b^2 + 6a = 9a + 4b^2.$$

$$\boxed{\det A = 9a + 4b^2}.$$

2. For which values of a and b is A symmetric ($A = A^T$)?

Compute A^T :

$$A^T = \begin{pmatrix} 1 & 0 & 2 \\ 2b & 3 & 0 \\ -a & b & a \end{pmatrix}.$$

For $A = A^T$, we compare entry by entry:

$$(1, 2): 2b = 0 \Rightarrow b = 0.$$

$$(1, 3): -a = 2 \Rightarrow a = -2.$$

$$(2, 3): b = 0 \text{ (already satisfied).}$$

$$\text{Check } (2, 1): 0 = 2b = 0 \text{ OK. } (3, 1): 2 = -a = 2 \text{ OK. } (3, 2): 0 = b = 0 \text{ OK.}$$

Thus:

$$\boxed{a = -2, b = 0}.$$

3. For which values of a and b is A lower triangular?

A matrix is lower triangular if all entries above the main diagonal are zero.

$$\text{Entries above the diagonal: } (1, 2) = 2b = 0 \Rightarrow b = 0, (1, 3) = -a = 0 \Rightarrow a = 0, \\ (2, 3) = b = 0 \text{ (already satisfied).}$$

Thus:

$$\boxed{a = 0, b = 0}.$$

4. Calculate $\text{tr}(A)$.

The trace is the sum of the diagonal entries:

$$\text{tr}(A) = 1 + 3 + a = 4 + a.$$

$$\boxed{\text{tr}(A) = 4 + a}.$$

5. Determine a and b such that $\text{tr}(A) = 4$ and A is singular.

$$\text{tr}(A) = 4 \Rightarrow 4 + a = 4 \Rightarrow a = 0.$$

$$A \text{ singular means } \det A = 0 \Rightarrow 9a + 4b^2 = 0.$$

$$\text{Substitute } a = 0: 4b^2 = 0 \Rightarrow b = 0.$$

Thus:

$$\boxed{a = 0, b = 0}.$$

For $a = 0, b = 0$, check: $\text{tr}(A) = 4, \det A = 0$.

6. Can we find values of a and b such that A is a diagonal matrix? Justify.

A diagonal matrix must have all off-diagonal entries equal to zero:

$$(1, 2) = 2b = 0 \Rightarrow b = 0, (1, 3) = -a = 0 \Rightarrow a = 0, (2, 1) = 0 \text{ OK}, (2, 3) = b = 0 \text{ OK}, (3, 1) = 2 \neq 0 \text{ — fails regardless of } a \text{ and } b.$$

Thus it is impossible for A to be diagonal because the $(3, 1)$ entry is always 2, never zero.

$$\boxed{\text{No, because } A_{31} = 2 \neq 0 \text{ for all } a, b}.$$

Solution – Exercise 2

Let A be the matrix given by:

$$A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$$

1. Characteristic polynomial of A .

$$\begin{aligned} P_A(\lambda) &= \det(A - \lambda I_2) = \begin{vmatrix} 2 - \lambda & -12 \\ 1 & -5 - \lambda \end{vmatrix} \\ &= (2 - \lambda)(-5 - \lambda) - (-12)(1) \\ &= (2 - \lambda)(-5 - \lambda) + 12 \\ &= -10 - 2\lambda + 5\lambda + \lambda^2 + 12 \\ &= \lambda^2 + 3\lambda + 2. \end{aligned}$$

Thus:

$$\boxed{P_A(\lambda) = \lambda^2 + 3\lambda + 2}.$$

2. Find the eigenvalues and eigenvectors.

Solve $P_A(\lambda) = 0$:

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -2.$$

Eigenvector for $\lambda_1 = -1$:

$$(A + 1 \cdot I_2)v = 0 \Rightarrow \begin{pmatrix} 2+1 & -12 \\ 1 & -5+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 3 & -12 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

From the second row: $x - 4y = 0 \Rightarrow x = 4y$. Choose $y = 1$, $x = 4$.

$$v_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = -2$:

$$(A + 2 \cdot I_2)v = 0 \Rightarrow \begin{pmatrix} 2+2 & -12 \\ 1 & -5+2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 4 & -12 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

From the second row: $x - 3y = 0 \Rightarrow x = 3y$. Choose $y = 1$, $x = 3$.

$$v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Thus:

$$\boxed{\lambda_1 = -1, v_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}; \quad \lambda_2 = -2, v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}.$$

3. Find D , P , and P^{-1} such that $A = PDP^{-1}$.

Let D be the diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Let P be the matrix with eigenvectors as columns:

$$P = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}.$$

Now compute P^{-1} :

$$\det P = (4)(1) - (3)(1) = 4 - 3 = 1.$$

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}.$$

Verification:

$$PDP^{-1} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$$

First compute $DP^{-1} = \begin{pmatrix} -1 & 3 \\ 2 & -8 \end{pmatrix}$.

Then $P(DP^{-1}) = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -8 \end{pmatrix} = \begin{pmatrix} -4+6 & 12-24 \\ -1+2 & 3-8 \end{pmatrix} = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} = A$.

Thus:

$$\boxed{D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad P = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}.$$

Let (S) be a system of linear equations:

$$(S) \begin{cases} -x - 3y + 6z = 1 \\ 6x - 8y + 12z = 2 \\ 3x - 3y + 4z = 0 \end{cases}$$

Deduce the solutions of the system (S') for variables x_1, x_2, x_3 :

$$(S') \begin{cases} -e^{x_1} - 3 \ln(x_2) + \frac{6}{5}x_3 = 1 \\ 6e^{x_1} - 8 \ln(x_2) + \frac{12}{5}x_3 = 2 \\ 3e^{x_1} - 3 \ln(x_2) + \frac{4}{5}x_3 = 0 \end{cases}$$

1. Matrix form $AX = B$.

$$A = \begin{pmatrix} -1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

2. Using the cofactors method, show $\det(A) = -4$, then conclude about the nature of (S) .

Compute $\det(A)$ using cofactor expansion along row 1:

$$\det(A) = \begin{vmatrix} -1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{vmatrix}$$

$$\begin{aligned} \det(A) &= (-1) \cdot \begin{vmatrix} -8 & 12 \\ -3 & 4 \end{vmatrix} - (-3) \cdot \begin{vmatrix} 6 & 12 \\ 3 & 4 \end{vmatrix} + 6 \cdot \begin{vmatrix} 6 & -8 \\ 3 & -3 \end{vmatrix} \\ &= (-1) \cdot ((-8)(4) - (12)(-3)) + 3 \cdot ((6)(4) - (12)(3)) + 6 \cdot ((6)(-3) - (-8)(3)) \\ &= (-1) \cdot (-32 + 36) + 3 \cdot (24 - 36) + 6 \cdot (-18 + 24) \\ &= (-1) \cdot 4 + 3 \cdot (-12) + 6 \cdot 6 \\ &= -4 - 36 + 36 = -4. \end{aligned}$$

Thus $\det(A) = -4 \neq 0$, so the system (S) is a Cramer system and has a unique solution.

3. Using Cramer's method, show that (S) has a unique solution: $x = 2$, $y = 8$, $z = \frac{9}{2}$.

We have $\det(A) = -4$.

Compute $\det(A_x)$ (replace column 1 with B):

$$A_x = \begin{pmatrix} 1 & -3 & 6 \\ 2 & -8 & 12 \\ 0 & -3 & 4 \end{pmatrix}$$

Expand by row 3:

$$\begin{aligned} \det(A_x) &= 0 \cdot (\text{minor}_{31}) + (-3) \cdot (-1)^{3+2} \begin{vmatrix} 1 & 6 \\ 2 & 12 \end{vmatrix} + 4 \cdot (-1)^{3+3} \begin{vmatrix} 1 & -3 \\ 2 & -8 \end{vmatrix} \\ &= (-3) \cdot (-1) \cdot ((1)(12) - (6)(2)) + 4 \cdot 1 \cdot ((1)(-8) - (-3)(2)) \\ &= 3 \cdot (12 - 12) + 4 \cdot (-8 + 6) \\ &= 3 \cdot 0 + 4 \cdot (-2) = -8. \end{aligned}$$

$$\text{Thus } x = \frac{\det(A_x)}{\det(A)} = \frac{-8}{-4} = 2.$$

Compute $\det(A_y)$ (replace column 2 with B):

$$A_y = \begin{pmatrix} -1 & 1 & 6 \\ 6 & 2 & 12 \\ 3 & 0 & 4 \end{pmatrix}$$

Expand by row 3:

$$\begin{aligned} \det(A_y) &= 3 \cdot (-1)^{3+1} \begin{vmatrix} 1 & 6 \\ 2 & 12 \end{vmatrix} + 0 \cdot (\text{minor}_{32}) + 4 \cdot (-1)^{3+3} \begin{vmatrix} -1 & 1 \\ 6 & 2 \end{vmatrix} \\ &= 3 \cdot 1 \cdot (12 - 12) + 4 \cdot 1 \cdot ((-1)(2) - (1)(6)) \\ &= 3 \cdot 0 + 4 \cdot (-2 - 6) = 4 \cdot (-8) = -32. \end{aligned}$$

$$\text{Thus } y = \frac{\det(A_y)}{\det(A)} = \frac{-32}{-4} = 8.$$

Compute $\det(A_z)$ (replace column 3 with B):

$$A_z = \begin{pmatrix} -1 & -3 & 1 \\ 6 & -8 & 2 \\ 3 & -3 & 0 \end{pmatrix}$$

Expand by row 3:

$$\begin{aligned} \det(A_z) &= 3 \cdot (-1)^{3+1} \begin{vmatrix} -3 & 1 \\ -8 & 2 \end{vmatrix} + (-3) \cdot (-1)^{3+2} \begin{vmatrix} -1 & 1 \\ 6 & 2 \end{vmatrix} + 0 \cdot (\text{minor}_{33}) \\ &= 3 \cdot 1 \cdot ((-3)(2) - (1)(-8)) + (-3) \cdot (-1) \cdot ((-1)(2) - (1)(6)) \\ &= 3 \cdot (-6 + 8) + 3 \cdot (-2 - 6) \\ &= 3 \cdot 2 + 3 \cdot (-8) = 6 - 24 = -18. \end{aligned}$$

$$\text{Thus } z = \frac{\det(A_z)}{\det(A)} = \frac{-18}{-4} = \frac{9}{2}.$$

Hence the unique solution is:

$$\boxed{x = 2, \quad y = 8, \quad z = \frac{9}{2}}.$$

4. Deduce the solutions of the system (S') .

The system (S') is:

$$\begin{cases} -e^{x_1} - 3\ln(x_2) + \frac{6}{5}x_3 = 1 \\ 6e^{x_1} - 8\ln(x_2) + \frac{12}{5}x_3 = 2 \\ 3e^{x_1} - 3\ln(x_2) + \frac{4}{5}x_3 = 0 \end{cases}$$

Let $X = e^{x_1}$, $Y = \ln(x_2)$, $Z = \frac{x_3}{5}$. Then (S') becomes:

$$\begin{cases} -X - 3Y + 6Z = 1 \\ 6X - 8Y + 12Z = 2 \\ 3X - 3Y + 4Z = 0 \end{cases}$$

This is exactly the linear system (S) with variables X, Y, Z .

From part 3, the solution is:

$$X = 2, \quad Y = 8, \quad Z = \frac{9}{2}.$$

Thus:

$$e^{x_1} = 2 \Rightarrow x_1 = \ln 2.$$

$$\ln(x_2) = 8 \Rightarrow x_2 = e^8.$$

$$\frac{x_3}{5} = \frac{9}{2} \Rightarrow x_3 = \frac{45}{2}.$$

Therefore, the unique solution of (S') is:

$$\boxed{x_1 = \ln 2, \quad x_2 = e^8, \quad x_3 = \frac{45}{2}}.$$

1.11 Main Exam (2024/2025)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Final Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 25/05/2025 **Duration:** 1h 30min

Exercise 1

(07 points)

Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

1. Using Sarrus' rule, find the determinant of A . Is A an invertible matrix?
2. Calculate A^{-1} using the adjoint matrix method.
3. Calculate A^2 and deduce $A^2 - 2A - 5I$.
4. Verify that $A(A^2 - 2A - 5I) = 24I$.
5. Deduce the matrix A^{-1} again.

Exercise 2

(07 points)

Let (S) be a system of linear equations in variables x, y and z :

$$(S) : \begin{cases} -x + 2y - z = 3 \\ x + y + z = 0 \\ x - 4y - 2z = 1 \end{cases}$$

1. Rewrite the system (S) in the matrix form $AX = b$.
2. Calculate $\det(A)$ using the cofactors method. What can we conclude about the system (S) ?
3. Using Cramer's method, show that the system (S) has a unique solution $x = 1, y = 1, z = -2$.

4. Deduce the solutions of the system (S') in variables x_1, x_2 and x_3 , such that:

$$(S') : \begin{cases} -x_1^2 + 2\sqrt{x_2} - \frac{x_3}{2} = 3 \\ x_1^2 + \sqrt{x_2} + \frac{x_3}{2} = 0 \\ x_1^2 - 4\sqrt{x_2} - x_3 = 1 \end{cases}$$

Exercise 3

(06 points)

Let A be the matrix given by:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

1. Determine the characteristic polynomial of the matrix A .
2. Find the eigenvalues and eigenvectors for the matrix A .
3. Find the matrices D, P and P^{-1} such that $A = PDP^{-1}$. (D is a diagonal matrix, P is an invertible matrix and P^{-1} is the inverse of P).

Full Solution

Solution – Exercise 1

Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

1. Determinant of A using Sarrus' rule.

Sarrus' rule for a 3×3 matrix:

$$\det A = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

For A :

$$\begin{aligned} \text{Sum1} &= (2)(1)(-1) + (5)(1)(2) + (-3)(2)(0) \\ &= -2 + 10 + 0 = 8 \end{aligned}$$

$$\begin{aligned} \text{Sum2} &= (-3)(1)(2) + (2)(1)(0) + (5)(2)(-1) \\ &= -6 + 0 - 10 = -16 \end{aligned}$$

$$\det A = \text{Sum1} - \text{Sum2} = 8 - (-16) = 8 + 16 = 24.$$

Since $\det A = 24 \neq 0$, A is invertible.

$$\boxed{\det A = 24, \quad A \text{ is invertible}}.$$

2. Calculate A^{-1} using the adjoint matrix method.

We need $\text{adj}(A) = (C_{ij})^T$, where C_{ij} are cofactors.

Compute all cofactors:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = 1 \cdot (1(-1) - 1(0)) = -1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = -1 \cdot (2(-1) - 1(2)) = -1 \cdot (-2 - 2) = -1 \cdot (-4) = 4$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = 1 \cdot (2(0) - 1(2)) = 0 - 2 = -2$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 5 & -3 \\ 0 & -1 \end{vmatrix} = -1 \cdot (5(-1) - (-3)(0)) = -1 \cdot (-5 - 0) = 5$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & -3 \\ 2 & -1 \end{vmatrix} = 1 \cdot (2(-1) - (-3)(2)) = -2 + 6 = 4$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 5 \\ 2 & 0 \end{vmatrix} = -1 \cdot (2(0) - 5(2)) = -1 \cdot (0 - 10) = 10$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 5 & -3 \\ 1 & 1 \end{vmatrix} = 1 \cdot (5(1) - (-3)(1)) = 5 + 3 = 8$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} = -1 \cdot (2(1) - (-3)(2)) = -1 \cdot (2 + 6) = -8$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} = 1 \cdot (2(1) - 5(2)) = 2 - 10 = -8$$

Cofactor matrix:

$$C = \begin{pmatrix} -1 & 4 & -2 \\ 5 & 4 & 10 \\ 8 & -8 & -8 \end{pmatrix}$$

Adjugate matrix (transpose of C):

$$\text{adj}(A) = C^T = \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}$$

Then:

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{24} \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}.$$

$$\boxed{A^{-1} = \frac{1}{24} \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}}.$$

3. Calculate A^2 and deduce $A^2 - 2A - 5I$.

$$A^2 = A \cdot A = \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$$

Compute:

Row 1:

$$\begin{aligned} (A^2)_{11} &= 2 \cdot 2 + 5 \cdot 2 + (-3) \cdot 2 = 4 + 10 - 6 = 8 \\ (A^2)_{12} &= 2 \cdot 5 + 5 \cdot 1 + (-3) \cdot 0 = 10 + 5 + 0 = 15 \\ (A^2)_{13} &= 2 \cdot (-3) + 5 \cdot 1 + (-3) \cdot (-1) = -6 + 5 + 3 = 2 \end{aligned}$$

Row 2:

$$\begin{aligned} (A^2)_{21} &= 2 \cdot 2 + 1 \cdot 2 + 1 \cdot 2 = 4 + 2 + 2 = 8 \\ (A^2)_{22} &= 2 \cdot 5 + 1 \cdot 1 + 1 \cdot 0 = 10 + 1 + 0 = 11 \\ (A^2)_{23} &= 2 \cdot (-3) + 1 \cdot 1 + 1 \cdot (-1) = -6 + 1 - 1 = -6 \end{aligned}$$

Row 3:

$$\begin{aligned} (A^2)_{31} &= 2 \cdot 2 + 0 \cdot 2 + (-1) \cdot 2 = 4 + 0 - 2 = 2 \\ (A^2)_{32} &= 2 \cdot 5 + 0 \cdot 1 + (-1) \cdot 0 = 10 + 0 + 0 = 10 \\ (A^2)_{33} &= 2 \cdot (-3) + 0 \cdot 1 + (-1) \cdot (-1) = -6 + 0 + 1 = -5 \end{aligned}$$

Thus:

$$A^2 = \begin{pmatrix} 8 & 15 & 2 \\ 8 & 11 & -6 \\ 2 & 10 & -5 \end{pmatrix}.$$

Now compute $2A$ and $5I$:

$$2A = \begin{pmatrix} 4 & 10 & -6 \\ 4 & 2 & 2 \\ 4 & 0 & -2 \end{pmatrix}, \quad 5I = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A^2 - 2A = \begin{pmatrix} 8-4 & 15-10 & 2-(-6) \\ 8-4 & 11-2 & -6-2 \\ 2-4 & 10-0 & -5-(-2) \end{pmatrix} = \begin{pmatrix} 4 & 5 & 8 \\ 4 & 9 & -8 \\ -2 & 10 & -3 \end{pmatrix}$$

$$A^2 - 2A - 5I = \begin{pmatrix} 4-5 & 5-0 & 8-0 \\ 4-0 & 9-5 & -8-0 \\ -2-0 & 10-0 & -3-5 \end{pmatrix} = \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}.$$

$$\boxed{A^2 - 2A - 5I = \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}}.$$

4. Verify that $A(A^2 - 2A - 5I) = 24I$.

Let $M = A^2 - 2A - 5I$. Compute $A \cdot M$:

$$A \cdot M = \begin{pmatrix} 2 & 5 & -3 \\ 2 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}$$

First row:

$$(AM)_{11} = 2(-1) + 5(4) + (-3)(-2) = -2 + 20 + 6 = 24$$

$$(AM)_{12} = 2(5) + 5(4) + (-3)(10) = 10 + 20 - 30 = 0$$

$$(AM)_{13} = 2(8) + 5(-8) + (-3)(-8) = 16 - 40 + 24 = 0$$

Second row:

$$(AM)_{21} = 2(-1) + 1(4) + 1(-2) = -2 + 4 - 2 = 0$$

$$(AM)_{22} = 2(5) + 1(4) + 1(10) = 10 + 4 + 10 = 24$$

$$(AM)_{23} = 2(8) + 1(-8) + 1(-8) = 16 - 8 - 8 = 0$$

Third row:

$$(AM)_{31} = 2(-1) + 0(4) + (-1)(-2) = -2 + 0 + 2 = 0$$

$$(AM)_{32} = 2(5) + 0(4) + (-1)(10) = 10 + 0 - 10 = 0$$

$$(AM)_{33} = 2(8) + 0(-8) + (-1)(-8) = 16 + 0 + 8 = 24$$

Thus:

$$A \cdot M = \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix} = 24I_3.$$

Verified: $A(A^2 - 2A - 5I) = 24I_3$.

5. Deduce the matrix A^{-1} again.

From $A \cdot M = 24I_3$, we have:

$$A \cdot \frac{1}{24}M = I_3.$$

Thus:

$$A^{-1} = \frac{1}{24}M = \frac{1}{24}(A^2 - 2A - 5I).$$

From step 3, $M = \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}$, so:

$$A^{-1} = \frac{1}{24} \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}.$$

This matches the result from step 2.

$$A^{-1} = \frac{1}{24} \begin{pmatrix} -1 & 5 & 8 \\ 4 & 4 & -8 \\ -2 & 10 & -8 \end{pmatrix}.$$

Solution – Exercise 2

Let (S) be a system of linear equations in variables x, y and z :

$$(S) : \begin{cases} -x + 2y - z = 3 \\ x + y + z = 0 \\ x - 4y - 2z = 1 \end{cases}$$

Deduce the solutions of the system (S') in variables x_1, x_2 and x_3 , such that:

$$(S') : \begin{cases} -x_1^2 + 2\sqrt{x_2} - \frac{x_3}{2} = 3 \\ x_1^2 + \sqrt{x_2} + \frac{x_3}{2} = 0 \\ x_1^2 - 4\sqrt{x_2} - x_3 = 1 \end{cases}$$

1. Matrix form $AX = b$.

$$A = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -4 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

The system is $AX = b$.

2. Calculate $\det(A)$ using the cofactors method.

$$\det A = \begin{vmatrix} -1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -4 & -2 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= (-1) \cdot \begin{vmatrix} 1 & 1 \\ -4 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} \\ &= (-1) \cdot ((1)(-2) - (1)(-4)) - 2 \cdot ((1)(-2) - (1)(1)) - 1 \cdot ((1)(-4) - (1)(1)) \\ &= (-1) \cdot (-2 + 4) - 2 \cdot (-2 - 1) - 1 \cdot (-4 - 1) \\ &= (-1) \cdot 2 - 2 \cdot (-3) - 1 \cdot (-5) \\ &= -2 + 6 + 5 = 9. \end{aligned}$$

Since $\det A = 9 \neq 0$, the system (S) has a unique solution.

$\det A = 9$, so (S) has a unique solution.

3. Using Cramer's method, show that the solution is $x = 1$, $y = 1$, $z = -2$.

We have $\det A = 9$.

Compute $\det A_1$ (replace column 1 with b):

$$A_1 = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & -4 & -2 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det A_1 &= 3 \cdot \begin{vmatrix} 1 & 1 \\ -4 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 0 & 1 \\ 1 & -4 \end{vmatrix} \\
 &= 3 \cdot ((1)(-2) - (1)(-4)) - 2 \cdot ((0)(-2) - (1)(1)) - 1 \cdot ((0)(-4) - (1)(1)) \\
 &= 3 \cdot (-2 + 4) - 2 \cdot (0 - 1) - 1 \cdot (0 - 1) \\
 &= 3 \cdot 2 - 2 \cdot (-1) - 1 \cdot (-1) \\
 &= 6 + 2 + 1 = 9.
 \end{aligned}$$

$$\text{Thus } x = \frac{\det A_1}{\det A} = \frac{9}{9} = 1.$$

Compute $\det A_2$ (replace column 2 with b):

$$A_2 = \begin{pmatrix} -1 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det A_2 &= (-1) \cdot \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\
 &= (-1) \cdot ((0)(-2) - (1)(1)) - 3 \cdot ((1)(-2) - (1)(1)) - 1 \cdot ((1)(1) - (0)(1)) \\
 &= (-1) \cdot (0 - 1) - 3 \cdot (-2 - 1) - 1 \cdot (1 - 0) \\
 &= (-1)(-1) - 3 \cdot (-3) - 1 \cdot 1 \\
 &= 1 + 9 - 1 = 9.
 \end{aligned}$$

$$\text{Thus } y = \frac{\det A_2}{\det A} = \frac{9}{9} = 1.$$

Compute $\det A_3$ (replace column 3 with b):

$$A_3 = \begin{pmatrix} -1 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & -4 & 1 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det A_3 &= (-1) \cdot \begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} \\
 &= (-1) \cdot ((1)(1) - (0)(-4)) - 2 \cdot ((1)(1) - (0)(1)) + 3 \cdot ((1)(-4) - (1)(1)) \\
 &= (-1) \cdot 1 - 2 \cdot 1 + 3 \cdot (-4 - 1) \\
 &= -1 - 2 + 3 \cdot (-5) = -3 - 15 = -18.
 \end{aligned}$$

$$\text{Thus } z = \frac{\det A_3}{\det A} = \frac{-18}{9} = -2.$$

Hence $x = 1$, $y = 1$, $z = -2$. Verified.

4. Deduce the solutions of the system (S') .

Notice that (S') is obtained from (S) by the substitution:

$$X = x_1^2, \quad Y = \sqrt{x_2}, \quad Z = \frac{x_3}{2}.$$

Indeed, with $x = X$, $y = Y$, $z = Z$, the first equation of (S) becomes:

$$-x + 2y - z = -X + 2Y - Z$$

which equals 3, matching the first equation of (S') .

From (S) we found $x = 1$, $y = 1$, $z = -2$.

Thus:

$$\begin{aligned} x_1^2 = 1 &\Rightarrow x_1 = 1 \text{ or } x_1 = -1, \\ \sqrt{x_2} = 1 &\Rightarrow x_2 = 1 \text{ (since } \sqrt{x_2} \geq 0, \text{ we take } x_2 = 1), \\ \frac{x_3}{2} = -2 &\Rightarrow x_3 = -4. \end{aligned}$$

Therefore (S') has two solutions:

$$(x_1, x_2, x_3) = (1, 1, -4) \quad \text{and} \quad (x_1, x_2, x_3) = (-1, 1, -4).$$

$$\boxed{(1, 1, -4) \text{ and } (-1, 1, -4)}.$$

Solution – Exercise 3

Let A be the matrix given by:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

1. Characteristic polynomial of A .

$$\begin{aligned} P_A(\lambda) &= \det(A - \lambda I_2) = \begin{vmatrix} 1 - \lambda & 0 \\ 1 & 2 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(2 - \lambda) - (0)(1) = (1 - \lambda)(2 - \lambda). \end{aligned}$$

Thus:

$$\boxed{P_A(\lambda) = (1 - \lambda)(2 - \lambda)}.$$

2. Find the eigenvalues and eigenvectors.

Set $P_A(\lambda) = 0$:

$$(1 - \lambda)(2 - \lambda) = 0 \quad \Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 2.$$

Eigenvector for $\lambda_1 = 1$:

$$(A - 1 \cdot I_2)v = 0 \quad \Rightarrow \quad \begin{pmatrix} 1 - 1 & 0 \\ 1 & 2 - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Second equation: $x + y = 0 \Rightarrow y = -x$. Choose $x = 1$, $y = -1$.

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = 2$:

$$(A - 2 \cdot I_2)v = 0 \quad \Rightarrow \quad \begin{pmatrix} 1-2 & 0 \\ 1 & 2-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

First equation: $-x = 0 \Rightarrow x = 0$. y is free. Choose $y = 1$.

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus:

$$\lambda_1 = 1, v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = 2, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Eigenvalues: 1 and 2, Eigenvectors: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

3. Find D, P, P^{-1} such that $A = PDP^{-1}$.

Since A has two distinct eigenvalues, it is diagonalizable.

Let P be the matrix with eigenvectors as columns:

$$P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Let D be the diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $A = PDP^{-1}$.

Now compute P^{-1} :

$$\det P = (1)(1) - (0)(-1) = 1.$$

$$P^{-1} = \frac{1}{1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Verification (optional):

$$PDP^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

First compute $DP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$.

Then $P(DP^{-1}) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1+2 & 0+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = A$.

Thus:

$$\boxed{D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}.$$

Chapter 2

Resit Examinations – Second Session

2.1 Resit Exam (2017/2018)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Resit Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 09/06/2018

Duration: 1h 30min

Exercise 1

(06 points)

I. Let A and B be two matrices given by:

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

1. Calculate $\det(A)$ and $\det(B)$. What can we remark?
2. Calculate AB and BA . What is the relationship between the two matrices?

II. Let the vectors

$$u_1 = \left(a, 1, -\frac{1}{2}\right), \quad u_2 = \left(\frac{1}{2}, 0, -a\right), \quad u_3 = \left(\frac{1}{2}, 1, a\right)$$

1. Find the value of a for which u_1, u_2, u_3 are linearly independent.

Exercise 2

(06 points)

Let the linear system

$$\begin{cases} -x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = -1 \\ x_1 - x_3 = 2 \end{cases}$$

1. Write the system in matrix form, then prove that it is a Cramer's system.
2. Solve the system using the inverse method.

Exercise 2

(08 points)

Let A be the matrix:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

1. Give the characteristic polynomial of A .
2. Find the eigenvalues and eigenvectors.
3. Prove that A is diagonalizable and write it in the form $A = PDP^{-1}$.
4. Calculate A^3 . What do you remark?

Full Solution – Resit Exam

Solution – Exercise 1

I. Let A and B be two matrices given by:

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

II. Let the vectors

$$u_1 = \left(a, 1, -\frac{1}{2}\right), \quad u_2 = \left(\frac{1}{2}, 0, -a\right), \quad u_3 = \left(\frac{1}{2}, 1, a\right)$$

Part I

1. Calculate $\det(A)$ and $\det(B)$.

$$\det A = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det A &= (-1) \cdot \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\
 &= (-1) \cdot ((1)(-1) - (2)(0)) - 1 \cdot ((0)(-1) - (2)(1)) + 1 \cdot ((0)(0) - (1)(1)) \\
 &= (-1) \cdot (-1 - 0) - 1 \cdot (0 - 2) + 1 \cdot (0 - 1) \\
 &= (-1)(-1) - 1 \cdot (-2) + (-1) \\
 &= 1 + 2 - 1 = 2.
 \end{aligned}$$

Now $\det B$:

$$\det B = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

Factor $\frac{1}{2}$ from row 1 and $-\frac{1}{2}$ from row 3? Better to compute directly: Expand by row 2 (contains zeros):

$$\det B = 1 \cdot (-1)^{2+1} \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} + 0 \cdot (\dots) + 1 \cdot (-1)^{2+3} \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

Simplify:

$$= 1 \cdot (-1) \cdot \left(\frac{1}{2} \cdot \left(-\frac{1}{2} \right) - \frac{1}{2} \cdot \frac{1}{2} \right) + 1 \cdot (-1) \cdot \left(\left(-\frac{1}{2} \right) \cdot \frac{1}{2} - \frac{1}{2} \cdot \left(-\frac{1}{2} \right) \right)$$

First term:

$$(-1) \cdot \left(-\frac{1}{4} - \frac{1}{4} \right) = (-1) \cdot \left(-\frac{1}{2} \right) = \frac{1}{2}.$$

Second term:

$$(-1) \cdot \left(-\frac{1}{4} + \frac{1}{4} \right) = (-1) \cdot 0 = 0.$$

Thus $\det B = \frac{1}{2}$.

Remark: $\det A = 2$, $\det B = \frac{1}{2} = \frac{1}{\det A}$.

2. Calculate AB and BA .

First compute AB :

$$AB = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Entry (1, 1): $(-1)(-\frac{1}{2}) + 1(1) + 1(-\frac{1}{2}) = \frac{1}{2} + 1 - \frac{1}{2} = 1$

(1, 2): $(-1)(\frac{1}{2}) + 1(0) + 1(\frac{1}{2}) = -\frac{1}{2} + 0 + \frac{1}{2} = 0$

(1, 3): $(-1)(\frac{1}{2}) + 1(1) + 1(-\frac{1}{2}) = -\frac{1}{2} + 1 - \frac{1}{2} = 0$

(2, 1): $0(-\frac{1}{2}) + 1(1) + 2(-\frac{1}{2}) = 0 + 1 - 1 = 0$

(2, 2): $0(\frac{1}{2}) + 1(0) + 2(\frac{1}{2}) = 0 + 0 + 1 = 1$

(2, 3): $0(\frac{1}{2}) + 1(1) + 2(-\frac{1}{2}) = 0 + 1 - 1 = 0$

$$(3, 1): 1(-\frac{1}{2}) + 0(1) + (-1)(-\frac{1}{2}) = -\frac{1}{2} + 0 + \frac{1}{2} = 0$$

$$(3, 2): 1(\frac{1}{2}) + 0(0) + (-1)(\frac{1}{2}) = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

$$(3, 3): 1(\frac{1}{2}) + 0(1) + (-1)(-\frac{1}{2}) = \frac{1}{2} + 0 + \frac{1}{2} = 1$$

Thus:

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

Similarly, compute BA (should also be I_3 if $B = A^{-1}$). Since $AB = I_3$, $BA = I_3$ automatically for square matrices.

Relationship: $B = A^{-1}$.

Part II

- Find a such that u_1, u_2, u_3 are linearly independent.

Form matrix M with u_1, u_2, u_3 as rows (or columns). Use rows:

$$M = \begin{pmatrix} a & 1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -a \\ \frac{1}{2} & 1 & a \end{pmatrix}$$

Vectors are linearly independent iff $\det M \neq 0$.

Compute $\det M$:

$$\det M = \begin{vmatrix} a & 1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -a \\ \frac{1}{2} & 1 & a \end{vmatrix}$$

Expand by row 2:

$$\begin{aligned} \det M &= \frac{1}{2} \cdot (-1)^{2+1} \begin{vmatrix} 1 & -\frac{1}{2} \\ 1 & a \end{vmatrix} + 0 \cdot (\dots) + (-a) \cdot (-1)^{2+3} \begin{vmatrix} a & 1 \\ \frac{1}{2} & 1 \end{vmatrix} \\ &= \frac{1}{2} \cdot (-1) \cdot \left(1 \cdot a - \left(-\frac{1}{2}\right) \cdot 1 \right) + (-a) \cdot (-1) \cdot \left(a \cdot 1 - 1 \cdot \frac{1}{2} \right) \\ &= -\frac{1}{2} \left(a + \frac{1}{2} \right) + a \left(a - \frac{1}{2} \right) \\ &= -\frac{a}{2} - \frac{1}{4} + a^2 - \frac{a}{2} \\ &= a^2 - a - \frac{1}{4}. \end{aligned}$$

Set $\det M \neq 0$:

$$a^2 - a - \frac{1}{4} \neq 0$$

$$4a^2 - 4a - 1 \neq 0$$

Solve $4a^2 - 4a - 1 = 0$:

$$a = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{4 \pm \sqrt{32}}{8} = \frac{4 \pm 4\sqrt{2}}{8} = \frac{1 \pm \sqrt{2}}{2}.$$

Thus u_1, u_2, u_3 are linearly independent for:

$$a \neq \frac{1 + \sqrt{2}}{2} \quad \text{and} \quad a \neq \frac{1 - \sqrt{2}}{2}.$$

Solution – Exercise 2

Let the linear system

$$\begin{cases} -x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = -1 \\ x_1 - x_3 = 2 \end{cases}$$

1. Matrix form and Cramer system verification.

The system can be written as $AX = B$ where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

A system is Cramer if n equations, n unknowns and $\det A \neq 0$. Here $n = 3$.

Compute $\det A$:

$$\det A = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= (-1) \cdot \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= (-1) \cdot ((1)(-1) - (2)(0)) - 1 \cdot ((0)(-1) - (2)(1)) + 1 \cdot ((0)(0) - (1)(1)) \\ &= (-1) \cdot (-1 - 0) - 1 \cdot (0 - 2) + 1 \cdot (0 - 1) \\ &= (-1)(-1) - 1 \cdot (-2) - 1 \\ &= 1 + 2 - 1 = 2. \end{aligned}$$

Since $\det A = 2 \neq 0$, the system is a Cramer system.

2. Solve using the inverse method.

Since A is invertible, $X = A^{-1}B$.

First find A^{-1} using the adjugate method.

Cofactor matrix of A :

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = 1 \cdot ((1)(-1) - (2)(0)) = -1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = -1 \cdot ((0)(-1) - (2)(1)) = -1 \cdot (0 - 2) = 2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot ((0)(0) - (1)(1)) = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 \cdot ((1)(-1) - (1)(0)) = -1 \cdot (-1 - 0) = 1$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 \cdot ((-1)(-1) - (1)(1)) = 1 - 1 = 0$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \cdot ((-1)(0) - (1)(1)) = -1 \cdot (0 - 1) = 1$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \cdot ((1)(2) - (1)(1)) = 2 - 1 = 1$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} = -1 \cdot ((-1)(2) - (1)(0)) = -1 \cdot (-2 - 0) = 2$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot ((-1)(1) - (1)(0)) = -1.$$

Thus the cofactor matrix is:

$$C = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}.$$

The adjugate matrix is C^T :

$$\text{adj}(A) = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{pmatrix}.$$

Then:

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Now compute $X = A^{-1}B$:

$$X = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

First component:

$$x_1 = \left(-\frac{1}{2}\right)(1) + \frac{1}{2}(-1) + \frac{1}{2}(2) = -\frac{1}{2} - \frac{1}{2} + 1 = -1 + 1 = 0.$$

Second component:

$$x_2 = 1(1) + 0(-1) + 1(2) = 1 + 0 + 2 = 3.$$

Third component:

$$x_3 = \left(-\frac{1}{2}\right)(1) + \frac{1}{2}(-1) + \left(-\frac{1}{2}\right)(2) = -\frac{1}{2} - \frac{1}{2} - 1 = -1 - 1 = -2.$$

Thus the solution is:

$$\boxed{x_1 = 0, \quad x_2 = 3, \quad x_3 = -2}.$$

Verification:

$$-x_1 + x_2 + x_3 = 0 + 3 - 2 = 1 \quad \checkmark$$

$$x_2 + 2x_3 = 3 + 2(-2) = 3 - 4 = -1 \quad \checkmark$$

$$x_1 - x_3 = 0 - (-2) = 2 \quad \checkmark$$

Solution – Exercise 3

Let A be the matrix:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

1. Characteristic polynomial of A .

$$\begin{aligned} P_A(\lambda) &= \det(A - \lambda I_2) = \begin{vmatrix} -1 - \lambda & 1 \\ 0 & 1 - \lambda \end{vmatrix} \\ &= (-1 - \lambda)(1 - \lambda) - (1)(0) \\ &= (-1 - \lambda)(1 - \lambda) \\ &= -(1 + \lambda)(1 - \lambda) = -(1 - \lambda^2) = \lambda^2 - 1. \end{aligned}$$

Thus:

$$\boxed{P_A(\lambda) = \lambda^2 - 1}.$$

2. Find the eigenvalues and eigenvectors.

Solve $P_A(\lambda) = 0$:

$$\lambda^2 - 1 = 0 \quad \Rightarrow \quad (\lambda - 1)(\lambda + 1) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = -1.$$

Eigenvector for $\lambda_1 = 1$:

$$(A - 1 \cdot I_2)v = 0 \Rightarrow \begin{pmatrix} -1-1 & 1 \\ 0 & 1-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

First equation: $-2x + y = 0 \Rightarrow y = 2x$. Choose $x = 1$, $y = 2$.

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = -1$:

$$(A + 1 \cdot I_2)v = 0 \Rightarrow \begin{pmatrix} -1+1 & 1 \\ 0 & 1+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

First equation: $0 \cdot x + 1 \cdot y = 0 \Rightarrow y = 0$. x is free. Choose $x = 1$, $y = 0$.

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus:

$$\lambda_1 = 1, v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = -1, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

3. Prove A is diagonalizable and write $A = PDP^{-1}$.

A is 2×2 with two distinct eigenvalues (1 and -1), so it is diagonalizable.

Let P be the matrix with eigenvectors as columns:

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

Let D be the diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $A = PDP^{-1}$.

Compute P^{-1} :

$$\det P = (1)(0) - (1)(2) = -2.$$

$$P^{-1} = \frac{1}{-2} \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

Verification (optional):

$$PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

First $DP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}$

Then $P(DP^{-1}) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0-1 & \frac{1}{2}+\frac{1}{2} \\ 0+0 & 1+0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = A.$

Thus:

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

4. Calculate A^3 . What do you remark?

Since $A = PDP^{-1}$, we have $A^3 = PD^3P^{-1}$.

$$D^3 = \begin{pmatrix} 1^3 & 0 \\ 0 & (-1)^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D.$$

Thus $A^3 = PDP^{-1} = A$.

$$\boxed{A^3 = A}.$$

Remark: $A^3 = A$, so A is an idempotent-like matrix (periodic with period 2: $A^{2k+1} = A$ for all $k \geq 0$). Also note $A^2 = I_2$? Let's check quickly:

$$A^2 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

So indeed $A^2 = I_2$, which explains $A^3 = A^2 \cdot A = I_2 \cdot A = A$.

2.2 Resit Exam (2018/2019)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Resit Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 29/09/2019 **Duration:** 1h 30min

Exercise 1

(06 points)

Answer by **True** or **False**, correcting the false statements:

1. The trace of a square matrix is the product of its main diagonal elements.
2. The determinant of any square matrix can be calculated using the following methods: Sarrus, Gauss, and Expansion.
3. Homogeneous systems have at least one unique solution.
4. A Cramer system is solved using the "Cramer", "Gauss" and "Rank" methods.

Exercise 2

(06 points)

Let the two matrices A and B be defined as:

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{pmatrix}$$

1. Calculate $|A|$ and $|B|$. What do you observe?
2. Calculate $A \times B$ and $B \times A$. Determine the relationship between A and B .

Exercise 3

(08 points)

Consider the following system of linear equations:

$$(S) : \begin{cases} -x_1 + x_2 + x_3 = 2 \\ x_2 + 2x_3 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

1. Is (S) a Cramer system?
2. Solve the system (S) using two methods: (a) Cramer's rule, and (b) the matrix inverse method.

Full Solution

Solution – Exercise 1

1. **False.** The trace of a square matrix is the **sum** of its main diagonal elements.
2. **True.** These are the standard computational methods for square matrices.
3. **False.** Homogeneous systems always have at least the **trivial solution** $(0, 0, 0)$. They have a unique solution only if the determinant is non-zero.
4. **False.** Both Cramer's rule and Gauss elimination are applicable for solving Cramer's systems and rank methods is used for non-Cramer's systems.

Solution – Exercise 2

1. **Determinants:** $|A| = -1(-1 - 0) - 1(0 - 2) + 1(0 - 1) = 1 + 2 - 1 = 2$.

$$|B| = \left(\frac{1}{2}\right)^3 \begin{vmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{vmatrix} = \frac{1}{8}[-1(0 - 2) - 1(-2 - (-2)) + 1(2 - 0)] = \frac{1}{8}[2 + 2] = \frac{1}{2}.$$

Observation: $|B| = \frac{1}{|A|}$.

2. **Matrix Products:** $A \times B = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \left[\frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$

I_3 .

$B \times A = I_3$.

Relationship: Since $A \times B = B \times A = I_3$, the matrix B is the inverse of A ($B = A^{-1}$).

Solution – Exercise 3

Consider the following system of linear equations:

$$(S) : \begin{cases} -x_1 + x_2 + x_3 = 2 \\ x_2 + 2x_3 = 0 \\ x_1 - x_3 = 1 \end{cases}$$

1. Is (S) a Cramer system?

A system is Cramer if it has n equations, n unknowns, and $\det A \neq 0$.

Write the system in matrix form $AX = B$:

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Here $n = 3$. Compute $\det A$:

$$\det A = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix}$$

Expand by row 2:

$$\begin{aligned} \det A &= 0 \cdot (\text{minor}_{21}) + 1 \cdot (-1)^{2+2} \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 2 \cdot (-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \\ &= 1 \cdot 1 \cdot ((-1)(-1) - (1)(1)) + 2 \cdot (-1) \cdot ((-1)(0) - (1)(1)) \\ &= 1 \cdot (1 - 1) - 2 \cdot (0 - 1) \\ &= 1 \cdot 0 - 2 \cdot (-1) = 0 + 2 = 2. \end{aligned}$$

Since $\det A = 2 \neq 0$, the system is a Cramer system.

2. Solve the system using two methods.

(a) Cramer's rule:

We have $\det A = 2$.

Compute $\det A_1$ (replace column 1 with B):

$$A_1 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_1 &= 2 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= 2 \cdot ((1)(-1) - (2)(0)) - 1 \cdot ((0)(-1) - (2)(1)) + 1 \cdot ((0)(0) - (1)(1)) \\ &= 2 \cdot (-1 - 0) - 1 \cdot (0 - 2) + 1 \cdot (0 - 1) \\ &= -2 - 1 \cdot (-2) - 1 = -2 + 2 - 1 = -1. \end{aligned}$$

Compute $\det A_2$ (replace column 2 with B):

$$A_2 = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

Expand by row 2:

$$\det A_2 = 0 \cdot (\text{minor}_{21}) + 0 \cdot (\text{minor}_{22}) + 2 \cdot (-1)^{2+3} \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$\begin{aligned}
&= 2 \cdot (-1) \cdot ((-1)(1) - (2)(1)) \\
&= -2 \cdot (-1 - 2) = -2 \cdot (-3) = 6.
\end{aligned}$$

Compute $\det A_3$ (replace column 3 with B):

$$A_3 = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Expand by row 2:

$$\begin{aligned}
\det A_3 &= 0 \cdot (\text{minor}_{21}) + 1 \cdot (-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} + 0 \cdot (\text{minor}_{23}) \\
&= 1 \cdot 1 \cdot ((-1)(1) - (2)(1)) = -1 - 2 = -3.
\end{aligned}$$

Now by Cramer's rule:

$$x_1 = \frac{\det A_1}{\det A} = \frac{-1}{2} = -\frac{1}{2}, \quad x_2 = \frac{\det A_2}{\det A} = \frac{6}{2} = 3, \quad x_3 = \frac{\det A_3}{\det A} = \frac{-3}{2} = -\frac{3}{2}.$$

Verification of solution:

$$\begin{aligned}
\text{Eq1: } -x_1 + x_2 + x_3 &= -\left(-\frac{1}{2}\right) + 3 + \left(-\frac{3}{2}\right) \\
&= \frac{1}{2} + 3 - \frac{3}{2} = 3 - 1 = 2 \quad \checkmark
\end{aligned}$$

$$\text{Eq2: } x_2 + 2x_3 = 3 + 2\left(-\frac{3}{2}\right) = 3 - 3 = 0 \quad \checkmark$$

$$\text{Eq3: } x_1 - x_3 = \left(-\frac{1}{2}\right) - \left(-\frac{3}{2}\right) = -\frac{1}{2} + \frac{3}{2} = 1 \quad \checkmark$$

Thus the solution is:

$$\boxed{x_1 = -\frac{1}{2}, \quad x_2 = 3, \quad x_3 = -\frac{3}{2}}.$$

(b) Matrix inverse method:

We have $X = A^{-1}B$.

First find A^{-1} using the adjugate method.

Cofactor matrix of A :

$$C_{11} = (-1)^2 \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = 1 \cdot (1(-1) - 2(0)) = -1$$

$$C_{12} = (-1)^3 \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = -1 \cdot (0(-1) - 2(1)) = -1 \cdot (0 - 2) = 2$$

$$C_{13} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 \cdot 0 - 1 \cdot 1) = -1$$

$$C_{21} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 \cdot (1(-1) - 1(0)) = -1 \cdot (-1) = 1$$

$$C_{22} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 \cdot ((-1)(-1) - 1(1)) = 1 - 1 = 0$$

$$C_{23} = (-1)^5 \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \cdot ((-1)(0) - 1(1)) = -1 \cdot (-1) = 1$$

$$C_{31} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \cdot (1 \cdot 2 - 1 \cdot 1) = 1$$

$$C_{32} = (-1)^5 \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} = -1 \cdot ((-1)(2) - 1(0)) = -1 \cdot (-2) = 2$$

$$C_{33} = (-1)^6 \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot ((-1)(1) - 1(0)) = -1$$

Cofactor matrix:

$$C = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

Adjugate matrix = C^T :

$$\text{adj}(A) = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{pmatrix}$$

Since $\det A = 2$,

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & -1 \end{pmatrix}.$$

Now compute $X = A^{-1}B$ with $B = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$:

First component:

$$x_1 = \frac{1}{2}((-1)(2) + 1(0) + 1(1)) = \frac{1}{2}(-2 + 0 + 1) = \frac{1}{2}(-1) = -\frac{1}{2}$$

Second component:

$$x_2 = \frac{1}{2}(2(2) + 0(0) + 2(1)) = \frac{1}{2}(4 + 0 + 2) = \frac{1}{2} \cdot 6 = 3$$

Third component:

$$x_3 = \frac{1}{2}((-1)(2) + 1(0) + (-1)(1)) = \frac{1}{2}(-2 + 0 - 1) = \frac{1}{2}(-3) = -\frac{3}{2}$$

Thus:

$$\boxed{x_1 = -\frac{1}{2}, \quad x_2 = 3, \quad x_3 = -\frac{3}{2}}$$

Both methods give the same solution.

2.3 Resit Exam (2019/2020)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Resit Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 25/11/2020 **Duration:** 1h 00min

Exercise 1

(8 points)

Let A, B and C three matrices: given by

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -2 \end{pmatrix}, C = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 4 \end{pmatrix}$$

Calculate

$$2A - B + 3I_3 = \dots\dots\dots$$

$$A^2 = \dots\dots\dots$$

$$\det(A) = \dots\dots\dots$$

$$C^T = \dots\dots\dots$$

$$\text{tr}(B) = \dots\dots\dots$$

Exercise 2

(12 points)

Let A be the matrix given in the first exercise:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

1. Calculate

$$A^3 - A = \dots$$

2. Deduce that A is invertible and calculate its inverse A^{-1} .

Full Solution

Solution – Exercise 1

Let A , B , and C be three matrices given by:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 4 \end{pmatrix}$$

1. Compute $2A - B + 3I_3$:

$$2A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & -2 & 2 \\ 2 & -4 & 0 \end{pmatrix}, \quad 3I_3 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

First compute $2A - B$:

$$2A - B = \begin{pmatrix} 2-1 & 0-2 & 4-4 \\ 0-0 & -2-(-1) & 2-3 \\ 2-0 & -4-0 & 0-(-2) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & -1 \\ 2 & -4 & 2 \end{pmatrix}$$

Then add $3I_3$:

$$2A - B + 3I_3 = \begin{pmatrix} 1+3 & -2+0 & 0+0 \\ 0+0 & -1+3 & -1+0 \\ 2+0 & -4+0 & 2+3 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 2 & -4 & 5 \end{pmatrix}$$

$$2A - B + 3I_3 = \begin{pmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 2 & -4 & 5 \end{pmatrix}.$$

2. Compute A^2 :

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

Compute each entry:

Row 1:

$$(A^2)_{11} = 1 \cdot 1 + 0 \cdot 0 + 2 \cdot 1 = 1 + 0 + 2 = 3$$

$$(A^2)_{12} = 1 \cdot 0 + 0 \cdot (-1) + 2 \cdot (-2) = 0 + 0 - 4 = -4$$

$$(A^2)_{13} = 1 \cdot 2 + 0 \cdot 1 + 2 \cdot 0 = 2 + 0 + 0 = 2$$

Row 2:

$$(A^2)_{21} = 0 \cdot 1 + (-1) \cdot 0 + 1 \cdot 1 = 0 + 0 + 1 = 1$$

$$(A^2)_{22} = 0 \cdot 0 + (-1) \cdot (-1) + 1 \cdot (-2) = 0 + 1 - 2 = -1$$

$$(A^2)_{23} = 0 \cdot 2 + (-1) \cdot 1 + 1 \cdot 0 = 0 - 1 + 0 = -1$$

Row 3:

$$(A^2)_{31} = 1 \cdot 1 + (-2) \cdot 0 + 0 \cdot 1 = 1 + 0 + 0 = 1$$

$$(A^2)_{32} = 1 \cdot 0 + (-2) \cdot (-1) + 0 \cdot (-2) = 0 + 2 + 0 = 2$$

$$(A^2)_{33} = 1 \cdot 2 + (-2) \cdot 1 + 0 \cdot 0 = 2 - 2 + 0 = 0$$

Thus:

$$A^2 = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}.$$

3. Compute $\det(A)$:

$$\det A = \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= 1 \cdot \begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & -1 \\ 1 & -2 \end{vmatrix} \\ &= 1 \cdot ((-1)(0) - (1)(-2)) + 2 \cdot ((0)(-2) - (-1)(1)) \\ &= 1 \cdot (0 + 2) + 2 \cdot (0 + 1) \\ &= 2 + 2 = 4. \end{aligned}$$

$$\boxed{\det(A) = 4}.$$

4. Compute C^T (transpose of C):

$$C = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & 3 \\ 1 & -1 & 4 \end{pmatrix}$$

$$C^T = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 3 & 4 \end{pmatrix}$$

$$\boxed{C^T = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 3 & 4 \end{pmatrix}}.$$

5. Compute $\text{tr}(B)$ (trace of B):

$$B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\text{tr}(B) = 1 + (-1) + (-2) = -2.$$

$$\boxed{\text{tr}(B) = -2}.$$

Solution – Exercise 2

Let A be the matrix given by:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

1. First, we already computed A^2 in the previous exercise:

$$A^2 = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Now compute $A^3 = A^2 \cdot A$:

$$A^3 = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

Compute each entry:

Row 1:

$$\begin{aligned} (A^3)_{11} &= 3 \cdot 1 + (-4) \cdot 0 + 2 \cdot 1 = 3 + 0 + 2 = 5 \\ (A^3)_{12} &= 3 \cdot 0 + (-4) \cdot (-1) + 2 \cdot (-2) = 0 + 4 - 4 = 0 \\ (A^3)_{13} &= 3 \cdot 2 + (-4) \cdot 1 + 2 \cdot 0 = 6 - 4 + 0 = 2 \end{aligned}$$

Row 2:

$$\begin{aligned} (A^3)_{21} &= 1 \cdot 1 + (-1) \cdot 0 + (-1) \cdot 1 = 1 + 0 - 1 = 0 \\ (A^3)_{22} &= 1 \cdot 0 + (-1) \cdot (-1) + (-1) \cdot (-2) = 0 + 1 + 2 = 3 \\ (A^3)_{23} &= 1 \cdot 2 + (-1) \cdot 1 + (-1) \cdot 0 = 2 - 1 + 0 = 1 \end{aligned}$$

Row 3:

$$\begin{aligned} (A^3)_{31} &= 1 \cdot 1 + 2 \cdot 0 + 0 \cdot 1 = 1 + 0 + 0 = 1 \\ (A^3)_{32} &= 1 \cdot 0 + 2 \cdot (-1) + 0 \cdot (-2) = 0 - 2 + 0 = -2 \\ (A^3)_{33} &= 1 \cdot 2 + 2 \cdot 1 + 0 \cdot 0 = 2 + 2 + 0 = 4 \end{aligned}$$

Thus:

$$A^3 = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix}.$$

Now compute $A^3 - A$:

$$A^3 - A = \begin{pmatrix} 5-1 & 0-0 & 2-2 \\ 0-0 & 3-(-1) & 1-1 \\ 1-1 & -2-(-2) & 4-0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I_3.$$

$$\boxed{A^3 - A = 4I_3}.$$

2. From $A^3 - A = 4I_3$, we have:

$$A^3 - A = 4I_3$$

$$A(A^2 - I_3) = 4I_3$$

$$A \cdot \frac{1}{4}(A^2 - I_3) = I_3.$$

Thus A is invertible and:

$$A^{-1} = \frac{1}{4}(A^2 - I_3).$$

Compute $A^2 - I_3$:

$$A^2 = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^2 - I_3 = \begin{pmatrix} 3-1 & -4-0 & 2-0 \\ 1-0 & -1-1 & -1-0 \\ 1-0 & 2-0 & 0-1 \end{pmatrix} = \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Therefore:

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}.$$

$$\boxed{A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix}}.$$

2.4 Resit Exam (2020/2021)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Resit Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 13/07/2021

Duration: 1h 30min

Exercise 1

(08 points)

Let A , B , C , and D be the matrices given by:

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$$

1. Give the size of the previous matrices.
2. Give the possible product operations (without calculation).
3. Calculate $\text{tr}(A)$ and $\det(A)$.
4. Complete the operations:

$$C^T = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \quad A \times \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = A, \quad B \times \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = 0_{2 \times 2}.$$

5. Show that B is the inverse of A .

Exercise 2

(12 points)

Let the linear system (S) :

$$(S) \begin{cases} 3x + 2y + 4z = -1 \\ 2x - y + 2z = -2 \\ -x + y + 2z = 2 \end{cases}$$

1. Write the system in matrix form, then prove that it is a Cramer system.
2. Solve the system using the Gauss method.
3. Deduce the solutions of the system:

$$(S') \begin{cases} 3x^3 + 2y^2 + 4z^2 = -1 \\ 2x^3 - y^2 + 2z^2 = -2 \\ -x^3 + y^2 + 2z^2 = 2 \end{cases}$$

Full Solution

Solution – Exercise 1

Let A , B , C , and D be the matrices given by:

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$$

1. Size of the matrices:

$$A : 2 \times 2, \quad B : 2 \times 2, \quad C : 3 \times 2, \quad D : 2 \times 3.$$

$$\boxed{A_{2 \times 2}, B_{2 \times 2}, C_{3 \times 2}, D_{2 \times 3}}.$$

2. Possible product operations (without calculation):

For two matrices X (size $m \times n$) and Y (size $p \times q$), the product XY is possible if $n = p$.

- $A \times A$: 2×2 possible, result 2×2
- $A \times B$: 2×2 possible, result 2×2
- $B \times A$: 2×2 possible, result 2×2
- $A \times C$: 2×3 ? No — C is 3×2 , so $A(2 \times 2)$ times $C(3 \times 2)$: $2 \neq 3$, not possible
- $C \times D$: C is 3×2 , D is 2×3 : $2 = 2$, possible, result 3×3
- $D \times C$: D is 2×3 , C is 3×2 : $3 = 3$, possible, result 2×2
- $A \times D$: A is 2×2 , D is 2×3 : $2 = 2$, possible, result 2×3
- $D \times A$: D is 2×3 , A is 2×2 : $3 \neq 2$, not possible
- $C \times A$: C is 3×2 , A is 2×2 : $2 = 2$, possible, result 3×2
- $B \times D$: B is 2×2 , D is 2×3 : $2 = 2$, possible, result 2×3
- $D \times B$: D is 2×3 , B is 2×2 : $3 \neq 2$, not possible

$$\boxed{\text{Possible: } AA, AB, BA, CD, DC, AD, CA, BD}.$$

3. Calculate $\text{tr}(A)$ and $\det(A)$.

$$\text{tr}(A) = 2 + 2 = 4.$$

$$\det(A) = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = (2)(2) - (3)(-1) = 4 + 3 = 7.$$

$$\boxed{\text{tr}(A) = 4, \quad \det(A) = 7}.$$

4. Complete the operations.

First, C^T : C is 3×2 , so C^T is 2×3 :

$$C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Second, $A \times (?) = A$: the identity matrix I_2 satisfies $AI_2 = A$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Third, $B \times (?) = 0_{2 \times 2}$: the zero matrix $0_{2 \times 2}$ satisfies $B \cdot 0 = 0$:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus:

$$\boxed{C^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}}, \quad \boxed{A \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A}, \quad \boxed{B \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{2 \times 2}}.$$

5. Show that B is the inverse of A .

We must show $AB = I_2$ and $BA = I_2$.

$$AB = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}$$

First row:

$$(AB)_{11} = 2 \cdot \frac{2}{7} + 3 \cdot \frac{1}{7} = \frac{4}{7} + \frac{3}{7} = \frac{7}{7} = 1$$

$$(AB)_{12} = 2 \cdot \left(-\frac{3}{7}\right) + 3 \cdot \frac{2}{7} = -\frac{6}{7} + \frac{6}{7} = 0$$

Second row:

$$(AB)_{21} = (-1) \cdot \frac{2}{7} + 2 \cdot \frac{1}{7} = -\frac{2}{7} + \frac{2}{7} = 0$$

$$(AB)_{22} = (-1) \cdot \left(-\frac{3}{7}\right) + 2 \cdot \frac{2}{7} = \frac{3}{7} + \frac{4}{7} = \frac{7}{7} = 1$$

Thus:

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Similarly, compute BA (or note that for square matrices, $AB = I$ implies $BA = I$):

$$BA = \begin{pmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$$

First row:

$$(BA)_{11} = \frac{2}{7} \cdot 2 + \left(-\frac{3}{7}\right) \cdot (-1) = \frac{4}{7} + \frac{3}{7} = 1$$

$$(BA)_{12} = \frac{2}{7} \cdot 3 + \left(-\frac{3}{7}\right) \cdot 2 = \frac{6}{7} - \frac{6}{7} = 0$$

Second row:

$$(BA)_{21} = \frac{1}{7} \cdot 2 + \frac{2}{7} \cdot (-1) = \frac{2}{7} - \frac{2}{7} = 0$$

$$(BA)_{22} = \frac{1}{7} \cdot 3 + \frac{2}{7} \cdot 2 = \frac{3}{7} + \frac{4}{7} = 1$$

Thus $BA = I_2$. Therefore $B = A^{-1}$.

$$\boxed{B = A^{-1}}.$$

Solution – Exercise 2

Let the linear system (S) :

$$(S) \begin{cases} 3x + 2y + 4z = -1 \\ 2x - y + 2z = -2 \\ -x + y + 2z = 2 \end{cases}$$

1. Matrix form and Cramer system verification.

The system can be written as $AX = B$ where:

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}.$$

A system is Cramer if it has n equations, n unknowns, and $\det A \neq 0$. Here $n = 3$.

Compute $\det A$:

$$\det A = \begin{vmatrix} 3 & 2 & 4 \\ 2 & -1 & 2 \\ -1 & 1 & 2 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= 3 \cdot \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} + 4 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} \\ &= 3 \cdot ((-1)(2) - (2)(1)) - 2 \cdot ((2)(2) - (2)(-1)) + 4 \cdot ((2)(1) - (-1)(-1)) \\ &= 3 \cdot (-2 - 2) - 2 \cdot (4 + 2) + 4 \cdot (2 - 1) \\ &= 3 \cdot (-4) - 2 \cdot 6 + 4 \cdot 1 \\ &= -12 - 12 + 4 = -20. \end{aligned}$$

Since $\det A = -20 \neq 0$, the system is a Cramer system.

2. Solve the system using the Gauss method.

Write the augmented matrix $[A|B]$:

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & -1 \\ 2 & -1 & 2 & -2 \\ -1 & 1 & 2 & 2 \end{array} \right].$$

Step 1: Swap rows 1 and 3 to get a leading 1 (optional but easier):

$$R_1 \leftrightarrow R_3 : \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 2 & -1 & 2 & -2 \\ 3 & 2 & 4 & -1 \end{array} \right].$$

Multiply R_1 by -1 :

$$R_1 \leftarrow -R_1 : \left[\begin{array}{ccc|c} 1 & -1 & -2 & -2 \\ 2 & -1 & 2 & -2 \\ 3 & 2 & 4 & -1 \end{array} \right].$$

Step 2: Eliminate x from rows 2 and 3: $R_2 \leftarrow R_2 - 2R_1$:

$$(2 - 2, -1 + 2, 2 + 4, -2 + 4) = (0, 1, 6, 2)$$

$R_3 \leftarrow R_3 - 3R_1$:

$$(3 - 3, 2 + 3, 4 + 6, -1 + 6) = (0, 5, 10, 5)$$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -2 \\ 0 & 1 & 6 & 2 \\ 0 & 5 & 10 & 5 \end{array} \right].$$

Step 3: Eliminate y from row 3: $R_3 \leftarrow R_3 - 5R_2$:

$$(0, 5 - 5, 10 - 30, 5 - 10) = (0, 0, -20, -5)$$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -2 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & -20 & -5 \end{array} \right].$$

Step 4: Back substitution.

$$\text{From } R_3: -20z = -5 \Rightarrow z = \frac{-5}{-20} = \frac{1}{4}.$$

$$\text{From } R_2: y + 6z = 2 \Rightarrow y + 6 \cdot \frac{1}{4} = 2 \Rightarrow y + \frac{3}{2} = 2 \Rightarrow y = 2 - \frac{3}{2} = \frac{1}{2}.$$

$$\text{From } R_1: x - y - 2z = -2 \Rightarrow x - \frac{1}{2} - 2 \cdot \frac{1}{4} = -2 \Rightarrow x - \frac{1}{2} - \frac{1}{2} = -2 \Rightarrow x - 1 = -2 \Rightarrow x = -1.$$

Thus the solution is:

$$\boxed{x = -1, \quad y = \frac{1}{2}, \quad z = \frac{1}{4}}.$$

Verification:

$$3(-1) + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) = -3 + 1 + 1 = -1 \quad \checkmark$$

$$2(-1) - \frac{1}{2} + 2\left(\frac{1}{4}\right) = -2 - \frac{1}{2} + \frac{1}{2} = -2 \quad \checkmark$$

$$-(-1) + \frac{1}{2} + 2\left(\frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 2 \quad \checkmark$$

3. Deduce the solutions of the system (S') .

Let $X = x^3$, $Y = y^2$, $Z = z^2$. Then (S') becomes:

$$\begin{cases} 3X + 2Y + 4Z = -1 \\ 2X - Y + 2Z = -2 \\ -X + Y + 2Z = 2 \end{cases}$$

This is exactly the same linear system as (S) , but with variables X, Y, Z . From part 2, the unique solution is:

$$X = -1, \quad Y = \frac{1}{2}, \quad Z = \frac{1}{4}.$$

Thus:

$$x^3 = -1 \Rightarrow x = -1 \quad (\text{real solution})$$

$$y^2 = \frac{1}{2} \Rightarrow y = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$$

$$z^2 = \frac{1}{4} \Rightarrow z = \pm \frac{1}{2}.$$

Therefore, the system (S') has $\boxed{1 \times 2 \times 2 = 4}$ real solutions:

$$(x, y, z) = \left(-1, \frac{\sqrt{2}}{2}, \frac{1}{2}\right), \quad \left(-1, \frac{\sqrt{2}}{2}, -\frac{1}{2}\right), \quad \left(-1, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right), \quad \left(-1, -\frac{\sqrt{2}}{2}, -\frac{1}{2}\right).$$

2.5 Resit Exam (2021/2022)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Resit Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 15/06/2022

Duration: 1h 30min

Exercise 1

(11 points)

I. Let A and B be two matrices given by:

$$A = \begin{pmatrix} 0 & -1 & 6 \\ 2 & 4 & -2 \\ 3 & 5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

1. Calculate $\text{tr}(A)$, A^T , $\det(B)$, $B - 2A$, and AB .

II. Let α be a real number and A the matrix:

$$A = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 1 & 0 \end{pmatrix}$$

1. Calculate $\det(A)$ in terms of α , then find the values of α for which A is invertible.

2. For $\alpha = 1$, calculate A^{-1} using the cofactors method.

Exercise 2

(09 points)

Let the linear system (S_1) :

$$(S_1) \begin{cases} -x_1 + 2x_2 + x_3 = 4 \\ 3x_1 - 5x_2 + 3x_3 = 1 \\ 2x_1 + 7x_2 - x_3 = -8 \end{cases}$$

1. Write the system (S_1) in matrix form, then prove that it is a Cramer system.

2. Solve the system using Cramer's method and Gauss's method.

Full Solution

Solution – Exercise 1

I. Let A and B be two matrices given by:

$$A = \begin{pmatrix} 0 & -1 & 6 \\ 2 & 4 & -2 \\ 3 & 5 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

II. Let α be a real number and A the matrix:

$$A = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 1 & 0 \end{pmatrix}$$

I.

1. Compute the required quantities.

$$\text{tr}(A) = 0 + 4 + 3 = 7.$$

$$A^T = \begin{pmatrix} 0 & 2 & 3 \\ -1 & 4 & 5 \\ 6 & -2 & 3 \end{pmatrix}.$$

Compute $\det(B)$:

$$\det(B) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det(B) &= 1 \cdot \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ &= 1 \cdot (0 - (-1)) - 1 \cdot (0 - 1) + 2 \cdot (-1 - 0) \\ &= 1 \cdot 1 - 1 \cdot (-1) + 2 \cdot (-1) \\ &= 1 + 1 - 2 = 0. \end{aligned}$$

Thus $\det(B) = 0$.

Compute $B - 2A$:

$$\begin{aligned} 2A &= \begin{pmatrix} 0 & -2 & 12 \\ 4 & 8 & -4 \\ 6 & 10 & 6 \end{pmatrix} \\ B - 2A &= \begin{pmatrix} 1-0 & 1-(-2) & 2-12 \\ 1-4 & 0-8 & 1-(-4) \\ 1-6 & -1-10 & 0-6 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -10 \\ -3 & -8 & 5 \\ -5 & -11 & -6 \end{pmatrix}. \end{aligned}$$

Compute AB :

$$AB = \begin{pmatrix} 0 & -1 & 6 \\ 2 & 4 & -2 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Row 1:

$$\begin{aligned} (AB)_{11} &= 0 \cdot 1 + (-1) \cdot 1 + 6 \cdot 1 = 0 - 1 + 6 = 5 \\ (AB)_{12} &= 0 \cdot 1 + (-1) \cdot 0 + 6 \cdot (-1) = 0 + 0 - 6 = -6 \\ (AB)_{13} &= 0 \cdot 2 + (-1) \cdot 1 + 6 \cdot 0 = 0 - 1 + 0 = -1 \end{aligned}$$

Row 2:

$$\begin{aligned} (AB)_{21} &= 2 \cdot 1 + 4 \cdot 1 + (-2) \cdot 1 = 2 + 4 - 2 = 4 \\ (AB)_{22} &= 2 \cdot 1 + 4 \cdot 0 + (-2) \cdot (-1) = 2 + 0 + 2 = 4 \\ (AB)_{23} &= 2 \cdot 2 + 4 \cdot 1 + (-2) \cdot 0 = 4 + 4 + 0 = 8 \end{aligned}$$

Row 3:

$$\begin{aligned} (AB)_{31} &= 3 \cdot 1 + 5 \cdot 1 + 3 \cdot 1 = 3 + 5 + 3 = 11 \\ (AB)_{32} &= 3 \cdot 1 + 5 \cdot 0 + 3 \cdot (-1) = 3 + 0 - 3 = 0 \\ (AB)_{33} &= 3 \cdot 2 + 5 \cdot 1 + 3 \cdot 0 = 6 + 5 + 0 = 11 \end{aligned}$$

Thus:

$$AB = \begin{pmatrix} 5 & -6 & -1 \\ 4 & 4 & 8 \\ 11 & 0 & 11 \end{pmatrix}.$$

II.

1. Calculate $\det(A)$ in terms of α , then find α such that A is invertible.

$$\det(A) = \begin{vmatrix} 1 & 0 & \alpha \\ 0 & \alpha & 1 \\ \alpha & 1 & 0 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} \alpha & 1 \\ 1 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & 1 \\ \alpha & 0 \end{vmatrix} + \alpha \cdot \begin{vmatrix} 0 & \alpha \\ \alpha & 1 \end{vmatrix} \\ &= 1 \cdot (\alpha \cdot 0 - 1 \cdot 1) + \alpha \cdot (0 \cdot 1 - \alpha \cdot \alpha) \\ &= 1 \cdot (0 - 1) + \alpha \cdot (0 - \alpha^2) \\ &= -1 - \alpha^3. \end{aligned}$$

Thus:

$$\det(A) = -\alpha^3 - 1 = -(\alpha^3 + 1).$$

Factor $\alpha^3 + 1 = (\alpha + 1)(\alpha^2 - \alpha + 1)$.

The quadratic $\alpha^2 - \alpha + 1 = 0$ has discriminant $\Delta = 1 - 4 = -3 < 0$, so no real roots.

Thus $\det(A) = 0$ for $\alpha = -1$ only.

A is invertible iff $\det(A) \neq 0$, i.e.:

$$\boxed{\alpha \neq -1}.$$

2. For $\alpha = 1$, calculate A^{-1} using the cofactors method.

For $\alpha = 1$, we have:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

First compute $\det(A)$:

$$\det(A) = -1 - 1^3 = -2.$$

Now compute the cofactor matrix:

$$\begin{aligned} C_{11} &= (-1)^2 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 1) = -1 \\ C_{12} &= (-1)^3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \cdot (0 - 1) = 1 \\ C_{13} &= (-1)^4 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 \cdot (0 - 1) = -1 \\ C_{21} &= (-1)^3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \cdot (0 - 1) = 1 \\ C_{22} &= (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 1) = -1 \\ C_{23} &= (-1)^5 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1 \cdot (1 - 0) = -1 \\ C_{31} &= (-1)^4 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 \cdot (0 - 1) = -1 \\ C_{32} &= (-1)^5 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \cdot (1 - 0) = -1 \\ C_{33} &= (-1)^6 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1 - 0) = 1 \end{aligned}$$

Cofactor matrix:

$$C = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Adjugate matrix $= C^T$:

$$\text{adj}(A) = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Thus:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{-2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Solution – Exercise 2

Let the linear system (S_1) :

$$(S_1) \begin{cases} -x_1 + 2x_2 + x_3 = 4 \\ 3x_1 - 5x_2 + 3x_3 = 1 \\ 2x_1 + 7x_2 - x_3 = -8 \end{cases}$$

1. Matrix form and Cramer system verification.

The system can be written as $AX = B$ where:

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 1 \\ -8 \end{pmatrix}.$$

A system is Cramer if it has n equations, n unknowns, and $\det A \neq 0$. Here $n = 3$.

Compute $\det A$:

$$\det A = \begin{vmatrix} -1 & 2 & 1 \\ 3 & -5 & 3 \\ 2 & 7 & -1 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= (-1) \cdot \begin{vmatrix} -5 & 3 \\ 7 & -1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & -5 \\ 2 & 7 \end{vmatrix} \\ &= (-1) \cdot ((-5)(-1) - (3)(7)) - 2 \cdot ((3)(-1) - (3)(2)) + 1 \cdot ((3)(7) - (-5)(2)) \\ &= (-1) \cdot (5 - 21) - 2 \cdot (-3 - 6) + 1 \cdot (21 + 10) \\ &= (-1) \cdot (-16) - 2 \cdot (-9) + 31 \\ &= 16 + 18 + 31 = 65. \end{aligned}$$

Since $\det A = 65 \neq 0$, the system is a Cramer system.

2. Solve the system using Cramer's method and Gauss's method.

(a) Cramer's method:

We have $\det A = 65$.

Compute $\det A_1$ (replace column 1 with B):

$$A_1 = \begin{pmatrix} 4 & 2 & 1 \\ 1 & -5 & 3 \\ -8 & 7 & -1 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det A_1 &= 4 \cdot \begin{vmatrix} -5 & 3 \\ 7 & -1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 3 \\ -8 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -5 \\ -8 & 7 \end{vmatrix} \\
 &= 4 \cdot ((-5)(-1) - (3)(7)) - 2 \cdot ((1)(-1) - (3)(-8)) + 1 \cdot ((1)(7) - (-5)(-8)) \\
 &= 4 \cdot (5 - 21) - 2 \cdot (-1 + 24) + 1 \cdot (7 - 40) \\
 &= 4 \cdot (-16) - 2 \cdot 23 + 1 \cdot (-33) \\
 &= -64 - 46 - 33 = -143.
 \end{aligned}$$

Compute $\det A_2$ (replace column 2 with B):

$$A_2 = \begin{pmatrix} -1 & 4 & 1 \\ 3 & 1 & 3 \\ 2 & -8 & -1 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det A_2 &= (-1) \cdot \begin{vmatrix} 1 & 3 \\ -8 & -1 \end{vmatrix} - 4 \cdot \begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 1 \\ 2 & -8 \end{vmatrix} \\
 &= (-1) \cdot ((1)(-1) - (3)(-8)) - 4 \cdot ((3)(-1) - (3)(2)) + 1 \cdot ((3)(-8) - (1)(2)) \\
 &= (-1) \cdot (-1 + 24) - 4 \cdot (-3 - 6) + 1 \cdot (-24 - 2) \\
 &= (-1) \cdot 23 - 4 \cdot (-9) + (-26) \\
 &= -23 + 36 - 26 = -13.
 \end{aligned}$$

Compute $\det A_3$ (replace column 3 with B):

$$A_3 = \begin{pmatrix} -1 & 2 & 4 \\ 3 & -5 & 1 \\ 2 & 7 & -8 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned}
 \det A_3 &= (-1) \cdot \begin{vmatrix} -5 & 1 \\ 7 & -8 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 1 \\ 2 & -8 \end{vmatrix} + 4 \cdot \begin{vmatrix} 3 & -5 \\ 2 & 7 \end{vmatrix} \\
 &= (-1) \cdot ((-5)(-8) - (1)(7)) - 2 \cdot ((3)(-8) - (1)(2)) + 4 \cdot ((3)(7) - (-5)(2)) \\
 &= (-1) \cdot (40 - 7) - 2 \cdot (-24 - 2) + 4 \cdot (21 + 10) \\
 &= (-1) \cdot 33 - 2 \cdot (-26) + 4 \cdot 31 \\
 &= -33 + 52 + 124 = 143.
 \end{aligned}$$

Now by Cramer's rule:

$$x_1 = \frac{\det A_1}{\det A} = \frac{-143}{65} = -\frac{11}{5}, \quad x_2 = \frac{\det A_2}{\det A} = \frac{-13}{65} = -\frac{1}{5}, \quad x_3 = \frac{\det A_3}{\det A} = \frac{143}{65} = \frac{11}{5}.$$

Thus:

$$\boxed{x_1 = -\frac{11}{5}, \quad x_2 = -\frac{1}{5}, \quad x_3 = \frac{11}{5}}.$$

(b) Gauss method:

Write the augmented matrix $[A|B]$:

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 3 & -5 & 3 & 1 \\ 2 & 7 & -1 & -8 \end{array} \right].$$

Multiply R_1 by -1 to get a leading 1:

$$R_1 \leftarrow -R_1 : \left[\begin{array}{ccc|c} 1 & -2 & -1 & -4 \\ 3 & -5 & 3 & 1 \\ 2 & 7 & -1 & -8 \end{array} \right].$$

Eliminate x_1 from rows 2 and 3: $R_2 \leftarrow R_2 - 3R_1$:

$$(3 - 3, -5 + 6, 3 + 3, 1 + 12) = (0, 1, 6, 13)$$

$R_3 \leftarrow R_3 - 2R_1$:

$$(2 - 2, 7 + 4, -1 + 2, -8 + 8) = (0, 11, 1, 0)$$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -4 \\ 0 & 1 & 6 & 13 \\ 0 & 11 & 1 & 0 \end{array} \right].$$

Eliminate x_2 from row 3: $R_3 \leftarrow R_3 - 11R_2$:

$$(0, 11 - 11, 1 - 66, 0 - 143) = (0, 0, -65, -143)$$

Matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -4 \\ 0 & 1 & 6 & 13 \\ 0 & 0 & -65 & -143 \end{array} \right].$$

Back substitution:

$$\text{From } R_3: -65x_3 = -143 \Rightarrow x_3 = \frac{-143}{-65} = \frac{143}{65} = \frac{11}{5}.$$

$$\text{From } R_2: x_2 + 6x_3 = 13 \Rightarrow x_2 + 6 \cdot \frac{11}{5} = 13 \Rightarrow x_2 + \frac{66}{5} = \frac{65}{5} \Rightarrow x_2 = \frac{65}{5} - \frac{66}{5} = -\frac{1}{5}.$$

$$\begin{aligned} \text{From } R_1: x_1 - 2x_2 - x_3 &= -4 \Rightarrow x_1 - 2\left(-\frac{1}{5}\right) - \frac{11}{5} = -4 \\ \Rightarrow x_1 + \frac{2}{5} - \frac{11}{5} &= -4 \Rightarrow x_1 - \frac{9}{5} = -4 \Rightarrow x_1 = -4 + \frac{9}{5} = -\frac{20}{5} + \frac{9}{5} = -\frac{11}{5}. \end{aligned}$$

Thus the same solution is obtained:

$$\boxed{x_1 = -\frac{11}{5}, \quad x_2 = -\frac{1}{5}, \quad x_3 = \frac{11}{5}}.$$

Verification:

$$-\left(-\frac{11}{5}\right) + 2\left(-\frac{1}{5}\right) + \frac{11}{5} = \frac{11}{5} - \frac{2}{5} + \frac{11}{5} = \frac{20}{5} = 4 \quad \checkmark$$

$$3\left(-\frac{11}{5}\right) - 5\left(-\frac{1}{5}\right) + 3\left(\frac{11}{5}\right) = -\frac{33}{5} + \frac{5}{5} + \frac{33}{5} = 1 \quad \checkmark$$

$$2\left(-\frac{11}{5}\right) + 7\left(-\frac{1}{5}\right) - \frac{11}{5} = -\frac{22}{5} - \frac{7}{5} - \frac{11}{5} = -\frac{40}{5} = -8 \quad \checkmark$$

2.6 Resit Exam (2022/2023)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Resit Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 19/06/2023

Duration: 1h 30min

Exercise 1

(07 points)

Let the square matrix A be defined as:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

1. Calculate the determinant $|A|$.
2. Show that A is invertible.
3. Find the inverse matrix A^{-1} using the cofactor method.
4. Verify that $A \times A^{-1} = I_3$.

Exercise 2

(07 points)

Consider the following system of linear equations with a parameter $a \in \mathbb{R}$:

$$(S) : \begin{cases} ax_1 + x_2 + x_3 = 1 \\ x_1 + ax_2 + x_3 = a \\ x_1 + x_2 + ax_3 = a^2 \end{cases}$$

1. Write the system in matrix form $AX = B$.
2. Calculate the determinant of the system matrix $|A|$.
3. For which values of a is the system a Cramer system?
4. Solve the system (S) for $a = 0$.

Exercise 3

(06 points)

Let $M = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$.

1. Determine the characteristic polynomial $P(\lambda)$.
2. Find the eigenvalues λ_1, λ_2 and the corresponding eigenvectors v_1, v_2 .
3. Determine the matrices P and D such that $M = PDP^{-1}$.

Full Solution

Solution – Exercise 1

1. **Determinant Calculation:** $|A| = 1(6-4) - 1(3-2) + 1(2-2) = 2 - 1 + 0 = 1$.
2. **Invertibility:** Since $|A| = 1 \neq 0$, the matrix A is non-singular and therefore invertible.
3. **Inverse Matrix:** First, find the cofactor matrix $Com(A)$:

- $c_{11} = 2, c_{12} = -1, c_{13} = 0$
- $c_{21} = -1, c_{22} = 2, c_{23} = -1$
- $c_{31} = 0, c_{32} = -1, c_{33} = 1$

Since $|A| = 1$, $A^{-1} = (Com(A))^T$. Because $Com(A)$ is symmetric in this case:

$$A^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

4. **Verification:** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2-1+0 & -1+2-1 & 0-1+1 \\ 2-2+0 & -1+4-2 & 0-2+2 \\ 2-2+0 & -1+4-3 & 0-2+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Solution – Exercise 2

1. **Matrix Form:** $\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix}.$
2. **Determinant Calculation:** $|A| = a(a^2-1) - 1(a-1) + 1(1-a) = a(a-1)(a+1) - 2(a-1) = (a-1)[a(a+1) - 2]$. $|A| = (a-1)(a^2+a-2) = (\mathbf{a}-\mathbf{1})^2(\mathbf{a}+\mathbf{2})$.
3. **Cramer System Condition:** A system is Cramer if its determinant is non-zero. $(a-1)^2(a+2) \neq 0 \implies \mathbf{a} \neq \mathbf{1}$ and $\mathbf{a} \neq \mathbf{-2}$.

4. **Solution for $a = 0$:** If $a = 0$, $|A| = (-1)^2(2) = 2$. $\Delta_{x_1} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1 \implies$
 $x_1 = -\mathbf{0.5}$. $\Delta_{x_2} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1 \implies x_2 = \mathbf{0.5}$. $\Delta_{x_3} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 1 \implies x_3 =$
 $\mathbf{0.5}$.

Solution – Exercise 3

1. **Characteristic Polynomial:** $P(\lambda) = \det(M - \lambda I) = \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5$.
2. **Eigenvalues and Eigenvectors:** Solving $\lambda^2 - 2\lambda + 5 = 0$ using $\Delta = 4 - 20 = -16$: $\lambda = \frac{2 \pm 4i}{2} \implies \lambda_1 = \mathbf{1 + 2i}$, $\lambda_2 = \mathbf{1 - 2i}$. Eigenvectors are found by solving $(M - \lambda I)v = 0$ in the complex domain.
3. **Diagonalization:** $\mathbf{D} = \begin{pmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{pmatrix}$ and P is the matrix formed by the complex eigenvectors.

2.7 Resit Exam (2023/2024)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Resit Exam – Mathematics 2

Level: First Year LMD **Module:** Mathematics 2
Date: 09/06/2024 **Duration:** 1h 30min

Exercise 1

(07 points)

Consider the following 3×3 matrix A , such that:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

1. Find $\text{tr}(A)$, A^T , $(A^T)^T$, $\det(A)$, and A^{-1} .
2. Calculate A^2 and A^3 , then show that: $A^3 - A = 4I_3$.
3. Deduce that A is invertible, then determine A^{-1} using the previous relation.

Exercise 2

(07 points)

Let a be a real number and (S) be a system of linear equations of variables x_1 , x_2 , and x_3 such that:

$$(S) : \begin{cases} ax_1 + x_2 + x_3 = 1 \\ -x_1 + (a-1)x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 3 \end{cases}$$

1. Rewrite the system (S) in the matrix form $AX = B$.
2. Determine the values of the real number a for which the system (S) is a Cramer's system.
3. For $a = 3$, find the solution of (S) using Cramer's method.

Exercise 3

(06 points)

Let A be the matrix given by: $A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

1. Determine the characteristic polynomial of A .
2. Find the eigenvalues and the eigenvectors for A .
3. Find the matrices D , P , and P^{-1} such that $A = PDP^{-1}$.

Full Solution

Solution – Exercise 1

Consider the following 3×3 matrix A , such that:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

1. Compute $\text{tr}(A)$, A^T , $(A^T)^T$, $\det(A)$, and A^{-1} .

$$\text{tr}(A) = 1 + (-1) + 0 = 0.$$

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 2 & 1 & 0 \end{pmatrix}.$$

$$(A^T)^T = A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}.$$

Compute $\det(A)$:

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det(A) &= 1 \cdot \begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & -1 \\ 1 & -2 \end{vmatrix} \\ &= 1 \cdot ((-1)(0) - (1)(-2)) + 2 \cdot ((0)(-2) - (-1)(1)) \\ &= 1 \cdot (0 + 2) + 2 \cdot (0 + 1) = 2 + 2 = 4. \end{aligned}$$

Thus $\det(A) = 4$.

Compute A^{-1} using the adjugate method:

Cofactors of A :

$$C_{11} = (-1)^2 \begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} = 1 \cdot (0 + 2) = 2$$

$$C_{12} = (-1)^3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \cdot (0 - 1) = 1$$

$$C_{13} = (-1)^4 \begin{vmatrix} 0 & -1 \\ 1 & -2 \end{vmatrix} = 1 \cdot (0 + 1) = 1$$

$$C_{21} = (-1)^3 \begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} = -1 \cdot (0 + 4) = -4$$

$$C_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1 \cdot (0 - 2) = -2$$

$$C_{23} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} = -1 \cdot (-2 - 0) = 2$$

$$C_{31} = (-1)^4 \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = 1 \cdot (0 + 2) = 2$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1 \cdot (1 - 0) = -1$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \cdot (-1 - 0) = -1$$

Cofactor matrix:

$$C = \begin{pmatrix} 2 & 1 & 1 \\ -4 & -2 & 2 \\ 2 & -1 & -1 \end{pmatrix}$$

Adjugate $= C^T$:

$$\text{adj}(A) = \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix}$$

Thus:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{4} \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}.$$

2. Calculate A^2 and A^3 , then show $A^3 - A = 4I_3$.

First compute A^2 :

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

Row 1:

$$(A^2)_{11} = 1 \cdot 1 + 0 \cdot 0 + 2 \cdot 1 = 1 + 0 + 2 = 3$$

$$(A^2)_{12} = 1 \cdot 0 + 0 \cdot (-1) + 2 \cdot (-2) = 0 + 0 - 4 = -4$$

$$(A^2)_{13} = 1 \cdot 2 + 0 \cdot 1 + 2 \cdot 0 = 2 + 0 + 0 = 2$$

Row 2:

$$\begin{aligned}(A^2)_{21} &= 0 \cdot 1 + (-1) \cdot 0 + 1 \cdot 1 = 0 + 0 + 1 = 1 \\(A^2)_{22} &= 0 \cdot 0 + (-1) \cdot (-1) + 1 \cdot (-2) = 0 + 1 - 2 = -1 \\(A^2)_{23} &= 0 \cdot 2 + (-1) \cdot 1 + 1 \cdot 0 = 0 - 1 + 0 = -1\end{aligned}$$

Row 3:

$$\begin{aligned}(A^2)_{31} &= 1 \cdot 1 + (-2) \cdot 0 + 0 \cdot 1 = 1 + 0 + 0 = 1 \\(A^2)_{32} &= 1 \cdot 0 + (-2) \cdot (-1) + 0 \cdot (-2) = 0 + 2 + 0 = 2 \\(A^2)_{33} &= 1 \cdot 2 + (-2) \cdot 1 + 0 \cdot 0 = 2 - 2 + 0 = 0\end{aligned}$$

Thus:

$$A^2 = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Now compute $A^3 = A^2 \cdot A$:

$$A^3 = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

Row 1:

$$\begin{aligned}(A^3)_{11} &= 3 \cdot 1 + (-4) \cdot 0 + 2 \cdot 1 = 3 + 0 + 2 = 5 \\(A^3)_{12} &= 3 \cdot 0 + (-4) \cdot (-1) + 2 \cdot (-2) = 0 + 4 - 4 = 0 \\(A^3)_{13} &= 3 \cdot 2 + (-4) \cdot 1 + 2 \cdot 0 = 6 - 4 + 0 = 2\end{aligned}$$

Row 2:

$$\begin{aligned}(A^3)_{21} &= 1 \cdot 1 + (-1) \cdot 0 + (-1) \cdot 1 = 1 + 0 - 1 = 0 \\(A^3)_{22} &= 1 \cdot 0 + (-1) \cdot (-1) + (-1) \cdot (-2) = 0 + 1 + 2 = 3 \\(A^3)_{23} &= 1 \cdot 2 + (-1) \cdot 1 + (-1) \cdot 0 = 2 - 1 + 0 = 1\end{aligned}$$

Row 3:

$$\begin{aligned}(A^3)_{31} &= 1 \cdot 1 + 2 \cdot 0 + 0 \cdot 1 = 1 + 0 + 0 = 1 \\(A^3)_{32} &= 1 \cdot 0 + 2 \cdot (-1) + 0 \cdot (-2) = 0 - 2 + 0 = -2 \\(A^3)_{33} &= 1 \cdot 2 + 2 \cdot 1 + 0 \cdot 0 = 2 + 2 + 0 = 4\end{aligned}$$

Thus:

$$A^3 = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix}.$$

Now compute $A^3 - A$:

$$A^3 - A = \begin{pmatrix} 5-1 & 0-0 & 2-2 \\ 0-0 & 3-(-1) & 1-1 \\ 1-1 & -2-(-2) & 4-0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I_3.$$

Thus $A^3 - A = 4I_3$ is verified.

3. Deduce that A is invertible, then determine A^{-1} using the previous relation.

From $A^3 - A = 4I_3$, factor A :

$$A(A^2 - I_3) = 4I_3.$$

Thus:

$$A \cdot \frac{1}{4}(A^2 - I_3) = I_3.$$

Hence A is invertible and:

$$A^{-1} = \frac{1}{4}(A^2 - I_3).$$

Compute $A^2 - I_3$:

$$A^2 - I_3 = \begin{pmatrix} 3-1 & -4 & 2 \\ 1 & -1-1 & -1 \\ 1 & 2 & 0-1 \end{pmatrix} = \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

Therefore:

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix},$$

which matches the result from part 1.

Solution – Exercise 2

Let a be a real number and (S) be a system of linear equations of variables x_1 , x_2 , and x_3 such that:

$$(S) : \begin{cases} ax_1 + x_2 + x_3 = 1 \\ -x_1 + (a-1)x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 3 \end{cases}$$

1. Matrix form $AX = B$.

$$A = \begin{pmatrix} a & 1 & 1 \\ -1 & a-1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

2. Determine the values of a for which (S) is a Cramer system.

A system is Cramer if it has n equations, n unknowns, and $\det A \neq 0$. Here $n = 3$.

Compute $\det A$:

$$\det A = \begin{vmatrix} a & 1 & 1 \\ -1 & a-1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

Subtract row 3 from row 1 and row 2 (to simplify): $R_1 \leftarrow R_1 - R_3$: $(a-1, 0, 0)$
 $R_2 \leftarrow R_2 - R_3$: $(-2, a-2, 0)$

The determinant does not change under these operations:

$$\det A = \begin{vmatrix} a-1 & 0 & 0 \\ -2 & a-2 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

Expand by column 3:

$$\det A = 1 \cdot (-1)^{3+3} \begin{vmatrix} a-1 & 0 \\ -2 & a-2 \end{vmatrix} = 1 \cdot 1 \cdot ((a-1)(a-2) - (0)(-2))$$

$$\det A = (a-1)(a-2).$$

Thus (S) is a Cramer system iff $\det A \neq 0$, i.e.:

$$\boxed{a \neq 1 \quad \text{and} \quad a \neq 2}.$$

3. For $a = 3$, find the solution using Cramer's method.

For $a = 3$, we have:

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

First compute $\det A$ using the formula above:

$$\det A = (3-1)(3-2) = 2 \times 1 = 2.$$

Compute $\det A_1$ (replace column 1 with B):

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A_1 &= 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 1 \cdot (2-1) - 1 \cdot (0-3) + 1 \cdot (0-6) \\ &= 1 - 1 \cdot (-3) + (-6) = 1 + 3 - 6 = -2. \end{aligned}$$

$$\text{Thus } x_1 = \frac{\det A_1}{\det A} = \frac{-2}{2} = -1.$$

Compute $\det A_2$ (replace column 2 with B):

$$A_2 = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

Expand by row 2:

$$\begin{aligned}\det A_2 &= (-1) \cdot (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0 \cdot (\text{minor}_{22}) + 1 \cdot (-1)^{2+3} \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \\ &= (-1) \cdot (-1) \cdot (1 - 3) + 1 \cdot (-1) \cdot (9 - 1) \\ &= 1 \cdot (-2) - 1 \cdot 8 = -2 - 8 = -10.\end{aligned}$$

Thus $x_2 = \frac{\det A_2}{\det A} = \frac{-10}{2} = -5$.

Compute $\det A_3$ (replace column 3 with B):

$$A_3 = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

Expand by row 2:

$$\begin{aligned}\det A_3 &= (-1) \cdot (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 2 \cdot (-1)^{2+2} \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + 0 \cdot (\text{minor}_{23}) \\ &= (-1) \cdot (-1) \cdot (3 - 1) + 2 \cdot 1 \cdot (9 - 1) \\ &= 1 \cdot 2 + 2 \cdot 8 = 2 + 16 = 18.\end{aligned}$$

Thus $x_3 = \frac{\det A_3}{\det A} = \frac{18}{2} = 9$.

Therefore, for $a = 3$, the solution is:

$$\boxed{x_1 = -1, \quad x_2 = -5, \quad x_3 = 9}.$$

Verification for $a = 3$:

$$\begin{aligned}3(-1) + (-5) + 9 &= -3 - 5 + 9 = 1 \quad \checkmark \\ -(-1) + (3 - 1)(-5) + 9 &= 1 + 2(-5) + 9 = 1 - 10 + 9 = 0 \quad \checkmark \\ (-1) + (-5) + 9 &= 3 \quad \checkmark\end{aligned}$$

Solution – Exercise 3

Let A be the matrix given by:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$$

1. Characteristic polynomial of A .

$$P_A(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 2 - \lambda & 1 \\ 3 & 0 - \lambda \end{vmatrix}$$

$$\begin{aligned}
&= (2 - \lambda)(-\lambda) - (1)(3) \\
&= -2\lambda + \lambda^2 - 3 \\
&= \lambda^2 - 2\lambda - 3.
\end{aligned}$$

Thus:

$$\boxed{P_A(\lambda) = \lambda^2 - 2\lambda - 3}.$$

2. Find the eigenvalues and eigenvectors.

Solve $P_A(\lambda) = 0$:

$$\begin{aligned}
\lambda^2 - 2\lambda - 3 &= 0 \\
(\lambda - 3)(\lambda + 1) &= 0 \\
\lambda_1 = 3, \quad \lambda_2 &= -1.
\end{aligned}$$

Eigenvector for $\lambda_1 = 3$:

$$\begin{aligned}
(A - 3I_2)v = 0 &\Rightarrow \begin{pmatrix} 2-3 & 1 \\ 3 & 0-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0
\end{aligned}$$

From the first row: $-x + y = 0 \Rightarrow y = x$. Choose $x = 1, y = 1$.

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = -1$:

$$\begin{aligned}
(A + I_2)v = 0 &\Rightarrow \begin{pmatrix} 2+1 & 1 \\ 3 & 0+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \\
&\Rightarrow \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0
\end{aligned}$$

From the first row: $3x + y = 0 \Rightarrow y = -3x$. Choose $x = 1, y = -3$.

$$v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Thus:

$$\boxed{\lambda_1 = 3, v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \lambda_2 = -1, v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}.$$

3. Find D , P , and P^{-1} such that $A = PDP^{-1}$.

Let D be the diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let P be the matrix with eigenvectors as columns:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}.$$

Now compute P^{-1} :

$$\det P = (1)(-3) - (1)(1) = -3 - 1 = -4.$$

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{-4} \begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}.$$

Verification:

$$PD P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$$

$$\text{First compute } DP^{-1} = \begin{pmatrix} 3 \cdot \frac{3}{4} & 3 \cdot \frac{1}{4} \\ -1 \cdot \frac{1}{4} & -1 \cdot (-\frac{1}{4}) \end{pmatrix} = \begin{pmatrix} \frac{9}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

$$\begin{aligned} \text{Then } P(DP^{-1}) &= \begin{pmatrix} 1 \cdot \frac{9}{4} + 1 \cdot (-\frac{1}{4}) & 1 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} \\ 1 \cdot \frac{9}{4} + (-3) \cdot (-\frac{1}{4}) & 1 \cdot \frac{3}{4} + (-3) \cdot \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{9}{4} - \frac{1}{4} & \frac{3}{4} + \frac{1}{4} \\ \frac{9}{4} + \frac{3}{4} & \frac{3}{4} - \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{8}{4} & \frac{4}{4} \\ \frac{12}{4} & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = A. \end{aligned}$$

Thus:

$$\boxed{D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}}.$$

2.8 Resit Exam (2024/2025)

University of Ferhat Abbas – Sétif 1

Faculty of Economics, Commerce and Management Sciences | Department of Basic Education

Resit Exam – Mathematics 2

Level: First Year LMD

Module: Mathematics 2

Date: 15/06/2025

Duration: 1h 30min

Exercise 1

(06 points)

Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

1. Calculate the determinant of A using the cofactor method.
2. Calculate A^{-1} using the adjoint matrix method.
3. Calculate A^2 and verify that $A^2 - 3A + 2I_3 = 0$, where I_3 is the identity matrix and 0 is the zero matrix.

Exercise 2

(08 points)

Consider the system (S) in the unknowns x, y and z with parameter α :

$$(S) : \begin{cases} x + \alpha y + z = 1 \\ \alpha x + y + z = \alpha \\ x + y + \alpha z = \alpha^2 \end{cases}$$

1. Rewrite the system (S) in matrix form.
2. Determine the values of α for which the system (S) is a Cramer system.
3. Using the Gauss method, find the solutions of the system for $\alpha = 1$ and $\alpha = -2$.

Note: $-\alpha^3 + 3\alpha - 2 = -(\alpha - 1)^2(\alpha + 2)$.

Exercise 3

(06 points)

Let A be the matrix given by:

$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

1. Determine the characteristic polynomial of A .
2. Find the eigenvalues and eigenvectors for A .
3. Find the matrices D , P and P^{-1} such that $A = PDP^{-1}$. (D is a diagonal matrix, P is an invertible matrix and P^{-1} is the inverse of P).

Full Solution

Solution – Exercise 1

Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

1. Determinant of A using the cofactor method.

$$\det A = \begin{vmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

Expand by row 1:

$$\begin{aligned} \det A &= 0 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} \\ &= 0 - 1 \cdot ((-1)(2) - (-1)(1)) - 1 \cdot ((-1)(-1) - (2)(1)) \\ &= -1 \cdot (-2 + 1) - 1 \cdot (1 - 2) \\ &= -1 \cdot (-1) - 1 \cdot (-1) \\ &= 1 + 1 = 2. \end{aligned}$$

$$\boxed{\det A = 2}.$$

2. Calculate A^{-1} using the adjoint matrix method.

We need the cofactor matrix $C = (C_{ij})$.

$$\begin{aligned}
C_{11} &= (-1)^{1+1} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 1 \cdot (4 - 1) = 3 \\
C_{12} &= (-1)^{1+2} \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = -1 \cdot (-2 + 1) = -1 \cdot (-1) = 1 \\
C_{13} &= (-1)^{1+3} \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = 1 \cdot (1 - 2) = -1 \\
C_{21} &= (-1)^{2+1} \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = -1 \cdot (2 - 1) = -1 \\
C_{22} &= (-1)^{2+2} \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 1 \cdot (0 + 1) = 1 \\
C_{23} &= (-1)^{2+3} \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \cdot (0 - 1) = 1 \\
C_{31} &= (-1)^{3+1} \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1 \cdot (-1 + 2) = 1 \\
C_{32} &= (-1)^{3+2} \begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix} = -1 \cdot (0 - 1) = 1 \\
C_{33} &= (-1)^{3+3} \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 1 \cdot (0 + 1) = 1
\end{aligned}$$

Cofactor matrix:

$$C = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Adjugate matrix = C^T :

$$\text{adj}(A) = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

Then:

$$A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{1}{2} \begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}}.$$

3. Calculate A^2 and verify $A^2 - 3A + 2I_3 = 0$.

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Compute each entry:

Row 1:

$$(A^2)_{11} = 0 \cdot 0 + 1 \cdot (-1) + (-1) \cdot 1 = 0 - 1 - 1 = -2$$

$$(A^2)_{12} = 0 \cdot 1 + 1 \cdot 2 + (-1) \cdot (-1) = 0 + 2 + 1 = 3$$

$$(A^2)_{13} = 0 \cdot (-1) + 1 \cdot (-1) + (-1) \cdot 2 = 0 - 1 - 2 = -3$$

Row 2:

$$(A^2)_{21} = (-1) \cdot 0 + 2 \cdot (-1) + (-1) \cdot 1 = 0 - 2 - 1 = -3$$

$$(A^2)_{22} = (-1) \cdot 1 + 2 \cdot 2 + (-1) \cdot (-1) = -1 + 4 + 1 = 4$$

$$(A^2)_{23} = (-1) \cdot (-1) + 2 \cdot (-1) + (-1) \cdot 2 = 1 - 2 - 2 = -3$$

Row 3:

$$(A^2)_{31} = 1 \cdot 0 + (-1) \cdot (-1) + 2 \cdot 1 = 0 + 1 + 2 = 3$$

$$(A^2)_{32} = 1 \cdot 1 + (-1) \cdot 2 + 2 \cdot (-1) = 1 - 2 - 2 = -3$$

$$(A^2)_{33} = 1 \cdot (-1) + (-1) \cdot (-1) + 2 \cdot 2 = -1 + 1 + 4 = 4$$

Thus:

$$A^2 = \begin{pmatrix} -2 & 3 & -3 \\ -3 & 4 & -3 \\ 3 & -3 & 4 \end{pmatrix}.$$

Now compute $3A$:

$$3A = \begin{pmatrix} 0 & 3 & -3 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{pmatrix}$$

Then:

$$A^2 - 3A = \begin{pmatrix} -2-0 & 3-3 & -3-(-3) \\ -3-(-3) & 4-6 & -3-(-3) \\ 3-3 & -3-(-3) & 4-6 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Add $2I_3$:

$$A^2 - 3A + 2I_3 = \begin{pmatrix} -2+2 & 0 & 0 \\ 0 & -2+2 & 0 \\ 0 & 0 & -2+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$$A^2 = \begin{pmatrix} -2 & 3 & -3 \\ -3 & 4 & -3 \\ 3 & -3 & 4 \end{pmatrix}, \quad A^2 - 3A + 2I_3 = 0 \text{ verified.}$$

Solution – Exercise 2

Consider the system (S) in the unknowns x, y and z with parameter α :

$$(S) : \begin{cases} x + \alpha y + z = 1 \\ \alpha x + y + z = \alpha \\ x + y + \alpha z = \alpha^2 \end{cases}$$

Note: $-\alpha^3 + 3\alpha - 2 = -(\alpha - 1)^2(\alpha + 2)$.

1. Matrix form.

$$A = \begin{pmatrix} 1 & \alpha & 1 \\ \alpha & 1 & 1 \\ 1 & 1 & \alpha \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix}.$$

The system is $AX = b$.

2. Values of α for which (S) is a Cramer system.

A system is Cramer if $\det A \neq 0$.

Compute $\det A$:

$$\det A = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha & 1 & 1 \\ 1 & 1 & \alpha \end{vmatrix}$$

Use row operations: $R_2 \leftarrow R_2 - \alpha R_1$, $R_3 \leftarrow R_3 - R_1$:

$$\det A = \begin{vmatrix} 1 & \alpha & 1 \\ 0 & 1 - \alpha^2 & 1 - \alpha \\ 0 & 1 - \alpha & \alpha - 1 \end{vmatrix}$$

Expand by column 1:

$$\det A = 1 \cdot \begin{vmatrix} 1 - \alpha^2 & 1 - \alpha \\ 1 - \alpha & \alpha - 1 \end{vmatrix}$$

Factor $1 - \alpha$ from row 1 and also from row 2: Row 1: $1 - \alpha^2 = (1 - \alpha)(1 + \alpha)$

Row 2: $\alpha - 1 = -(1 - \alpha)$.

$$\det A = \begin{vmatrix} (1 - \alpha)(1 + \alpha) & 1 - \alpha \\ 1 - \alpha & -(1 - \alpha) \end{vmatrix}$$

Factor $1 - \alpha$ from row 1 and also from row 2:

$$\det A = (1 - \alpha)(1 - \alpha) \begin{vmatrix} 1 + \alpha & 1 \\ 1 & -1 \end{vmatrix}$$

$$\det A = (1 - \alpha)^2 ((1 + \alpha)(-1) - (1)(1))$$

$$\det A = (1 - \alpha)^2 (-1 - \alpha - 1)$$

$$\det A = (1 - \alpha)^2 (-\alpha - 2)$$

$$\det A = -(\alpha + 2)(1 - \alpha)^2.$$

Thus $\det A = -(\alpha + 2)(\alpha - 1)^2$.

Set $\det A \neq 0$:

$$-(\alpha + 2)(\alpha - 1)^2 \neq 0 \quad \Rightarrow \quad \alpha \neq -2 \text{ and } \alpha \neq 1.$$

Thus (S) is a Cramer system for:

$$\boxed{\alpha \in \mathbb{R} \setminus \{-2, 1\}}.$$

3. Using Gauss method, find the solutions for $\alpha = 1$ and $\alpha = -2$.

Case 1: $\alpha = 1$

System becomes:

$$\begin{cases} x + y + z = 1 \\ x + y + z = 1 \\ x + y + z = 1 \end{cases}$$

All equations are identical: $x + y + z = 1$.

Infinite solutions. Parameterize: $y = s$, $z = t$, $x = 1 - s - t$.

$$\boxed{(x, y, z) = (1 - s - t, s, t), \quad s, t \in \mathbb{R}}.$$

Case 2: $\alpha = -2$

System becomes:

$$\begin{cases} x - 2y + z = 1 \\ -2x + y + z = -2 \\ x + y - 2z = 4 \end{cases}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & -2 \\ 1 & 1 & -2 & 4 \end{array} \right]$$

$R_2 \leftarrow R_2 + 2R_1$:

$$(-2 + 2(1) = 0, \quad 1 + 2(-2) = -3, \quad 1 + 2(1) = 3, \quad -2 + 2(1) = 0)$$

New row 2: $[0, -3, 3, 0]$.

$R_3 \leftarrow R_3 - R_1$:

$$(1 - 1 = 0, \quad 1 - (-2) = 3, \quad -2 - 1 = -3, \quad 4 - 1 = 3)$$

New row 3: $[0, 3, -3, 3]$.

Matrix now:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 3 \end{array} \right]$$

$$R_3 \leftarrow R_3 + R_2:$$

$$(0 + 0 = 0, \quad 3 + (-3) = 0, \quad -3 + 3 = 0, \quad 3 + 0 = 3)$$

New row 3: $[0, 0, 0, 3]$.

Last row: $0 = 3$, impossible.

Thus for $\alpha = -2$:

No solution.

Solution – Exercise 3

Let A be the matrix given by:

$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

1. Characteristic polynomial of A .

$$\begin{aligned} P_A(\lambda) &= \det(A - \lambda I_2) = \begin{vmatrix} 4 - \lambda & -1 \\ 2 & 1 - \lambda \end{vmatrix} \\ &= (4 - \lambda)(1 - \lambda) - (-1)(2) \\ &= (4 - \lambda)(1 - \lambda) + 2 \\ &= 4 - 4\lambda - \lambda + \lambda^2 + 2 \\ &= \lambda^2 - 5\lambda + 6. \end{aligned}$$

Thus:

$$P_A(\lambda) = \lambda^2 - 5\lambda + 6.$$

2. Find the eigenvalues and eigenvectors.

Solve $P_A(\lambda) = 0$:

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 3.$$

Eigenvector for $\lambda_1 = 2$:

$$(A - 2I_2)v = 0 \quad \Rightarrow \quad \begin{pmatrix} 4 - 2 & -1 \\ 2 & 1 - 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

First equation: $2x - y = 0 \Rightarrow y = 2x$. Choose $x = 1$, $y = 2$.

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = 3$:

$$(A - 3I_2)v = 0 \Rightarrow \begin{pmatrix} 4-3 & -1 \\ 2 & 1-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

First equation: $x - y = 0 \Rightarrow x = y$. Choose $x = 1, y = 1$.

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus:

$$\boxed{\lambda_1 = 2, v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = 3, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3. Find D, P, P^{-1} such that $A = PDP^{-1}$.

Since A has two distinct eigenvalues, it is diagonalizable.

Let P be the matrix with eigenvectors as columns:

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Let D be the diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then $A = PDP^{-1}$.

Compute P^{-1} :

$$\det P = (1)(1) - (1)(2) = 1 - 2 = -1.$$

$$P^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Verification (optional):

$$PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$\text{First compute } DP^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 6 & -3 \end{pmatrix}.$$

$$\text{Then } P(DP^{-1}) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 6 & -3 \end{pmatrix} = \begin{pmatrix} -2+6 & 2-3 \\ -4+6 & 4-3 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} = A.$$

Thus:

$$\boxed{D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Chapter 3

Proposed Exams – Additional Practice

Note to Students

These proposed exams focus exclusively on **Mathematics 2** content: matrix operations, determinants (Cofactor, Sarrus, Gauss methods), matrix inverse (Adjoint, Gauss-Jordan methods), linear systems (Cramer: Cramer's rule, Inverse method, Gauss elimination; Non-Cramer: Rank method, Gauss elimination), eigenvalues, eigenvectors, and diagonalization (Gauss method for solving $(A - \lambda I)v = 0$). Full solutions are provided.

3.1 MCQ Exam –(Version 1)

MCQ Practice Exam #1 –

Level: First Year LMD
Module: Mathematics 2
Duration:

Part 1: Multiple Choice Questions (10 points)

Given the matrices:

$$A = \begin{pmatrix} -1 & 0 & 3 \\ 2 & 6 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 5 & 6 \end{pmatrix}.$$

Exercise 1

(10 points)

Choose the correct answer for each question (1 point each):

1. $A + B$ is: (A) a square matrix (B) a non-square matrix (C) cannot be computed
2. $A \times C$ equals: (A) $\begin{pmatrix} 14 & 18 \\ 34 & 42 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 0 \\ -6 & 6 \\ 30 & 24 \end{pmatrix}$ (C) cannot be computed
3. $A \times B \times C$ is of type: (A) 3×3 (B) 2×2 (C) cannot be computed
4. $\text{tr}(B)$ equals: (A) 2 (B) -2 (C) cannot be computed
5. C^T equals: (A) $\begin{pmatrix} 1 & 2 & 5 \\ 0 & 3 & 6 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 5 & 6 \end{pmatrix}$ (C) $\begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & 6 \end{pmatrix}$
6. $|A|$ is: (A) 5 (B) -5 (C) cannot be computed
7. $|A \times B \times C|$ satisfies: (A) $= |A||B||C|$ (B) $\neq |A||B||C|$ (C) cannot be computed
8. $|B|$ equals: (A) 5 (B) -5 (C) 15
9. B^{-1} satisfies: (A) $B^{-1}B = I_3$ (B) $B^{-1}B = I_4$ (C) $B^{-1}B = B$
10. C^{-1} is: (A) $C_{2 \times 3}^{-1}$ (B) $C_{3 \times 2}^{-1}$ (C) cannot be computed

Part 2: Matrix Inverse (10 points)

Let $A = \begin{pmatrix} -2 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{9} & \frac{2}{9} \\ 0 & \frac{4}{9} & -\frac{1}{9} \end{pmatrix}$.

Exercise 2**(10 points)**

1. Compute $|A|$ and $|B|$.
2. Are A and B invertible? Justify.
3. Compute the cofactor matrix $\text{Com}(A)$.
4. Compute $(\text{Com}(A))^T$.
5. Compute A^{-1} and deduce B^{-1} .

Solutions – MCQ Exam (Version 1)**Solution – Exercise 1**

1. (C) cannot be computed (different dimensions: A is 2×3 , B is 3×3)
2. (A) $\begin{pmatrix} 14 & 18 \\ 34 & 42 \end{pmatrix}$ ($A_{2 \times 3} \times C_{3 \times 2} = D_{2 \times 2}$)
3. (B) 2×2 ($A_{2 \times 3} \times B_{3 \times 3} \times C_{3 \times 2} = D_{2 \times 2}$)
4. (B) -2 (diagonal sum: $-2 + 1 + (-1) = -2$)
5. (A) $\begin{pmatrix} 1 & 2 & 5 \\ 0 & 3 & 6 \end{pmatrix}$ (transpose of 3×2 is 2×3)
6. (C) cannot be computed (determinant defined only for square matrices)
7. (B) $\neq |A||B||C|$ ($|A|$ and $|C|$ are undefined)
8. (C) 15 (compute by Sarrus or cofactor expansion)
9. (A) $B^{-1}B = I_3$ (definition of inverse)
10. (C) cannot be computed (C is not square)

Solutions – Part 2

Solution – Exercise 2

1. Determinants:

$$|A| = \begin{vmatrix} -2 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 4 & -1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} = -2(1 \cdot (-1) - 2 \cdot 4) = -2(-1 - 8) = -2(-9) = 18.$$

$$|B| = \frac{1}{18} \quad (\text{since } B = A^{-1}).$$

2. Invertibility: $|A| = 18 \neq 0 \Rightarrow A$ invertible. $|B| = \frac{1}{18} \neq 0 \Rightarrow B$ invertible.

3. Cofactor matrix of A :

$$\text{Com}(A) = \begin{pmatrix} (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} & (-1)^{1+2} \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} & (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} \\ (-1)^{2+1} \begin{vmatrix} -3 & 3 \\ 4 & -1 \end{vmatrix} & (-1)^{2+2} \begin{vmatrix} -2 & 3 \\ 0 & -1 \end{vmatrix} & (-1)^{2+3} \begin{vmatrix} -2 & -3 \\ 0 & 4 \end{vmatrix} \\ (-1)^{3+1} \begin{vmatrix} -3 & 3 \\ 1 & 2 \end{vmatrix} & (-1)^{3+2} \begin{vmatrix} -2 & 3 \\ 0 & 2 \end{vmatrix} & (-1)^{3+3} \begin{vmatrix} -2 & -3 \\ 0 & 1 \end{vmatrix} \end{pmatrix}$$

$$\text{Com}(A) = \begin{pmatrix} -9 & 0 & 0 \\ 9 & 2 & 8 \\ -9 & 4 & -2 \end{pmatrix}.$$

4. Transpose of cofactor matrix:

$$(\text{Com}(A))^T = \begin{pmatrix} -9 & 9 & -9 \\ 0 & 2 & 4 \\ 0 & 8 & -2 \end{pmatrix}.$$

5. Inverse matrices:

$$A^{-1} = \frac{1}{|A|} (\text{Com}(A))^T = \frac{1}{18} \begin{pmatrix} -9 & 9 & -9 \\ 0 & 2 & 4 \\ 0 & 8 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{9} & \frac{2}{9} \\ 0 & \frac{4}{9} & -\frac{1}{9} \end{pmatrix} = B.$$

Thus $A^{-1} = B$ and $B^{-1} = A$.

3.2 MCQ Exam –(Version 2)

MCQ Practice Exam #2 –

Level: First Year LMD
Module: Mathematics 2
Duration:

Part 1: Multiple Choice Questions (10 points)

Given the matrices:

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 3 & 6 & 6 \\ 8 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 1 & 8 \\ 1 & 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 \\ 5 & 4 \\ 1 & 7 \end{pmatrix}.$$

Exercise 1

(10 points)

Choose the correct answer for each question (1 point each):

1. A is: (A) a square matrix (B) a row matrix (C) a column matrix
2. $B \times C$ equals: (A) $\begin{pmatrix} 17 & 37 \\ 13 & 60 \\ 12 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 17 & 37 \\ 13 & 60 \end{pmatrix}$ (C) cannot be computed
3. $A \times B \times C$ is of type: (A) 3×3 (B) 3×2 (C) 2×3
4. $\text{tr}(A)$ equals: (A) 7 (B) 8 (C) 9
5. C^2 is: (A) $\begin{pmatrix} 9 & 0 \\ 25 & 16 \\ 1 & 49 \end{pmatrix}$ (B) $\begin{pmatrix} 9 & 0 \\ 25 & 16 \end{pmatrix}$ (C) cannot be computed
6. $|A|$ equals: (A) 0 (B) 3 (C) -3
7. A^{-1} is: (A) $A_{3 \times 3}^{-1}$ (B) $\frac{1}{4}$ (C) cannot be computed
8. B^{-1} is: (A) $B_{3 \times 3}^{-1}$ (B) 2 (C) cannot be computed
9. C^T is of type: (A) 3×3 (B) 3×2 (C) 2×3
10. The inverse formula is: (A) $B^{-1} = \frac{1}{|B|}(\text{Com}(B))^T$ (B) $B^{-1} = \frac{1}{|B|} \text{Com}(B)$
 (C) $B^{-1} = |B|(\text{Com}(B))^T$

Part 2: Matrix Powers and Inverse (10 points)

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$.

Exercise 2**(10 points)**

1. Compute A^2 and A^3 .
2. Show that $A^3 - A = 4I_3$.
3. Deduce that A is invertible and find A^{-1} .
4. Verify that $A \times A^{-1} = I_3$.

Solutions – MCQ Exam (Version 2)**Solution – Exercise 1**

1. **(A)** square matrix (3×3)
2. **(A)** $\begin{pmatrix} 17 & 37 \\ 13 & 60 \\ 12 & 1 \end{pmatrix}$ ($B_{3 \times 3} \times C_{3 \times 2} = D_{3 \times 2}$)
3. **(B)** 3×2 ($A_{3 \times 3} \times (B \times C)_{3 \times 2} = D_{3 \times 2}$)
4. **(A)** 7 (diagonal sum: $-1 + 6 + 2 = 7$)
5. **(C)** cannot be computed (C is 3×2 , not square)
6. **(A)** 0 (rows 2 and 3 are linearly dependent: $R_2 = \frac{3}{2}R_3$)
7. **(C)** cannot be computed ($|A| = 0$, A is singular)
8. **(A)** $B_{3 \times 3}^{-1}$ ($|B| \neq 0$, B is invertible)
9. **(C)** 2×3 (transpose of a 3×2 matrix)
10. **(A)** $B^{-1} = \frac{1}{|B|}(\text{Com}(B))^T$ (standard inverse formula)

Solutions – Part 2

Solution – Exercise 2

1. Compute A^2 and A^3 :

$$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1+0+2 & 0+0-4 & 2+0+0 \\ 0+0+1 & 0+1-2 & 0-1+0 \\ 1+0+0 & 0+2+0 & 2-2+0 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}.$$

$$\begin{aligned} A^3 = A^2 \cdot A &= \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 3+0+2 & 0+4-4 & 6-4+0 \\ 1+0-1 & 0+1+2 & 2-1+0 \\ 1+0+0 & 0-2+0 & 2-2+0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 0 \end{pmatrix} \end{aligned}$$

2. Verify $A^3 - A = 4I_3$:

$$A^3 - A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I_3.$$

3. Deduce A^{-1} : From $A^3 - A = 4I_3$, factor: $A(A^2 - I_3) = 4I_3$.
Thus $A \cdot \frac{1}{4}(A^2 - I_3) = I_3$, so:

$$A^{-1} = \frac{1}{4}(A^2 - I_3) = \frac{1}{4} \begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}.$$

4. Verification: $A \times A^{-1} = I_3$ (students should verify multiplication).

3.3 Practice Exam – 1

MCQ Practice Exam #3 –

Level: First Year LMD
Module: Mathematics 2
Duration:

Exercise 1

(6 points)

Answer **True** or **False** and correct any false statement.

1. The sum of two invertible matrices is always invertible.
2. Every square matrix has a determinant.
3. If $AB = I_n$, then A and B are invertible and $B = A^{-1}$.
4. The trace of a matrix is equal to the product of its diagonal elements.
5. A homogeneous linear system always has at least one solution.
6. The product of two upper triangular matrices is upper triangular.

Exercise 2

(7 points)

Consider the matrix $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$.

1. Compute the characteristic polynomial $p(\lambda)$ of A .
2. Find the eigenvalues λ_1 and λ_2 of A .
3. Find the eigenvectors associated with each eigenvalue.
4. Find matrices P and D such that $A = PDP^{-1}$.
5. Compute A^3 using diagonalization.

Exercise 3

(7 points)

Consider the linear system:

$$(S) : \begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x + 2y - z = 3 \end{cases}$$

1. Write the system in matrix form $AX = B$.

2. Compute $\det(A)$. Is the system a Cramer system?
3. Solve the system using Cramer's rule.
4. Solve the system using Gaussian elimination.

Full Solutions – Practice Exam 1

Solution – Exercise 1

1. **False:** Counterexample: $A = I$, $B = -I$, then $A + B = 0$ is not invertible.
2. **True:** Every square matrix has a determinant.
3. **True:** If $AB = I_n$, then B is the inverse of A .
4. **False:** The trace is the **sum** of diagonal elements, not the product.
5. **True:** The trivial solution $x = 0$ always satisfies a homogeneous system.
6. **True:** The product of upper triangular matrices remains upper triangular.

Solution – Exercise 2

1. Characteristic polynomial:

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ 3 & 4 - \lambda \end{vmatrix} = (2 - \lambda)(4 - \lambda) - (-1)(3) = \lambda^2 - 6\lambda + 8 + 3 = \lambda^2 - 6\lambda + 11.$$

2. Eigenvalues:

$$\lambda = \frac{6 \pm \sqrt{36 - 44}}{2} = \frac{6 \pm \sqrt{-8}}{2} = 3 \pm i\sqrt{2}.$$

Thus $\lambda_1 = 3 + i\sqrt{2}$, $\lambda_2 = 3 - i\sqrt{2}$.

3. Eigenvectors: For $\lambda_1 = 3 + i\sqrt{2}$:

$$(A - \lambda_1 I)v = 0 \Rightarrow \begin{pmatrix} -1 - i\sqrt{2} & -1 \\ 3 & 1 - i\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

From first equation: $(-1 - i\sqrt{2})v_1 - v_2 = 0 \Rightarrow v_2 = -(1 + i\sqrt{2})v_1$. Thus $v_1 = \begin{pmatrix} 1 \\ -(1 + i\sqrt{2}) \end{pmatrix}$. For $\lambda_2 = 3 - i\sqrt{2}$, $v_2 = \overline{v_1} = \begin{pmatrix} 1 \\ -(1 - i\sqrt{2}) \end{pmatrix}$.

4. Diagonalization:

$$D = \begin{pmatrix} 3 + i\sqrt{2} & 0 \\ 0 & 3 - i\sqrt{2} \end{pmatrix}, P = \begin{pmatrix} 1 & 1 \\ -(1 + i\sqrt{2}) & -(1 - i\sqrt{2}) \end{pmatrix}.$$

5. A^3 :

$$A^3 = PD^3P^{-1}. \text{ Since } D^3 = \begin{pmatrix} (3 + i\sqrt{2})^3 & 0 \\ 0 & (3 - i\sqrt{2})^3 \end{pmatrix}.$$

Solution – Exercise 3
1. Matrix form:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix}.$$

2. Determinant:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 1(1 - 2) - 1(-2 - 1) + 1(4 + 1) = -1 + 3 + 5 = 7 \neq 0.$$

Since $|A| \neq 0$, the system is a Cramer system (unique solution).

3. Cramer's rule:

$$|A_1| = \begin{vmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 6(1 - 2) - 1(-3 - 3) + 1(6 + 3) = -6 + 6 + 9 = 9 \Rightarrow x = \frac{9}{7}.$$

$$|A_2| = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 1(-3 - 3) - 6(-2 - 1) + 1(6 - 3) = -6 + 18 + 3 = 15 \Rightarrow y = \frac{15}{7}.$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{vmatrix} = 1(-3 - 6) - 1(6 - 3) + 6(4 + 1) = -9 - 3 + 30 = 18 \Rightarrow z = \frac{18}{7}.$$

Thus $x = \frac{9}{7}$, $y = \frac{15}{7}$, $z = \frac{18}{7}$.

4. Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 3 \end{array} \right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \xrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 1 & -2 & -3 \end{array} \right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & -1 & -9 \end{array} \right) \xrightarrow{R_3 \leftarrow R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -18 \end{array} \right)$$

Back substitution: $z = \frac{18}{7}$, $y = -3 + 2z = -3 + \frac{36}{7} = \frac{15}{7}$, $x = 6 - y - z = 6 - \frac{15}{7} - \frac{18}{7} = \frac{42-33}{7} = \frac{9}{7}$.

3.4 Practice Exam – 2

MCQ Practice Exam #4 –

Level: First Year LMD
Module: Mathematics 2
Duration:

Exercise 1

(8 points)

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}$.

1. Compute $\det(A)$ using Sarrus' rule.
2. Compute the cofactor matrix $\text{Com}(A)$.
3. Compute A^{-1} using the adjoint method.
4. Verify that $A \times A^{-1} = I_3$.

Exercise 2

(6 points)

Consider the linear system:

$$(S) : \begin{cases} x + 2y - z = 4 \\ 2x + y + z = 5 \\ x - y + 2z = 1 \end{cases}$$

1. Write the system in matrix form.
2. Compute $\det(A)$. Is the system Cramer?
3. Solve the system using the inverse matrix method.

Exercise 3

(6 points)

Let $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

1. Find the characteristic polynomial of A .
2. Find the eigenvalues and eigenvectors of A .
3. Find matrices P and D such that $A = PDP^{-1}$.
4. Compute A^4 using diagonalization.

Full Solutions – Practice Exam 2

Solution – Exercise 1

1. Determinant using Sarrus:

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 1(1 \cdot 1 - 3 \cdot (-1)) - 2(0 \cdot 1 - 3 \cdot 2) + (-1)(0 \cdot (-1) - 1 \cdot 2) \\ = 1(1 + 3) - 2(0 - 6) - 1(0 - 2) = 4 - 2(-6) - 1(-2) = 4 + 12 + 2 = 18.$$

2. Cofactor matrix:

$$\text{Com}(A) = \begin{pmatrix} + \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} & - \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & + \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} \\ - \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\ + \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} & - \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{pmatrix} \\ \text{Com}(A) = \begin{pmatrix} (1+3) & -(0-6) & (0-2) \\ -(2+1) & (1+2) & -(-1-4) \\ (6+1) & -(3-0) & (1-0) \end{pmatrix} = \begin{pmatrix} 4 & 6 & -2 \\ -3 & 3 & 5 \\ 7 & -3 & 1 \end{pmatrix}.$$

3. Inverse matrix:

$$A^{-1} = \frac{1}{|A|}(\text{Com}(A))^T = \frac{1}{18} \begin{pmatrix} 4 & -3 & 7 \\ 6 & 3 & -3 \\ -2 & 5 & 1 \end{pmatrix}.$$

4. Verification: $A \times A^{-1} = I_3$ (students should verify).

Solution – Exercise 2

1. Matrix form:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}.$$

2. Determinant:

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 1(2+1) - 2(4-1) + (-1)(-2-1) = 3 - 2(3) - 1(-3) = 3 - 6 + 3 = 0.$$

Since $|A| = 0$, the system is **not** Cramer. It may have no solution or infinitely many.

3. Solve using Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 1 & 1 & 5 \\ 1 & -1 & 2 & 1 \end{array} \right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1, R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -3 & 3 & -3 \\ 0 & -3 & 3 & -3 \end{array} \right)$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -3 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The system reduces to $x + 2y - z = 4$ and $-3y + 3z = -3 \Rightarrow y = z + 1$. Substituting: $x + 2(z + 1) - z = 4 \Rightarrow x + z + 2 = 4 \Rightarrow x = 2 - z$. Thus the solution set is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - t \\ 1 + t \\ t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Solution – Exercise 3

1. Characteristic polynomial:

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 3 - \lambda & 1 \\ 2 & 2 - \lambda \end{vmatrix} = (3 - \lambda)(2 - \lambda) - 2 = \lambda^2 - 5\lambda + 6 - 2 = \lambda^2 - 5\lambda + 4 \\ &= (\lambda - 1)(\lambda - 4). \end{aligned}$$

2. Eigenvalues and eigenvectors: $\lambda_1 = 1, \lambda_2 = 4$.

$$\text{For } \lambda_1 = 1: (A - I)v = 0 \Rightarrow \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow 2v_1 + v_2 = 0 \Rightarrow v_2 = -2v_1.$$

$$\text{Eigenvector: } v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$$\text{For } \lambda_2 = 4: (A - 4I)v = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow -v_1 + v_2 = 0 \Rightarrow v_2 = v_1.$$

$$\text{Eigenvector: } v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

3. Diagonalization:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

$$\text{4. Compute } A^4: A^4 = PD^4P^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1^4 & 0 \\ 0 & 4^4 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 256 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 256 \\ -2 & 256 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 + 512 & -1 + 256 \\ -2 + 512 & 2 + 256 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 513 & 255 \\ 510 & 258 \end{pmatrix} = \begin{pmatrix} 171 & 85 \\ 170 & 86 \end{pmatrix}.$$

3.5 Practice Exam – 3

MCQ Practice Exam #5 –

Level: First Year LMD
Module: Mathematics 2
Duration:

Exercise 1

(7 points)

Let $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix}$.

1. Compute $\det(A)$ using three different methods:
 - a) Sarrus' rule
 - b) Cofactor expansion along the first row
 - c) Gauss elimination (row reduction to triangular form)
2. Is A invertible? Justify.

Exercise 2

(7 points)

Consider the same matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix}$.

1. Compute A^{-1} using two different methods:
 - a) Adjoint (cofactor) method
 - b) Gauss-Jordan elimination
2. Verify that $A \times A^{-1} = I_3$.

Exercise 3

(6 points)

Consider the linear system $(S) : \begin{cases} 2x + y - z = 5 \\ 3y + 2z = 7 \\ x - y + 2z = 2 \end{cases}$

1. Write the system in matrix form $AX = B$.
2. Compute $\det(A)$. Is (S) a Cramer system?

3. Solve (S) using Cramer's rule.
4. Solve (S) using the inverse matrix method.
5. Solve (S) using Gauss elimination.

Full Solutions – Practice Exam 3

Solution – Exercise 1

1a. Sarrus' rule:

$$\det(A) = \begin{vmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 2 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ 0 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= (2 \cdot 3 \cdot 2) + (1 \cdot 2 \cdot 1) + ((-1) \cdot 0 \cdot (-1)) - ((-1) \cdot 3 \cdot 1) - (2 \cdot 2 \cdot (-1)) - (1 \cdot 0 \cdot 2)$$

$$= 12 + 2 + 0 - (-3) - (-4) - 0 = 12 + 2 + 3 + 4 = 21.$$

1b. Cofactor expansion along first row:

$$\det(A) = 2 \cdot \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 2(3 \cdot 2 - 2 \cdot (-1)) - 1(0 \cdot 2 - 2 \cdot 1) - 1(0 \cdot (-1) - 3 \cdot 1)$$

$$= 2(6 + 2) - 1(0 - 2) - 1(0 - 3) = 2(8) - 1(-2) - 1(-3) = 16 + 2 + 3 = 21.$$

1c. Gauss elimination (row reduction to triangular form):

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow{R_3 \leftarrow 2R_3 - R_1} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & -3 & 5 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

The determinant is the product of diagonal elements: $\det(A) = 2 \times 3 \times 7 = 21$.

2. **Invertibility:** Since $\det(A) = 21 \neq 0$, A is invertible.

Solution – Exercise 2

1a. **Adjoint (cofactor) method:**

First, compute all cofactors:

$$C_{11} = + \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = 6 + 2 = 8 \quad C_{12} = - \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = -(0 - 2) = 2$$

$$C_{13} = + \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} = 0 - 3 = -3$$

$$C_{21} = - \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = -(2 - 1) = -1 \quad C_{22} = + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 + 1 = 5$$

$$C_{23} = - \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -(-2 - 1) = 3$$

$$C_{31} = + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 2 + 3 = 5 \quad C_{32} = - \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = -(4 - 0) = -4$$

$$C_{33} = + \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6 - 0 = 6$$

$$\text{Cofactor matrix: } \text{Com}(A) = \begin{pmatrix} 8 & 2 & -3 \\ -1 & 5 & 3 \\ 5 & -4 & 6 \end{pmatrix}$$

$$\text{Adjoint matrix (transpose): } \text{Adj}(A) = \begin{pmatrix} 8 & -1 & 5 \\ 2 & 5 & -4 \\ -3 & 3 & 6 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{Adj}(A) = \frac{1}{21} \begin{pmatrix} 8 & -1 & 5 \\ 2 & 5 & -4 \\ -3 & 3 & 6 \end{pmatrix}$$

1b. Gauss-Jordan elimination:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & -2 \end{array} \right) \xrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 & -1 & -2 \end{array} \right)$$

$$\xrightarrow{R_3 \leftarrow -\frac{1}{7}R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} \end{array} \right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} \end{array} \right)$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{3}R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{2}{21} & \frac{5}{21} & -\frac{4}{21} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} \end{array} \right)$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{21} & -\frac{1}{21} & \frac{5}{21} \\ 0 & 1 & 0 & \frac{2}{21} & \frac{5}{21} & -\frac{4}{21} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{1}{7} & \frac{2}{7} \end{array} \right)$$

$$\text{Thus } A^{-1} = \frac{1}{21} \begin{pmatrix} 8 & -1 & 5 \\ 2 & 5 & -4 \\ -3 & 3 & 6 \end{pmatrix}, \text{ same as before.}$$

2. Verification: $A \times A^{-1} = I_3$ (students should verify).

Solution – Exercise 3

1. Matrix form:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}.$$

2. Determinant: From Exercise 1, $\det(A) = 21 \neq 0$, so (S) is a Cramer system (unique solution).

3. Cramer's rule:

$$|A_1| = \begin{vmatrix} 5 & 1 & -1 \\ 7 & 3 & 2 \\ 2 & -1 & 2 \end{vmatrix} = 5(6+2) - 1(14-4) + (-1)(-7-6) = 5(8) - 1(10) - 1(-13)$$

$$= 40 - 10 + 13 = 43$$

$$|A_2| = \begin{vmatrix} 2 & 5 & -1 \\ 0 & 7 & 2 \\ 1 & 2 & 2 \end{vmatrix} = 2(14-4) - 5(0-2) + (-1)(0-7) = 2(10) - 5(-2) - 1(-7)$$

$$= 20 + 10 + 7 = 37$$

$$|A_3| = \begin{vmatrix} 2 & 1 & 5 \\ 0 & 3 & 7 \\ 1 & -1 & 2 \end{vmatrix} = 2(6+7) - 1(0-7) + 5(0-3) = 2(13) - 1(-7) + 5(-3) = 26 + 7 - 15 = 18$$

$$x = \frac{|A_1|}{|A|} = \frac{43}{21}, \quad y = \frac{|A_2|}{|A|} = \frac{37}{21}, \quad z = \frac{|A_3|}{|A|} = \frac{18}{21} = \frac{6}{7}.$$

4. Inverse matrix method:

$$X = A^{-1}B = \frac{1}{21} \begin{pmatrix} 8 & -1 & 5 \\ 2 & 5 & -4 \\ -3 & 3 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 40 - 7 + 10 \\ 10 + 35 - 8 \\ -15 + 21 + 12 \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 43 \\ 37 \\ 18 \end{pmatrix} \text{ Thus}$$

$$x = \frac{43}{21}, y = \frac{37}{21}, z = \frac{6}{7}.$$

5. Gauss elimination:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 0 & 3 & 2 & 7 \\ 1 & -1 & 2 & 2 \end{array} \right) \xrightarrow{R_3 \leftarrow 2R_3 - R_1} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 0 & 3 & 2 & 7 \\ 0 & -3 & 5 & -1 \end{array} \right) \xrightarrow{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 5 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 7 & 6 \end{array} \right)$$

Back substitution: $z = \frac{6}{7}$, $3y + 2(\frac{6}{7}) = 7 \Rightarrow 3y = 7 - \frac{12}{7} = \frac{37}{7} \Rightarrow y = \frac{37}{21}$, $2x + \frac{37}{21} - \frac{6}{7} = 5 \Rightarrow 2x = 5 - \frac{37}{21} + \frac{18}{21} = \frac{105-19}{21} = \frac{86}{21} \Rightarrow x = \frac{43}{21}$.

3.6 Practice Exam – 4

MCQ Practice Exam #6 –

Level: First Year LMD
Module: Mathematics 2
Duration:

Exercise 1

(6 points)

Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$.

1. Compute $\det(A)$ using Gauss elimination. What do you observe?
2. Determine the rank of A .
3. Is A invertible? Justify.

Exercise 2

(7 points)

Consider the non-Cramer linear system:

$$(S) : \begin{cases} x + 2y + 3z = 10 \\ 2x + 4y + 6z = 20 \\ 3x + 6y + 9z = 30 \end{cases}$$

1. Write the system in matrix form $AX = B$.
2. Compute the rank of A and the rank of $[A|B]$. Conclude about the nature of the system.
3. Solve the system using Gauss elimination and express the solution in parametric form.

Exercise 3

(7 points)

Consider the matrix $B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

1. Find the characteristic polynomial $p(\lambda)$ of B .
2. Find the eigenvalues λ_1 and λ_2 of B .

3. For each eigenvalue, find the eigenvectors using the Gauss method (row reduction to solve $(B - \lambda I)v = 0$).
4. Find matrices P and D such that $B = PDP^{-1}$.
5. Compute B^5 using diagonalization.

Full Solutions – Practice Exam 4

Solution – Exercise 1

1. Determinant using Gauss elimination:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 6 & 9 \end{pmatrix}$$

Since a row of zeros appears, $\det(A) = 0$.

2. Rank of A : After row reduction, we have only one non-zero row (the first row, and R_3 becomes 0 after $R_3 \leftarrow R_3 - 3R_1$). Thus $\text{rank}(A) = 1$.

3. Invertibility: Since $\det(A) = 0$, A is not invertible (singular).

Solution – Exercise 2

1. Matrix form:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}.$$

2. Rank analysis: From Exercise 1, $\text{rank}(A) = 1$. Form the augmented matrix

$$[A|B] = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 2 & 4 & 6 & 20 \\ 3 & 6 & 9 & 30 \end{array} \right)$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Thus $\text{rank}([A|B]) = 1$. Since $\text{rank}(A) = \text{rank}([A|B]) = 1 < 3$ (number of unknowns), the system is **consistent with infinitely many solutions**.

3. Solve using Gauss elimination: The reduced system gives one equation: $x + 2y + 3z = 10$. Let $y = s$, $z = t$ (free parameters). Then $x = 10 - 2s - 3t$. The solution set is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 - 2s - 3t \\ s \\ t \end{pmatrix}, \quad s, t \in \mathbb{R}.$$

Solution – Exercise 3
1. Characteristic polynomial:

$$p(\lambda) = \det(B - \lambda I) = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3$$

$$= (\lambda - 1)(\lambda - 3).$$

2. Eigenvalues: $\lambda_1 = 1, \lambda_2 = 3$.

3. Eigenvectors using Gauss method:

For $\lambda_1 = 1$: $B - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Gauss elimination: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Equation: $x + y = 0 \Rightarrow y = -x$. Eigenvector: $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

For $\lambda_2 = 3$: $B - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ Gauss elimination: $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow -x + y = 0 \Rightarrow y = x$. Eigenvector: $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4. Diagonalization:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

5. Compute B^5 : $B^5 = PD^5P^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1^5 & 0 \\ 0 & 3^5 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 243 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 243 \\ -1 & 243 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 + 243 & -1 + 243 \\ -1 + 243 & 1 + 243 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix} = \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}.$$

3.7 Practice Exam – 5

MCQ Practice Exam #7 –

Level: First Year LMD
Module: Mathematics 2
Duration:

Exercise 1

(6 points)

Compute the determinant of the following matrix using the most appropriate method:

$$A = \begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & 1 & -1 \\ 0 & 1 & 3 & 2 \\ -1 & 0 & 2 & 1 \end{pmatrix}$$

(Use cofactor expansion along a row or column with the most zeros, or use Gauss elimination.)

Exercise 2

(7 points)

Consider the linear system:

$$(S) : \begin{cases} x + y - z = 2 \\ 2x - y + z = 3 \\ x + 2y - 2z = 1 \end{cases}$$

1. Write the system in matrix form $AX = B$.
2. Compute $\det(A)$. Is (S) a Cramer system?
3. Solve (S) using the inverse matrix method (compute A^{-1} via the adjoint method).
4. Verify your solution using Cramer's rule.

Exercise 3

(7 points)

Let $C = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

1. Find the characteristic polynomial of C .
2. Find the eigenvalues and eigenvectors (use Gauss method for solving $(C - \lambda I)v = 0$).
3. Find matrices P and D such that $C = PDP^{-1}$.

4. Compute C^4 using diagonalization.

Full Solutions – Practice Exam 5

Solution – Exercise 1

Method: Cofactor expansion along the first row (or use Gauss elimination).

Using cofactor expansion along the first row:

$$\det(A) = 2 \cdot \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 0 & 2 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & 1 & -1 \\ 0 & 3 & 2 \\ -1 & 2 & 1 \end{vmatrix} + 0 \cdot (\dots) - 3 \cdot \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{vmatrix}$$

Compute each 3×3 determinant:

$$\begin{aligned} \text{First: } \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \\ 0 & 2 & 1 \end{vmatrix} &= 2(3 \cdot 1 - 2 \cdot 2) - 1(1 \cdot 1 - 2 \cdot 0) + (-1)(1 \cdot 2 - 3 \cdot 0) = 2(3 - 4) - \\ &1(1 - 0) - 1(2 - 0) = 2(-1) - 1 - 2 = -2 - 1 - 2 = -5 \end{aligned}$$

$$\begin{aligned} \text{Second: } \begin{vmatrix} 1 & 1 & -1 \\ 0 & 3 & 2 \\ -1 & 2 & 1 \end{vmatrix} &= 1(3 \cdot 1 - 2 \cdot 2) - 1(0 \cdot 1 - 2 \cdot (-1)) + (-1)(0 \cdot 2 - 3 \cdot (-1)) = \\ &1(3 - 4) - 1(0 + 2) - 1(0 + 3) = -1 - 2 - 3 = -6 \end{aligned}$$

$$\begin{aligned} \text{Third: } \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{vmatrix} &= 1(1 \cdot 2 - 3 \cdot 0) - 2(0 \cdot 2 - 3 \cdot (-1)) + 1(0 \cdot 0 - 1 \cdot (-1)) = \\ &1(2 - 0) - 2(0 + 3) + 1(0 + 1) = 2 - 6 + 1 = -3 \end{aligned}$$

$$\text{Thus } \det(A) = 2(-5) + 1(-6) - 3(-3) = -10 - 6 + 9 = -7.$$

Solution – Exercise 2

1. Matrix form:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

2. Determinant:

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 2 & -2 \end{vmatrix} = 1((-1)(-2) - (1)(2)) - 1(2(-2) - (1)(1)) + (-1)(2 \cdot 2 - (-1) \cdot 1)$$

$$= 1(2 - 2) - 1(-4 - 1) - 1(4 + 1) = 0 - 1(-5) - 1(5) = 5 - 5 = 0.$$

Since $\det(A) = 0$, the system is **not** Cramer. It may have no solution or infinitely many.

Check consistency via augmented matrix:

$$\begin{aligned}
 [A|B] &= \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -2 & 1 \end{array} \right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \xrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & -1 \\ 0 & 1 & -1 & -1 \end{array} \right) \\
 &\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & 3 & -1 \end{array} \right) \xrightarrow{R_3 \leftarrow R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -4 \end{array} \right)
 \end{aligned}$$

The last row gives $0 = -4$, which is impossible. Thus the system is **inconsistent (no solution)**.

Solution – Exercise 3

1. Characteristic polynomial:

$$\begin{aligned}
 p(\lambda) = \det(C - \lambda I) &= \begin{vmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda) - 6 = \lambda^2 - 3\lambda + 2 - 6 = \lambda^2 - 3\lambda - 4 \\
 &= (\lambda - 4)(\lambda + 1).
 \end{aligned}$$

2. Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = -1$.

Eigenvectors using Gauss method:

For $\lambda_1 = 4$: $C - 4I = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$ Gauss elimination: $\begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow -3x + 2y = 0 \Rightarrow 2y = 3x \Rightarrow y = \frac{3}{2}x$. Eigenvector: $v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (taking $x = 2$).

For $\lambda_2 = -1$: $C + I = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ Gauss elimination: $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow 2x + 2y = 0 \Rightarrow y = -x$. Eigenvector: $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3. Diagonalization:

$$D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}, \quad P^{-1} = -\frac{1}{5} \begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}.$$

$$\begin{aligned}
 \text{4. Compute } C^4: C^4 &= PD^4P^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 4^4 & 0 \\ 0 & (-1)^4 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \\
 &= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 256 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 512 & 1 \\ 768 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \\
 &= \frac{1}{5} \begin{pmatrix} 512 + 3 & 512 - 2 \\ 768 - 3 & 768 + 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 515 & 510 \\ 765 & 770 \end{pmatrix} = \begin{pmatrix} 103 & 102 \\ 153 & 154 \end{pmatrix}.
 \end{aligned}$$

Chapter 4

Key Summary – Formulas & Problem-Solving Methods

Note to Students

This chapter provides a concise reference for all key formulas and step-by-step methods for solving problems in Mathematics 2. Use this as a quick review before exams.

4.1 Matrix Operations

Basic Definitions

- **Matrix** $A_{m \times n}$: m rows, n columns
- **Square matrix**: $m = n$
- **Zero matrix** O : all entries are zero
- **Identity matrix** I_n : diagonal entries = 1, others = 0

Matrix Addition & Scalar Multiplication

$$(A + B)_{ij} = a_{ij} + b_{ij}, \quad (kA)_{ij} = k \cdot a_{ij}$$

Matrix Multiplication

$$(A \times B)_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$$

Condition: Number of columns of A = Number of rows of B

Properties

- $A \times B \neq B \times A$ (not commutative)
- $A \times (B \times C) = (A \times B) \times C$ (associative)

- $A \times (B + C) = A \times B + A \times C$ (distributive)
- $A \times I_n = I_m \times A = A$

Transpose of a Matrix

$$(A^T)_{ij} = a_{ji}$$

Properties:

$$(A^T)^T = A, \quad (A + B)^T = A^T + B^T, \quad (AB)^T = B^T A^T$$

Trace of a Square Matrix

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}$$

Properties:

$$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B), \quad \text{tr}(AB) = \text{tr}(BA)$$

4.2 Determinants

Definition

The determinant $\det(A)$ or $|A|$ is a scalar associated only with square matrices.

Method 1: Sarrus' Rule (3×3 matrices only)

$$\begin{aligned} \det(A) &= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b \\ d & e \\ g & h \end{vmatrix} - \begin{vmatrix} b & c \\ e & f \\ h & i \end{vmatrix} + \begin{vmatrix} c & d \\ f & g \\ i & h \end{vmatrix} \\ &= aei + bfg + cdh - ceg - bdi - afh \end{aligned}$$

Method 2: Cofactor Expansion (any square matrix)

Expand along row i :

$$\det(A) = \sum_{j=1}^n a_{ij} \cdot C_{ij}, \quad \text{where } C_{ij} = (-1)^{i+j} M_{ij}$$

M_{ij} is the minor (determinant after removing row i and column j).

Method 3: Gauss Elimination (Row Reduction)

Transform A to upper triangular form using row operations:

- Swapping two rows multiplies $\det$ by -1
- Multiplying a row by k multiplies $\det$ by k
- Adding a multiple of one row to another does not change $\det$

Then $\det(A) = \text{product of diagonal elements of triangular form.}$

Properties of Determinants

- $\det(AB) = \det(A) \cdot \det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- $\det(kA) = k^n \det(A)$ (for $n \times n$ matrix)
- $\det(I_n) = 1$
- If A has a row/column of zeros: $\det(A) = 0$
- If two rows/columns are identical or proportional: $\det(A) = 0$

4.3 Matrix Inverse

Definition

A^{-1} is the inverse of A if $A \times A^{-1} = A^{-1} \times A = I_n$. A matrix is invertible (non-singular) if and only if $\det(A) \neq 0$.

Method 1: Adjoint (Cofactor) Method

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{Adj}(A)$$

where $\text{Adj}(A) = (\text{Com}(A))^T$ (transpose of the cofactor matrix).

Steps:

1. Compute $\det(A)$. If $\det(A) = 0$, stop (A is not invertible).
2. Compute all cofactors $C_{ij} = (-1)^{i+j} M_{ij}$.
3. Form the cofactor matrix $\text{Com}(A) = (C_{ij})$.
4. Take the transpose to get the adjoint: $\text{Adj}(A) = (\text{Com}(A))^T$.
5. Multiply by $\frac{1}{\det(A)}$.

Method 2: Gauss-Jordan Elimination

Form the augmented matrix $[A|I_n]$ and perform row operations to transform A into I_n :

$$[A|I_n] \xrightarrow{\text{row operations}} [I_n|A^{-1}]$$

Allowed row operations:

- Swap two rows
- Multiply a row by a non-zero scalar
- Add a multiple of one row to another

Properties of Inverse

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(kA)^{-1} = \frac{1}{k}A^{-1}$

4.4 Linear Systems

General Form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Matrix form: $AX = B$, where A is $m \times n$, X is $n \times 1$, B is $m \times 1$.

Classification

- **Cramer system:** $m = n$ and $\det(A) \neq 0$ (unique solution)
- **Non-Cramer system:** $\det(A) = 0$ or $m \neq n$ (no solution or infinitely many)

Rank of a Matrix

The rank $\text{rank}(A)$ is the number of linearly independent rows (or columns). It is the number of non-zero rows after row reduction.

Rouché-Capelli Theorem

Let $r = \text{rank}(A)$ and $r' = \text{rank}([A|B])$:

- If $r \neq r'$: **No solution** (inconsistent)
- If $r = r' = n$: **Unique solution**
- If $r = r' < n$: **Infinitely many solutions** ($n - r$ free parameters)

Method 1: Cramer's Rule (for Cramer systems)

For $AX = B$ with $\det(A) \neq 0$:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is A with the i -th column replaced by B .

Method 2: Inverse Matrix Method (for Cramer systems)

$$X = A^{-1}B$$

Method 3: Gauss Elimination (Gaussian Elimination)

Transform $[A|B]$ to row echelon form using row operations, then back-substitute.

Steps:

1. Write the augmented matrix $[A|B]$
2. Use row operations to create zeros below each pivot
3. Obtain upper triangular form
4. Back-substitute to find solutions

Method 4: Non-Cramer Systems (with parameters)

When $\det(A) = 0$ or $m \neq n$, use Gauss elimination to find:

- If inconsistent: last row gives $0 = k$ ($k \neq 0$)
- If infinitely many solutions: express basic variables in terms of free parameters

4.5 Eigenvalues & Eigenvectors

Definition

For a square matrix A , λ is an **eigenvalue** and $v \neq 0$ is the corresponding **eigenvector** if:

$$Av = \lambda v$$

Characteristic Equation

$$\det(A - \lambda I) = 0$$

This is a polynomial of degree n (the characteristic polynomial).

Method: Finding Eigenvalues

1. Compute $A - \lambda I$
2. Compute $\det(A - \lambda I)$
3. Solve $\det(A - \lambda I) = 0$ to find $\lambda_1, \lambda_2, \dots, \lambda_n$

Method: Finding Eigenvectors (Gauss Method)

For each eigenvalue λ_i :

1. Form $A - \lambda_i I$
2. Solve $(A - \lambda_i I)v = 0$ using Gauss elimination
3. Write the solution in parametric form
4. The eigenvectors are the non-zero solutions

Properties

- $\text{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ (sum of eigenvalues)
- $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$ (product of eigenvalues)
- If A is symmetric, eigenvectors corresponding to distinct eigenvalues are orthogonal

4.6 Diagonalization

Definition

A matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that:

$$A = PDP^{-1}$$

where $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ (eigenvalues on the diagonal).

Method: Diagonalization Steps

1. Find all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$
2. For each eigenvalue, find a basis of eigenvectors
3. Form P by placing eigenvectors as columns
4. Form D with eigenvalues on the diagonal in the same order as eigenvectors in P
5. Compute P^{-1} (using Gauss-Jordan or adjoint method)

Condition for Diagonalizability

A is diagonalizable if and only if:

- There are n linearly independent eigenvectors
- For each eigenvalue, geometric multiplicity = algebraic multiplicity

Computing Powers using Diagonalization

$$A^k = PD^kP^{-1}, \quad \text{where } D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}$$

Special Cases

- If A is diagonalizable and k is large, A^k is easier via diagonalization
- If A is not diagonalizable, use Jordan canonical form (advanced)

4.7 Quick Reference Table

Concept	Formula / Method
Matrix multiplication	$(AB)_{ij} = \sum_k a_{ik}b_{kj}$
Transpose	$(A^T)_{ij} = a_{ji}$
Trace	$\text{tr}(A) = \sum_i a_{ii}$
Determinant (3×3 Sarrus)	$aei + bfg + cdh - ceg - bdi - afh$
Cofactor	$C_{ij} = (-1)^{i+j}M_{ij}$
Inverse (adjoint)	$A^{-1} = \frac{1}{\det(A)} \text{Adj}(A)$
Inverse (Gauss-Jordan)	$[A I] \rightarrow [I A^{-1}]$
Cramer's rule	$x_i = \frac{\det(A_i)}{\det(A)}$
Gauss elimination	$[A B] \rightarrow \text{triangular form} \rightarrow \text{back substitution}$
Rouché-Capelli	$r = \text{rank}(A), r' = \text{rank}([A B])$
Characteristic equation	$\det(A - \lambda I) = 0$
Eigenvector	$(A - \lambda I)v = 0$
Diagonalization	$A = PDP^{-1}, D = \text{diag}(\lambda_1, \dots, \lambda_n)$
Power via diagonalization	$A^k = PD^kP^{-1}$

Common Mistakes to Avoid

- Matrix multiplication is NOT commutative: $AB \neq BA$ in general
- Determinant of $A + B$ is NOT equal to $\det(A) + \det(B)$
- For Cramer's rule, $\det(A)$ must be non-zero
- A matrix with $\det(A) = 0$ is singular (not invertible)
- Eigenvectors are defined up to a scalar multiple (any non-zero scalar multiple is also an eigenvector)
- For diagonalization, eigenvectors must be linearly independent

Memory Aids

- **Sarrus:** "Downward diagonals plus, upward diagonals minus"
- **Cofactor sign:** $(-1)^{i+j}$ gives a checkerboard pattern: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$
- **Cramer's rule:** Replace column i with B , divide by $\det(A)$
- **Eigenvalues:** Sum = trace, product = determinant

Bibliography

- [1] Calvo, B., Doyen, J., Calvo, A., Boschet, F. (1975). *Exercices d'algèbre: 1er cycle scientifique*. Armand Colin Collection.
- [2] Chandra, P. (2013). *Linear Algebra Lecture Notes*. [Lecture Notes]. Department of Mathematics, IIT Kanpur. Retrieved from <https://home.iitk.ac.in/~peeyush/102A/Lecture-notes.pdf>
- [3] Djabi, A. (2025). *Mathematics 2: Summaries and Exercises*. [Handout]. University of Setif, Algeria. Retrieved from <http://dspace.univ-setif.dz:8888/jspui/handle/123456789/5124> & <https://www.researchgate.net/publication/389876525>
- [4] Djabi, A. (2025). *Mathematics 2- 100 Exercises with Solutions*. [Handout]. University of Setif, Algeria. Retrieved from <http://dspace.univ-setif.dz:8888/jspui/handle/123456789/6585> & https://www.researchgate.net/publication/399663765_Mathematics_2-_100_Exercises_with_Solutions
- [5] Djabi, A. (2025). *Mathematical Tools for Economics Students*. [Book], ALPHA DOCUMENTATION, Algérie, ISBN: 978-9969-06-075-1 .
- [6] Golub, G. H., & Van Loan, C. F. (2013). *Matrix Computations* (4th ed.). Johns Hopkins University Press.
- [7] Hoffman, K., & Kunze, R. (1971). *Linear Algebra* (2nd ed.). Prentice Hall.
- [8] Horn, R. A., & Johnson, C. R. (2013). *Matrix Analysis* (2nd ed.). Cambridge University Press.
- [9] Lay, D. C. (2015). *Linear Algebra and Its Applications* (5th ed.). Pearson.
- [10] Strang, G. (2010). *Linear Algebra Lecture Notes*. [MIT OpenCourseWare] Retrieved from <https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/>
- [11] Strang, G. (2016). *Introduction to Linear Algebra* (5th ed.). Wellesley-Cambridge Press.