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Contact Problems in Plasticity Involving Complex Boundary Conditions

Presented by

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Contents

Contents iii

Notations v

Introduction 1

1 Mathematical Modeling and Formulation of Mechanical Problems 4

1.1 Constitutive laws 5

1.1.1 Fundamental Mechanical Constitutive Laws 5

1.1.2 Associated Constitutive Laws 7

1.1.3 Thermal, Electrical, and Multi-Physical Constitutive Laws . . 8

1.1.4 Mechanical Phenomena 11

1.1.5 Multi-Physical Constitutive Laws with Contact Phenomena . 12

1.1.6 Time Fractional-Order Constitutive Laws 14

1.2 Boundary Conditions 15

1.2.1 Linear Boundary Conditions 16

1.2.2 Nonlinear Boundary Conditions 17

1.3 Modeling and Problem Formulation 21

1.3.1 Physical Frameworks 21

1.3.2 Mathematical Models 23

2 Resolution of Electromechanical Problems with Wear and Internal State Variables 26

2.1 Mechanical Problem Formulation and Assumptions 27

2.2 Existence and uniqueness of a solution 32

3 Resolution of Thermomechanical Problems with Fractional Time Derivative **43**

3.1 Mechanical Problem Formulation and Assumptions 43

3.2 Variational Formulation 48

3.3 Existence and uniqueness of a solution 50

Appendices **66**

A Functional Setting and Theoretical Preliminaries **66**

A.1 Functional spaces for vector fields 66

A.2 Sobolev Spaces in Contact Modeling 67

A.3 Foundations of Nonlinear Analysis 70

Bibliography **76**

Notations

Let Ω be a domain in $\mathbb{R}^m$ $(m = 2, 3)$. We denote:

- $\overline{\Omega}$: the closure of Ω,
- Γ : the boundary of Ω, assumed to be sufficiently regular,
- Γ_i : a partition of the boundary Γ,
- ν : the unit outward normal to Γ,
- v_ν, v_τ : the normal and tangential components of a vector field v,
- σ_ν, σ_τ : the normal and tangential components of a tensor field $\boldsymbol{\sigma}$.

Functional spaces:

- $C^k(\Omega)$: space of k-times continuously differentiable functions on Ω,
- $AC[a, b]$: space of all absolutely continuous functions on $[a, b]$,
- $H = L^2(\Omega)^m$,
- $\mathcal{H} = L^2(\Omega)^{m\times m}$,
- $\mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} \,|\, \mathrm{Div}\, \boldsymbol{\tau} = (\tau_{ij,j}) \in H\}$,
- $H_\Gamma = H^{1/2}(\Gamma)^m$,
- H_Γ^* : the dual space of H_Γ.

Let E be a real Hilbert space or, more generally, a Banach space. We will adhere to the following notational conventions:

- $(\cdot, \cdot)_E$: the inner product in H,
- $(\cdot, \cdot)_{E^*\times E}$: the duality pairing between E^* and E,
- $\|\cdot\|_E$: the norm in E,
- $z_n \rightharpoonup z$: weak convergence of (z_n) to z in E,

- $z_n \to z$: strong convergence of (z_n) to z in E,
- $\mathcal{L}(E)$: space of continuous linear operators from E into E.

Moreover, if $I \equiv [0,T]$ is a time interval, $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$, we denote:

- $C(I;E)$: the space of all continuous functions from I to E,
- $C^1(I;E)$: the space of all continuously differentiable functions from I to E,
- $L^p(I;E)$: the space of strongly measurable functions on I with values in E,
- $\|\cdot\|_{L^p(I;E)}$: the norm in $L^p(I;E)$,
- $W^{k,p}(I;E)$: Sobolev space with parameters k and p,
- $\|\cdot\|_{W^{k,p}(I;E)}$: the norm in $W^{k,p}(I;E)$.

Operators and symbols:

- $\dot{f}, \ddot{f}$: first and second derivatives of f with respect to time,
- $\partial_i f = f_i$: partial derivative of f with respect to the ith component x_i,
- ∇f : gradient of f,
- $\mathrm{Div}\, f$: divergence of f,
- $\varepsilon(f)$: symmetric part of the gradient of f,
- $\mathbb{S}^m$: space of symmetric second-order tensors on $\mathbb{R}^m$,
- $\|\cdot\|$: Euclidean norm on $\mathbb{R}^m$ or $\mathbb{S}^m$,
- C : generic positive constants.

Introduction

Contact problems represent a fundamental area of research within the mechanics of deformable bodies. They describe the interaction between surfaces subjected to mechanical loads, including the transfer of forces, deformation under pressure, and frictional effects that may result in either sliding or sticking. These phenomena are characterized by significant complexity, owing to the simultaneous presence of elastic, viscous, and plastic effects, leading to nonlinear boundary conditions that pose major challenges for both modeling and numerical analysis. Applications of contact phenomena are widespread across various industrial fields, such as aerospace and automotive engineering, as well as crash testing. They also arise in biomechanical and medical contexts, including bone and artificial joint contact, needles, and surgical tools. Moreover, civil engineering provides further examples, where such problems appear in estimating the interaction forces between structural materials, such as those in bridges and skyscrapers. Due to this diversity of applications, the modeling and numerical study of contact phenomena have attracted extensive research interest, resulting in a vast and diversified body of literature (see, e.g., [28], [10]). From a mathematical standpoint, theoretical and numerical investigations of contact problems are relatively recent developments, primarily due to their nonlinear characteristics, which complicate their analysis. Signorini [33] made one of the earliest mathematical contributions to this field by formulating the unilateral contact problem between a rigid foundation and a linearly elastic body. Fischer [17] later employed elliptic variational inequality techniques to solve this problem. Duvaut and Lions [12] made a significant advancement by developing the differential formulation of several contact problems and proving the existence and uniqueness of solutions. Since these pioneering studies, considerable efforts have been undertaken

to consolidate and enhance contact models, yielding a substantial and extensive body of literature, as evidenced by the seminal references [11], [15], [32]. This field has experienced a significant transformation in recent years with the incorporation of time-fractional derivatives into classical mechanical models. This approach allows for a more precise depiction of memory-dependent and nonlocal temporal dynamics. Many viscoelastic contact models have been reformulated within the fractional framework, providing a more realistic description of material responses that depend on the temporal history of stress or strain. Recent studies [37], [5] have proposed dynamic fractional models, which have significantly broadened the theoretical and numerical analysis of complex contact phenomena.

In general, contact processes can be classified into three main categories: dynamic processes, modeled by the equation of motion (see [7, 16]); quasi-static processes, described by an equilibrium equation involving at least one time derivative of displacement or stress (see [19, 30]); and static processes, in which time derivatives are entirely neglected ([13, 21]).

Among the various contact conditions, the normal compliance condition is one of the most widely used in mechanical modeling. This condition assumes that the foundation is deformable, such that the contact pressure depends on the amount of interpenetration between the rough contacting surfaces. It was first introduced by Klarbring et al. [23], who demonstrated the existence of weak solutions for frictional contact problems under normal compliance. Further details can be found in [2–4, 34]. When frictional effects are taken into account, the problem becomes a frictional contact problem. The classical results in this area are due to Carter and Fromm, who employed Coulomb's dry friction law to describe the relationship between slip and tangential force, both for a cylinder on a flat surface and for two cylinders in steady rolling contact (see [24]). Frictional contact problems cover a wide range of engineering applications, such as brake pads and discs, pistons and engines, and braking systems in trains and aircraft. Owing to their practical significance, considerable research efforts have been devoted to their modeling, numerical simulation, and mathematical analysis (see [19, 31, 32]). From another perspective, wear in mechanics refers to the set of mechanical and physical processes that lead to

the gradual removal of material from a solid surface as a result of repeated contact or relative motion between two bodies. Wear is one of the principal causes of efficiency loss and performance degradation in mechanical components, as it alters surface geometry and reduces the service life of mechanical systems. The rate of wear depends on several factors, including surface roughness, applied load, sliding velocity, material properties, and the presence of lubricants or contaminants (see [6, 29]).

In this work, we address contact problems between a deformable body and a deformable foundation, where the contact is described either by a normal compliance condition coupled with wear, or by a multivalued normal compliance condition. The study focuses on the existence and uniqueness of weak solutions within a quasi-static framework. The analysis relies on results from the theory of elliptic variational inequalities, hemivariational inequalities, and fractional differential equations, in addition to fixed point theorems. The work also includes a detailed physical modeling of the problem, specifying the physical medium, the material behavior, and the boundary contact conditions.

The thesis is organized into three chapters and an appendix as follows:

Chapter 1 provides the physical background, constitutive laws, and boundary conditions, along with the mathematical frameworks that will serve as the basis for subsequent studies.

Chapter 2 deals with a quasi-static mechanical contact problem for an electromechanical body involving friction and wear, and establishes existence and uniqueness results for the weak solution.

Chapter 3 is devoted to the study of a thermo-frictional contact problem for a viscoplastic material that incorporates time-fractional derivatives in both the mechanical and thermal constitutive laws. The contact is assumed to satisfy a multivalued normal compliance condition, described by the Clarke subdifferential, and corresponding existence and uniqueness results are established. The analysis combines auxiliary results obtained via the Rothe method with a Galerkin approximation scheme, fixed-point arguments, and techniques from the theory of hemivariational inequalities.

Chapter 1

Mathematical Modeling and Formulation of Mechanical Problems

This chapter introduces the quasi-static evolution of a deformable body that exhibits thermo-mechanical or electro-mechanical behavior under external mechanical, thermal, and electrical actions. From a mathematical viewpoint, the response of such a system is described by a set of partial differential equations (PDEs) incorporating balance laws, constitutive relations, and the corresponding boundary and initial conditions.

The objective is to establish a consistent theoretical framework for analyzing the coupled interaction between mechanical, thermal, and electrical effects within the material. The analysis focuses on the equilibrium configuration achieved once external forces cease and the system reaches a steady state governed by these coupled field equations.

Modeling Framework

The formulation of a mechanical problem typically involves three essential stages:

1. **Mathematical Modeling**

   Development of the governing equations that represent the physical behavior of the material, based on suitable mechanical, thermal, and electrical assumptions. These equations are complemented by initial and boundary conditions

defining the system's evolution.

2. **Variational Formulation**

   Transformation of the strong (differential) form into its weak form, allowing the use of functional analysis tools to investigate the existence, uniqueness, stability, and regularity of the solution.

3. **Numerical Approximation**

   Design of computational schemes and discretization methods that provide reliable numerical simulations and serve to validate theoretical predictions.

Purpose and Scope

This chapter lays the physical and mathematical foundations for the thermo-mechanical and electro-thermo-mechanical models explored in subsequent chapters. Particular attention is given to constitutive relations that incorporate Caputo fractional derivatives in stress and temperature evolution laws. The fractional formulation enables the modeling of memory and hereditary effects characteristic of viscoplastic and complex materials, thereby extending the classical continuum framework to account for time-dependent and nonlocal behaviors. The theoretical framework presented here is mainly based on previous works reported in [18, 27, 34, 37].

The next section presents the governing equations and constitutive relations defining the proposed model.

1.1 Constitutive laws

The constitutive laws define the relationships connecting the displacement vector u, the stress tensor σ, and the linearized strain tensor ε, thereby describing the mechanical response of the material.

1.1.1 Fundamental Mechanical Constitutive Laws

Elastic Constitutive Law. A material is said to be elastic if it can sustain external loads that produce deformation and, upon removal of these loads, recover its original

shape and dimensions. In a general framework, the elastic constitutive relation can be expressed as

$$\sigma = \mathcal{M}\varepsilon(u),$$

where $\mathcal{M}$ denotes the elasticity operator, which characterizes the material's mechanical response.

In the particular case of a linear elastic material, the constitutive relation reduces to the classical Hooke's law, which is given by

$$\sigma = E\varepsilon(u),$$

where E is the fourth-order elasticity tensor describing the linear stress–strain relationship.

Viscous Constitutive Laws. A material is said to be viscous when its deformation depends not only on the applied stresses but also on the rate at which these stresses are applied. In other words, the material exhibits time-dependent behavior. The general viscous constitutive law can be expressed as

$$\sigma = \mathcal{L}\boldsymbol{\varepsilon}(\dot{u}),$$

where $\mathcal{L}$ denotes the viscosity operator, which characterizes the material's resistance to flow or rate-dependent deformation.

Plastic Constitutive Laws. A material is said to be plastic when it undergoes irreversible deformations after the applied stresses exceed a certain threshold known as the yield limit. In such a case, part of the deformation remains even after the removal of the external loads. The general plastic constitutive law can be written as

$$\sigma = \mathcal{P}(\boldsymbol{\varepsilon}(u), \varepsilon^p),$$

where $\mathcal{P}$ denotes the plasticity operator, and $\boldsymbol{\varepsilon}^p$ represents the plastic strain tensor, which characterizes the permanent part of the total strain.

1.1.2 Associated Constitutive Laws

A material is said to obey a general constitutive law when its mechanical response results from the combined effects of elasticity, viscosity, and plasticity, where elasticity characterizes the ability of the material to resist deformation and to recover its original shape once the applied stress is removed, viscosity describes the time-dependent response in which the stress depends on the rate of deformation and leads to gradual strain under constant load, and plasticity accounts for the irreversible deformations that occur when the applied stress exceeds the yield limit. Such a formulation provides a unified framework for describing real materials that exhibit simultaneously elastic recovery, viscous flow, and permanent plastic deformation.

Types of associations

Series Association. A series association is a configuration in which all components are subjected to the same stress, while the total strain is the sum of the individual strains, that is

$$\begin{cases} \sigma = \sigma_1 = \sigma_2 = \cdots, \\ \varepsilon = \sum_i \varepsilon_i. \end{cases}$$

Parallel Association. A parallel association is a configuration in which all components experience the same strain, while the total stress corresponds to the sum of the individual stresses, that is

$$\begin{cases} \varepsilon = \varepsilon_1 = \varepsilon_2 = \cdots, \\ \sigma = \sum_i \sigma_i. \end{cases}$$

This thesis focuses on associable constitutive laws, which characterize materials whose stress–strain behavior is derived from an associated potential.

Viscoplastic Constitutive Laws. Viscoplastic constitutive laws describe materials that exhibit both viscous and plastic behavior, meaning that their deformation depends simultaneously on the loading rate and on the irreversible strain that de-

velops once the yield stress is exceeded.

The general form of a viscoplastic constitutive law can be expressed as follows

$$\dot{\sigma}(t) = \mathcal{L}\,\boldsymbol{\varepsilon}(\dot{u}(t)) + \mathcal{S}(\sigma(t), \boldsymbol{\varepsilon}(u(t))),$$

where $\mathcal{L}$ and $\mathcal{S}$ are material constitutive functions

Viscoelastic Constitutive Laws. Viscoelastic materials exhibit both elastic and viscous behavior, meaning that their deformation depends on the applied stress and on the rate of loading. A general viscoelastic constitutive relation can be expressed as

$$\sigma(t) = \mathcal{L}\,\boldsymbol{\varepsilon}(\dot{u}(t)) + \mathcal{M}\,\boldsymbol{\varepsilon}(u(t)),$$

where $\mathcal{M}$ is the elasticity tensor describing the instantaneous elastic response, and $\mathcal{L}$ is the viscosity tensor, describing the material's resistance to deformation that depends on the strain rate.

Elasto-Viscoplastic Constitutive Laws. Elasto-viscoplastic materials exhibit elastic, viscous, and plastic behavior, meaning that their deformation depends on the applied stress, the rate of loading, and the accumulated irreversible strain. A general constitutive relation can be written in the integral form as

$$\sigma(t) = \mathcal{L}\varepsilon(\dot{u}(t)) + \mathcal{M}\varepsilon(u(t)) + \int_0^t \mathcal{S}\big(\sigma(r) - \mathcal{L}\varepsilon(\dot{u}(r)), \varepsilon(u(r))\big)\, dr.$$

Here, $\mathcal{L}$ and $\mathcal{M}$ denote two nonlinear operators representing the viscous and elastic behaviors of the material, respectively. The function $\mathcal{S}$ is a nonlinear constitutive function used to describe the plastic response in the model.

1.1.3 Thermal, Electrical, and Multi-Physical Constitutive Laws

Thermal Constitutive Law. The temperature field θ evolves in time according to the energy balance principle. This leads to the heat conduction equation, which

takes the form

$$\dot{\theta} = \operatorname{div}(\mathcal{K}\nabla\theta) + \psi(\sigma, \varepsilon(u), \theta) + q_t,$$

where $\mathcal{K}$ is the thermal conductivity tensor, q_t denotes the volumetric heat source density. This equation describes how the temperature field evolves in time under the combined effects of conduction, internal heat sources, and coupling with mechanical deformation.

Electrical Constitutive Law. The electrical constitutive law describes the relationship between the electric field and the electric displacement in a material. For a linear dielectric material, it can be expressed as:

$$D = \epsilon E,$$

where D is the electric displacement vector, E is the electric field vector, and ϵ is the permittivity of the material.
In the general case of time-dependent fields, the behavior of the electric field is governed by Maxwell's equations. In the quasi-static approximation (neglecting magnetic effects), the governing differential equation for the electric potential ϕ reads

$$\operatorname{div}(\epsilon\nabla\phi) = -\rho,$$

where ρ is the free charge density, and $E = -\nabla\phi$. This equation describes how the electric potential and field distribute in the material under the presence of charges and dielectric properties.

Multi-Physical Constitutive Laws. Multi-physical constitutive laws describe the behavior of materials that are simultaneously influenced by several physical phenomena, such as mechanical deformation, heat transfer, and electrical or magnetic effects. They define how these different fields interact within the same material and how changes in one field (for example, temperature or electric potential) can affect the mechanical or electrical response of the system. Such laws are fundamental for

modeling and analyzing coupled thermo-electro-mechanical processes in advanced or smart materials.

In order to better understand the coupling mechanisms involved in multi-physical materials, it is essential to recall the fundamental constitutive laws that describe each physical phenomenon separately. In what follows, we present the classical forms of the thermal, electrical, and mechanical constitutive laws that are adopted in this thesis.

Thermo-Elasto-Viscoplastic Constitutive Laws. The thermo-elasto-viscoplastic constitutive laws describe the behavior of materials whose mechanical response combines elastic, viscous, and plastic effects under the influence of temperature variations. The general constitutive relation can be expressed in the following integral form

$$\sigma(t) = \mathcal{L}\varepsilon(\dot{u}(t)) + \mathcal{M}\varepsilon(u(t)) + \int_0^t \mathcal{S}\big(\sigma(r) - \mathcal{L}(\dot{\varepsilon}(u(r))), \varepsilon(u(r)), \theta(r)\big)\, dr,$$

where $\mathcal{L}$ and $\mathcal{M}$ denote two nonlinear operators representing the viscous and elastic behaviors of the material, respectively. The function $\mathcal{S}$ is a nonlinear constitutive function used to describe the plastic response in the model.

Electro-Elasto-Viscoplastic Constitutive Laws. The electro-elasto-viscoplastic constitutive laws describe the behavior of materials whose mechanical response combines elastic, viscous, and plastic effects under the influence of an electric field. The general constitutive relation can be expressed as follows

$$\sigma(t) = \mathcal{L}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{M}\boldsymbol{\varepsilon}(\mathbf{u}(t)) - \mathcal{E}^* E(\phi) + \int_0^t \mathcal{S}\big(\sigma(r) - \mathcal{L}\boldsymbol{\varepsilon}(\dot{u}(r)) + \mathcal{E}^* E(\phi), \boldsymbol{\varepsilon}(u(r))\big)\, \mathrm{d}r,$$

where $\mathcal{L}$ and $\mathcal{M}$ represent, respectively, the purely viscous and elastic properties of the material, while $\mathcal{S}$ denotes a general nonlinear constitutive function that characterizes the viscoplastic behavior influenced by the applied electric field. The term $\mathcal{E}^* E(\phi)$ represents the electromechanical coupling contribution within the material. Additional assumptions are taken into account in the modeling process to repre-

sent the complex and time-dependent surface behavior of materials, which plays an important role in contact mechanics.

1.1.4 Mechanical Phenomena

Mechanical phenomena refer to the different processes that govern the deformation and interaction of materials under mechanical loading. These processes often occur at material interfaces and involve complex coupling between mechanical, thermal, and electrical effects. They play a crucial role in determining the global behavior of contact systems. In what follows, we present the mechanical and surface phenomena considered in this work.

Wear. Wear is a mechanical phenomenon characterized by the progressive removal or degradation of material from solid surfaces in contact, usually caused by frictional interaction or repeated mechanical loading. It results in surface damage, loss of material, and changes in the geometry or functionality of the components, which can significantly affect the performance and durability of engineering systems.

Heat Conduction. Thermal conduction constitutes the fundamental mechanism by which thermal energy is transferred within a material due to local temperature variations. This process plays a central role in characterizing the thermal response at contact interfaces and in analyzing the overall thermo-mechanical coupling of materials.

Piezoelectric Effect. The piezoelectric effect refers to the coupling between the electrical and mechanical fields within certain crystalline materials. When such a material is subjected to mechanical stress, it generates an electric polarization, and conversely, an applied electric field induces mechanical deformation. This bidirectional coupling plays a fundamental role in the modeling of electro-mechanical materials and devices, enabling energy conversion between mechanical and electrical forms.

1.1.5 Multi-Physical Constitutive Laws with Contact Phenomena

The multi-physical constitutive laws describe the behavior of materials subjected to coupled mechanical, thermal, and electrical effects, as well as other interacting physical fields. These couplings become particularly significant in contact problems, where surface phenomena such as friction, adhesion, wear, and interfacial heat or charge transfer strongly affect the global response of the system. The following section presents the general constitutive framework adopted in this work to represent these complex interactions.

Electro-Elasto-Viscoplastic Law with Wear. The material loss in the vertical direction induced by friction is described by the wear function

$$\omega : \Gamma_3 \times (0, T) \to \mathbb{R}^+,$$

which quantifies the evolution of wear depth on the contact surface. It is postulated that the wear process obeys a modified form of Archard's law

$$\dot{\omega} = -k_\omega \, \sigma_\nu \, |\dot{u}_\tau - w^*|,$$

where $k_\omega > 0$ denotes the wear coefficient, w^* represents the tangential velocity of the foundation, and $|\dot{u}_\tau - w^*|$ is the relative sliding velocity between the contacting surfaces. Introducing $\alpha^* = \|w^*\| > 0$, and assuming w^* is sufficiently large, the contribution of $\dot{u}_\tau$ can be neglected, yielding the simplified form

$$\dot{\omega} = -k_\omega \, \alpha^* \, \sigma_\nu.$$

The electro-elasto-viscoplastic law with wear is described as follows

$$\begin{aligned}
\boldsymbol{\sigma}(t) &= \mathcal{L}\dot{u}(t) + \mathcal{M}\varepsilon(u(t)) - \mathcal{E}^* E\big(\phi(t)\big) \\
&+ \int_0^t \mathcal{S}\Big(\sigma(r) - \mathcal{L}\dot{u}(r) + \mathcal{E}^* E(\phi(r)), \boldsymbol{\varepsilon}(u(r))\Big)\, dr, \\
\mathbf{D}(t) &= \mathcal{E}(u(t)) + \mathbf{B}E(\phi(t)), \\
\dot{\omega} &= -k_\omega\, \alpha^* \sigma_\nu,
\end{aligned}$$

where $\mathcal{L}$ and $\mathcal{M}$ represent, respectively, the purely viscous and elastic properties of the material, $\mathcal{S}$ is a general nonlinear operator describing the viscoplastic behavior, and $\mathcal{E}^* E(\phi)$ accounts for the electromechanical coupling. The operator $\boldsymbol{B}$ captures additional electromechanical effects, while k_ω and α^* are material parameters controlling the evolution of surface wear.

Electro-Elasto-Viscoplastic Law with Wear and Internal State Variables. An internal state variable is a mathematical quantity introduced in constitutive models to represent the internal state of a material, which cannot be directly observed but significantly affects its mechanical response. These variables capture history-dependent phenomena such as plastic deformation, viscous effects, and damage accumulation.

Let $\chi(t)$ denote a set of internal state variables. Their evolution is generally governed by

$$\dot{\chi} = F(\sigma, \varepsilon, \chi),$$

where F is a nonlinear operator describing how the internal state evolves depending on the stress σ, strain ε, internal state χ, and possibly the electric potential ϕ.

The introduction of the internal state variable $\boldsymbol{\chi}$ enables the constitutive model to retain the history of applied loads and to accurately represent the material's viscous, plastic, electromechanical, and damage-dependent responses. The model

can then be described as follows

$$\sigma(t) = \mathcal{L}\dot{u}(t) + \mathcal{M}\boldsymbol{\varepsilon}(u(t)) - \mathcal{E}^* E\big(\boldsymbol{\phi}(t)\big)$$
$$+ \int_0^t \mathcal{S}\Big(\sigma(r) - \mathcal{L}\dot{u}(r) + \mathcal{E}^* E(\phi(r)), \boldsymbol{\varepsilon}(u(r)), \chi(r)\Big)\, dr,$$
$$\mathbf{D}(t) = \mathcal{E}(u(t)) + \mathbf{B}E(\boldsymbol{\phi}(t)),$$
$$\dot{\chi}\,(t) = F\big(\sigma,\varepsilon(u), \chi\big),$$
$$\dot{\omega} = -\kappa_\mu\, \alpha^* \sigma_\nu.$$

1.1.6 Time Fractional-Order Constitutive Laws

The fundamental fractional element, called the Scott–Blair dashpot, follows the constitutive relation

$$\sigma(t) = \eta \left({}_0^C D_t^\alpha\, \varepsilon(t)\right), \quad 0 < \alpha < 1,\ t \geq 0,$$

where σ is the stress, η is the Newtonian viscosity coefficient, and ε is the strain of the element. The term ${}_0^C D_t^\alpha \varepsilon$ represents the time-fractional derivative of order α of the strain in the sense of Caputo.

This model seeks to generalize classical constitutive laws by incorporating time-fractional derivatives, effectively encapsulating memory effects and intricate time-dependent behaviors of materials, including viscoelasticity, stress relaxation, and history-dependent responses. Consequently, the subsequent formulations broaden the applicability of classical elasto-viscoplastic laws to include these fractional temporal effects, yielding a more precise and comprehensive characterization of the material response under diverse loading conditions.

Fractional-Order Elasto-Viscoplastic Constitutive Models. The fractional constitutive law is of the form

$$\sigma(t) = \mathcal{L}\,\varepsilon\big({}_0^C D_t^\alpha u(t)\big) + \mathcal{M}\ \varepsilon(u(t)) + \int_0^t S\big(\sigma(r) - \mathcal{L}\,\mathcal{E}({}_0^C D_t^\alpha u(r)),\ \varepsilon(u(r))\big)\, dr,$$

where the notation ${}_0^C D_t^\alpha \mathbf{u}(t)$ denotes the time-fractional derivative of order α of the function u, in the sense of Caputo.

Fractional-Order Thermal Constitutive Law. The fractional constitutive law is of the form

$$ {}_0^C D_t^\alpha \theta(t) = \operatorname{div} \mathcal{K} \nabla \theta(t) + \psi\big(\sigma(t), \varepsilon(u(t)), \theta(t)\big) + q_t(t). $$

In this setting, $\mathcal{K}$ denotes the tensor governing the conductive properties of the material, q_t represents the density of volumetric heat sources, and ψ is a nonlinear constitutive function that accounts for the thermal contribution generated by the material's internal mechanisms.

1.2 Boundary Conditions

Boundary conditions define how a system behaves at its edges. They are essential for solving differential equations and ensuring that the solutions accurately reflect the physical behavior of the system.

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with outer Lipschitz boundary Γ. The boundary Γ is divided into three disjoint relatively open parts

$$ \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \quad \Gamma_i \cap \Gamma_j = \emptyset, \text{ for } i \neq j. $$

Furthermore, the union of the first two parts is decomposed into two disjoint open subsets

$$ \Gamma_1 \cup \Gamma_2 = \Gamma_a \cup \Gamma_b, \quad \Gamma_a \cap \Gamma_b = \emptyset. $$

It is assumed that

$$ |\Gamma_1| > 0, \quad |\Gamma_a| > 0. $$

The time interval under consideration is denoted by $I \equiv [0, T]$ with $T > 0$. We shall

use the following notation:

$$\Sigma = \Omega \times \mathrm{I}, \quad \Xi = \Gamma \times \mathrm{I}, \quad \Xi_0 = (\Gamma_1 \cup \Gamma_2) \times \mathrm{I}$$
$$\Xi_1 = \Gamma_1 \times \mathrm{I}, \quad \Xi_2 = \Gamma_2 \times \mathrm{I}, \quad \Xi_3 = \Gamma_3 \times \mathrm{I},$$
$$\Xi_a = \Gamma_a \times \mathrm{I}, \quad \Xi_b = \Gamma_b \times \mathrm{I}.$$

1.2.1 Linear Boundary Conditions

Mechanical Boundary Conditions. The linear mechanical boundary conditions are of the form

$$\begin{cases} u = g \quad \text{on } \Xi_1, \\ \sigma\nu = h \quad \text{on } \Xi_2, \end{cases}$$

where ν indicates the outward-directed unit normal to the boundary surface, g is a prescribed displacement on Γ_1, and h is a prescribed traction on Γ_2.

Electrical Boundary Conditions. The linear electrical boundary conditions are of the form

$$\begin{cases} \phi = \rho_a \quad \text{on } \Xi_a, \\ D\nu = \rho_b \quad \text{on } \Xi_b, \end{cases}$$

where ν indicates the outward-directed unit normal to the boundary surface, ρ_a is the prescribed potential on Γ_a, and ρ_b is the prescribed current flux on Γ_b.

Thermal Boundary Conditions. The thermal boundary conditions are given by

$$\begin{cases} \theta = \alpha \quad \text{on } \Xi_1, \\ q\nu = \beta \quad \text{on } \Xi_2, \end{cases}$$

where ν is the outward unit normal vector to the boundary, α is the prescribed temperature on Γ_1, and β is the prescribed heat flux on Γ_2.

1.2.2 Nonlinear Boundary Conditions

Contact Conditions

Unilateral Contact Conditions (Signorini Conditions). Before applying external forces, the initial gap at a point $x \in \Gamma_3$ along the normal direction $\nu(x)$ is known and denoted by $g(x)$. The Signorini unilateral contact conditions are expressed in a combined form as

$$\begin{cases} u_\nu \leq g(x), \\ \sigma_\nu \leq 0, \\ \sigma_\nu \left(u_\nu - g(x)\right) = 0 \quad \text{on } \Gamma_3, \end{cases}$$

where u_ν is the normal displacement of the body, σ_ν is the normal stress, and $g(x)$ is the initial gap between the body and the contact surface Γ_3.
These conditions enforce that the body cannot penetrate the contact surface ($u_\nu \leq g(x)$), that only compressive normal stresses are allowed ($\sigma_\nu \leq 0$), and that contact occurs only where the gap is closed, as expressed by the complementarity condition ($\sigma_\nu \left(u_\nu - g(x)\right) = 0$).

Bilateral Contact Conditions. Bilateral contact occurs when there is no separation between the body and the foundation, meaning that the contact is maintained throughout the process. This condition can be expressed as

$$u_\nu = 0 \quad \text{on } \Xi_3.$$

Contact with Normal Compliance. This formulation refers to a slightly deformable foundation, where the normal stress σ_ν is related to the normal displacement u_ν by

$$-\sigma_\nu = p_\nu \left(u_\nu - g\right),$$

where g denotes the initial gap between the body and the foundation, and p_ν is a prescribed positive function, known as the normal compliance function, which char-

acterizes the mechanical response of the foundation under normal loading.
When the surface wear is taken into account, this relation can be modified by introducing a normalized compliance function, which is expressed as follows

$$-\sigma_\nu = p_\nu(u_\nu - \omega - g).$$

When the initial gap between the body and the foundation is zero, we take $g = 0$.
As an illustrative example, the normal compliance function p_ν may be defined by

$$p_\nu = c_\nu\, r^+,$$

where c_ν denotes a positive constant and $r^+ = \max(0, r)$.
Initially, the normal compliance function was introduced as a single-valued relation between the normal stress and displacement. Later studies extended this concept to a multivalued formulation, expressed through the following subdifferential inclusion

$$-\sigma_\nu \in \partial j_\nu(\dot{u}_\nu),$$

where ∂j_ν denotes the Clarke subdifferential of j_ν.

Furthermore, this formulation can be generalized by introducing a fractional derivative, which enables the model to capture memory and hereditary effects in the normal contact response, as follows

$$-\sigma_\nu \in \partial j_\nu({}_0^C D_t^\alpha u_\nu).$$

Friction Laws

Tresca-Type Friction Law. The friction is modeled according to the Tresca law, expressed by the following boundary conditions.

$$\textbf{Static Case:} \begin{cases} u_\nu = 0, \quad |\sigma_\tau| \le g, \\ |\sigma_\tau| < g \Rightarrow u_\tau = 0 \qquad \text{on } \Xi_3, \\ |\sigma_\tau| = g \Rightarrow \exists\, \lambda \ge 0 \ \text{ s.t. } \sigma_\tau = -\lambda\, u_\tau, \end{cases}$$

$$\textbf{Dynamic Case:} \begin{cases} u_\nu = 0, \quad |\sigma_\tau| \le g, \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{u}_\tau = 0 \qquad \text{on } \Xi_3, \\ |\sigma_\tau| = g \Rightarrow \exists\, \lambda \ge 0 \ \text{ s.t. } \sigma_\tau = -\lambda\, \dot{u}_\tau, \end{cases}$$

where $g > 0$ is the fixed sliding threshold, u_τ denotes the tangential displacement, and σ_τ is the tangential stress on the contact surface Γ_3.

The condition $u_\nu = 0$ indicates that the contact is bilateral, meaning that the body remains in continuous contact with the foundation without separation. The inequality $|\sigma_\tau| \le g$ expresses that the frictional stress is bounded by the threshold g. When the inequality is strict ($|\sigma_\tau| < g$), no sliding occurs and the tangential displacement vanishes ($u_\tau = 0$); in this case, the body adheres to the foundation. When equality holds ($|\sigma_\tau| = g$), sliding becomes possible, and the frictional stress σ_τ acts in the opposite direction of the tangential velocity, opposing motion along the contact surface.

Coulomb Friction Law. According to Coulomb's law of friction, the tangential traction σ_τ at the contact interface cannot exceed the friction threshold F_b, which defines the maximum frictional resistance between the interacting surfaces

$$\sigma_\tau \le F_b, \quad -\sigma_\tau = F_b \frac{v_\tau}{\|v_\tau\|} \quad \text{for} \quad v_\tau \neq 0.$$

Here, $v_\tau = \dot{u}_\tau$ denotes the relative tangential velocity. When sliding occurs, the tangential traction reaches the limiting value F_b and is directed opposite to the motion.

In engineering applications, the friction bound F_b is often expressed as a function of the normal stress σ_ν

$$F_b = F_b(\sigma_\nu) = p_\tau(u_\nu - g),$$

where p_τ is a nonnegative function that vanishes when there is no contact.

When the wear of the surface is considered, the frictional behavior can be described by a modified form of the Coulomb law, which can be expressed as follows

$$-\sigma_\tau = p_\tau(u_\nu - \omega - g)\frac{\dot{u}_\tau}{\|\dot{u}_\tau\|}.$$

In some cases, especially when the applied loads are relatively low or the friction coefficient is large, the friction threshold F_b may be treated as a constant, giving rise to the classical Tresca friction model:

$$F_b = \text{const.}$$

The studies have extended this concept to include a multivalued formulation, expressed through the following differential inclusion

$$-\sigma_\tau \in \partial j_\tau(\dot{u}_\tau),$$

where ∂j_τ denotes the Clarke subdifferential of j_τ.

Furthermore, this formulation can be generalized by introducing a fractional derivative, which allows the model to capture memory and hereditary effects in the tangential contact response, as follows

$$-\sigma_\tau \in \partial j_\tau({}_0^C D_t^\alpha u_\tau).$$

Thermal Boundary Conditions

They represent the heat exchanges between the body and its environment. Various types of thermal boundary conditions may be considered. The following boundary condition, associated with Fourier's law, is given by

$$\mathcal{K}_0 \frac{\partial \theta}{\partial \nu} = \kappa_e(\theta - \theta_F), \quad \text{on } \Xi_3,$$

where κ_e is the heat transfer coefficient at the body–obstacle interface, and θ_F is the foundation temperature.

Electrical Boundary Conditions

They characterize the electrical interactions occurring at the interface between the body and the foundation. Such conditions specify how the electric field and the

electric potential behave across the contact surface, depending on the electrical properties of the materials in contact. Several models of electrical boundary conditions can be found in the literature, and the one adopted in this work is expressed as follows

$$\mathbf{D} \cdot \nu = \psi(u_\nu - g)\, \Phi_l(\phi - \phi_0) \quad \text{on } \Xi_3,$$

where $\mathbf{D}$ denotes the electric displacement field, ν is the unit outward normal to the surface, and ϕ and ϕ_0 represent the electric potentials of the body and the foundation on the contact interface, respectively. Here, u_ν is the normal displacement, g represents the initial gap, ψ is a regular function describing the electrical conductance at the contact interface, and Φ_l is a truncation function ensuring the boundedness of the potential difference.

1.3 Modeling and Problem Formulation

This section introduces the physical frameworks that serve as the foundation of the present study and, consequently, the corresponding mathematical formulations derived from them. The aim is to establish a coherent link between the underlying physical principles and their mathematical representations, which form the basis for the subsequent modeling and analysis developed throughout this thesis.

1.3.1 Physical Frameworks

The phenomena studied in this thesis are described by the following physical frameworks.

Physical Framework 1

The considered material is assumed to exhibit a coupled elasto–viscoplastic and electromechanical behavior. It occupies a bounded domain $\Omega \subset \mathbb{R}^m$ (where $m = 2$ or 3), enclosed by a Lipschitz boundary Γ. The boundary Γ is divided into three open and disjoint parts: Γ_1, Γ_2, and Γ_3. The union $\Gamma_1 \cup \Gamma_2$ is further subdivided into two open subsets, Γ_a and Γ_b, with the assumptions that $|\Gamma_1| > 0$ and $|\Gamma_a| > 0$.

These boundary segments correspond to distinct physical phenomena: the body is mechanically clamped along Γ_1, while surface tractions act on Γ_2. Contact with the foundation occurs along Γ_3, where the interaction is governed by a normal compliance condition that accounts for frictional effects and surface wear. Electrically, a prescribed potential is imposed on Γ_a, whereas surface electric charges are applied on Γ_b. Throughout the time interval $I = [0, T]$, the body is subjected to body forces of density f_0 and volumetric electric charges of density q_0, resulting in a coupled electro-mechanical response involving both contact and conduction phenomena.

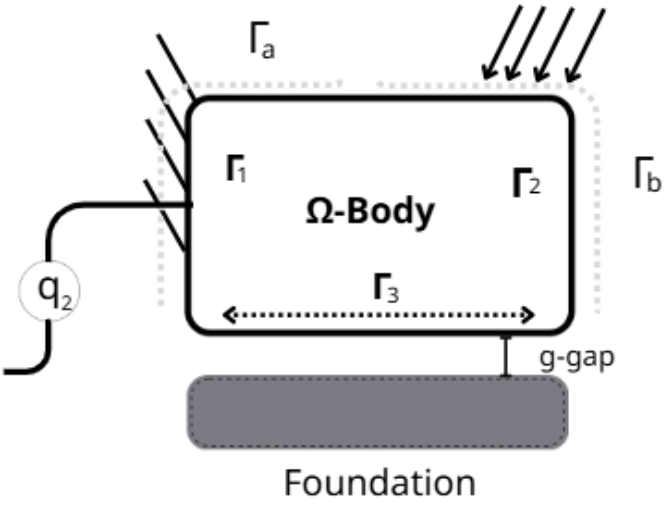


Figure 1.1: The Physical Framework 1; Γ_3 is the contact surface.

Physical Framework 2

The studied configuration concerns a body exhibiting thermo–elasto–viscoplastic behavior interacting with its surrounding medium. The body occupies a bounded domain $\Omega \subset \mathbb{R}^m$ (where $m = 2$ or 3) with a Lipschitz continuous boundary Γ. This boundary is partitioned into three relatively open and disjoint subsets: Γ_1, representing the clamped part of the body; Γ_2, where an external surface traction is applied; and Γ_3, corresponding to the possible contact surface with a deformable foundation, where the frictional contact is governed by a multi-valued normal compliance condition involving a fractional derivative.

The evolution of the system is studied over a finite time interval $I = [0, T]$, with $T > 0$. The body is subjected to a distributed body force of density f_0 and an internal heat source of intensity q_t.

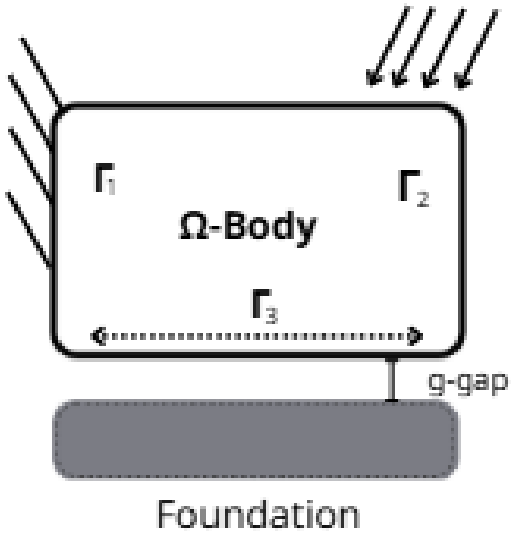


Figure 1.2: The Physical Framework 2; Γ_3 is the contact surface.

1.3.2 Mathematical Models

This section introduces complete mathematical models to represent the physical phenomena being studied.

Mathematical Model 1

Based on the assumptions stated in Physical Framework 1, the mathematical model is formulated as follows.

Problem $\mathcal{P}^1$. Find a displacement field $u : \Sigma \to \mathbb{R}^m$, a stress field $\sigma : \Sigma \to \mathbb{S}^m$, an electric potential $\phi : \Sigma \to \mathbb{R}$, an electric displacement field $\mathbf{D} : \Sigma \to \mathbb{R}^m$, a wear

$\omega : \Xi_3 \to \mathbb{R}^+$ and an internal state variable $\chi : \Sigma \to \mathbb{R}^n$ such that

$$\sigma = \mathcal{L}\varepsilon(\dot{u}\,(t)) + \mathcal{M}\varepsilon(u\,(t)) - \mathcal{E}^* E\,(\phi)$$
$$+ \int_0^t \mathcal{S}\big(\sigma\,(r) - \mathcal{L}\varepsilon(\dot{u}\,(r)) + \mathcal{E}^* E\,(\phi)\,, \varepsilon(u\,(r)), \chi\,(r)\,\big) dr \quad \text{in } \Sigma, \tag{1.1}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \boldsymbol{B} E\,(\phi\,(t))\,, \quad \text{in } \Sigma, \tag{1.2}$$

$$\dot{\chi}\,(t) = F\big(\sigma - \mathcal{L}\varepsilon(\dot{u}) + \mathcal{E}^* E\,(\phi)\,, \varepsilon(u), \chi\big) \quad \text{in } \Sigma, \tag{1.3}$$

$$-\operatorname{Div}\sigma = \mathbf{f}_0 \quad \text{in } \Sigma, \tag{1.4}$$

$$\operatorname{div} D = q_0 \quad \text{in } \Sigma, \tag{1.5}$$

$$u = 0 \quad \text{on } \Xi_1, \tag{1.6}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Xi_2, \tag{1.7}$$

$$-\sigma_\nu = p_\nu(u_\nu - \omega - g) \quad \text{on } \Xi_3, \tag{1.8}$$

$$\sigma_\tau = -p_\tau(u_\nu - \omega - g)\frac{w^*}{\|w^*\|}, \quad \text{on } \Xi_3, \tag{1.9}$$

$$\dot{\omega} = -\kappa_\omega \alpha^* \sigma_\nu = \kappa_\omega \alpha^* p_\nu(u_\nu - g - \omega), \quad \text{on } \Xi_3, \tag{1.10}$$

$$\phi = 0 \quad \text{on } \Xi_a, \tag{1.11}$$

$$D \cdot \nu = q_2 \quad \text{on } \Xi_b, \tag{1.12}$$

$$D \cdot \nu = \psi(u_\nu - g)\Phi_l(\phi - \phi_0) \quad \text{on } \Xi_3, \tag{1.13}$$

$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega, \tag{1.14}$$

$$\omega(0) = \omega_0, \quad \text{in } \Gamma_3. \tag{1.15}$$

Mathematical Model 2

Based on the assumptions of Physical Framework 2, the following mathematical model is proposed.

Problem $\mathcal{P}^2$. Find a displacement field $u : \Sigma \to \mathbb{R}^m$, a stress field $\sigma : \Sigma \longrightarrow \mathbb{S}^m$ and a temperature $\theta : \Sigma \to \mathbb{R}$ such that

$$\sigma = \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u(t)\right) + \mathcal{M}\boldsymbol{\varepsilon}(u(t))$$
$$+ \int_0^t \mathcal{S}\big(\sigma(t) - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u(r)\right), \boldsymbol{\varepsilon}(u(r)), \theta(r)\big)dr \quad \text{in } \Sigma, \tag{1.16}$$

$$-\operatorname{Div}\sigma = \mathbf{f}_0 \quad \text{in } \Sigma, \tag{1.17}$$

$${}_0^C D_t^\alpha \theta = \operatorname{div}\mathcal{K}\nabla\theta + \psi\big(\sigma - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right), \boldsymbol{\varepsilon}(u), \theta\big) + q_t \quad \text{in } \Sigma, \tag{1.18}$$

$$u = 0 \quad \text{on } \Xi_1, \tag{1.19}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Xi_2, \tag{1.20}$$

$$\theta = 0 \quad \text{on } \Xi_0, \tag{1.21}$$

$$-\sigma_\nu \in \partial j_\nu\left({}_0^C D_t^\alpha u_\nu\right) \quad \text{on } \Xi_3, \tag{1.22}$$

$$-\sigma_\tau \in \partial j_\tau\left({}_0^C D_t^\alpha u_\tau\right) \quad \text{on } \Xi_3, \tag{1.23}$$

$$\mathcal{K}\frac{\partial\theta}{\partial\nu} = \kappa_e(\theta - \theta_F) \quad \text{on } \Xi_3, \tag{1.24}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{1.25}$$

$$\theta(0) = \theta_0 \quad \text{in } \Omega. \tag{1.26}$$

These mathematical models form the basis for the subsequent analysis. Their solutions will be discussed in the following chapters, where the mechanical aspects of the problem are examined in detail.

Chapter 2

Resolution of Electromechanical Problems with Wear and Internal State Variables

In this chapter, we are concerned with the study of a frictional electromechanical contact problem, in which a body with electro-elasto-viscoplastic properties interacts with a deformable conductive foundation. The considered framework describes a quasi-static process and incorporates the material evolution through internal state variables that accurately capture the coupled and dissipative behavior of the material. Moreover, surface wear effects are incorporated into the normal compliance contact law.

Based on these assumptions, a comprehensive formulation of the governing problem is developed, followed by a rigorous analytical treatment relying on the theory of elliptic variational inequalities to establish the existence and uniqueness of a weak solution. This approach provides a solid theoretical foundation for the study of this class of coupled physico-engineering systems.

2.1 Mechanical Problem Formulation and Assumptions

The physical system under consideration is a body exhibiting electro-elasto-viscoplastic behavior, occupying a bounded domain $\Omega \subset \mathbb{R}^m$ (with $m \in \{2, 3\}$), whose Lipschitz boundary Γ is decomposed into the portions Γ_1, Γ_2, and Γ_3. Over the time interval $I = [0, T]$, the domain is subjected to body forces f_0 and a charge density q_0, together with a set of mixed boundary conditions.

Mechanically, the body is clamped (zero displacement) on Γ_1 and subjected to a traction force f_2 on Γ_2. Electrically, the potential vanishes on Γ_a, while a charge density q_2 is prescribed on Γ_b, where

$$\Gamma_a \cup \Gamma_b = \Gamma_1 \cup \Gamma_2, \qquad |\Gamma_1| > 0,\ |\Gamma_a| > 0.$$

Most importantly, the portion Γ_3 represents the frictional contact surface with a conductive foundation assumed to undergo rigid sliding motion. The interaction on this surface is governed by a normal compliance law that incorporates the physical effects of surface wear.

Based on this geometric configuration and the specified loading and boundary conditions, we now proceed to formulate the corresponding mathematical model, identifying the governing equations and constitutive relations for this quasi-static contact system.

The governing equations of the system are presented as follows:

Problem $\mathcal{P}^1$. Find a displacement field $u : \Sigma \to \mathbb{R}^m$, a stress field $\sigma : \Sigma \to \mathbb{S}^m$, an electric potential $\phi : \Sigma \to \mathbb{R}$, an electric displacement field $\mathbb{D} : \Sigma \to \mathbb{R}^m$, a wear

$\omega : \Xi_3 \to \mathbb{R}^+$ and an internal state variable $\chi : \Sigma \to \mathbb{R}^n$ such that

$$\sigma = \mathcal{L}\varepsilon(\dot{u}\,(t)) + \mathcal{M}\varepsilon(u\,(t)) - \mathcal{E}^* E\,(\phi)$$
$$+ \int_0^t \mathcal{S}\big(\sigma\,(r) - \mathcal{L}\varepsilon(\dot{u}\,(r)) + \mathcal{E}^* E\,(\phi)\,, \varepsilon(u\,(r)), \chi\,(r)\,\big) dr \quad \text{in } \Sigma, \tag{2.1}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \boldsymbol{B} E\,(\phi\,(t))\,, \quad \text{in } \Sigma, \tag{2.2}$$

$$\dot{\chi}\,(t) = F\big(\sigma - \mathcal{L}\varepsilon(\dot{u}) + \mathcal{E}^* E\,(\phi)\,, \varepsilon(u), \chi\big) \quad \text{in } \Sigma, \tag{2.3}$$

$$-\operatorname{Div}\sigma = \mathbf{f}_0 \quad \text{in } \Sigma, \tag{2.4}$$

$$\operatorname{div} D = q_0 \quad \text{in } \Sigma, \tag{2.5}$$

$$u = 0 \quad \text{on } \Xi_1, \tag{2.6}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Xi_2, \tag{2.7}$$

$$-\sigma_\nu = p_\nu(u_\nu - \omega - g) \quad \text{on } \Xi_3, \tag{2.8}$$

$$\sigma_\tau = -p_\tau(u_\nu - \omega - g)\frac{w^*}{\|w^\star\|}, \quad \text{on } \Xi_3, \tag{2.9}$$

$$\dot{\omega} = -\kappa_\omega \alpha^* \sigma_\nu = \kappa_\omega \alpha^* p_\nu(u_\nu - g - \omega), \quad \text{on } \Xi_3, \tag{2.10}$$

$$\phi = 0 \quad \text{on } \Xi_a, \tag{2.11}$$

$$D \cdot \nu = q_2 \quad \text{on } \Xi_b, \tag{2.12}$$

$$D \cdot \nu = \psi(u_\nu - g)\Phi_l(\phi - \phi_0) \quad \text{on } \Xi_3, \tag{2.13}$$

$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega, \tag{2.14}$$

$$\omega(0) = \omega_0, \quad \text{in } \Gamma_3. \tag{2.15}$$

In the problem defined by equations (2.1)–(2.15), the constitutive law is specified by equations (2.1)–(2.3). This law incorporates the combined effects of viscosity, plasticity, elasticity, and electrical phenomena, along with an internal state variable capturing the material's structural evolution.

The equilibrium equations governing the stress field and the electric displacement field are given in equations (2.4) and (2.5), respectively. Boundary conditions for displacement and surface traction are prescribed in equations (2.6) and (2.7).

Bilateral frictional contact, including the influence of surface wear, is described by equations (2.8) and (2.9). The temporal evolution of the wear function is governed

by the ordinary differential equation (2.10). Boundary conditions for the electric potential on Γ_a and for the electric charge density on Γ_b are given in equations (2.11) and (2.12), respectively. The electrical contact condition applied on Γ_3 is specified in equation (2.13).

Finally, the initial conditions of the system, including the initial displacement u_0, the initial value of the internal state variable χ_0, and the initial wear amount ω_0, are defined by equation (2.14) and (2.15), respectively.

Before formulating the variational framework of the problem, it is important to specify the assumptions regarding the physical setting and the material's constitutive behavior. These assumptions play a crucial role in guaranteeing the well-posedness of the model and facilitate the subsequent mathematical analysis. In the study of problem $\mathcal{P}^1$, we adopt the following set of assumptions.

The operators $\mathcal{L}, \mathcal{M}$ and $\mathcal{S}$, as well as the tensors $\mathcal{E}$ and $\mathbf{B}$, satisfy the following assumptions

$$\begin{cases} \text{(a) } \mathcal{L} = (l_{ijlk}) : \Omega \times S^m \longrightarrow S^m,\ l_{ijkl} = l_{lkij} = l_{ijlk} \in L^\infty(\Omega), \text{and} \\ \exists\, L_{\mathcal{L}} > 0,\ \text{such that } \|\mathcal{L}(.,\varpi_1) - \mathcal{L}(.,\varpi_2)\| \leq L_{\mathcal{L}}\|\varpi_1 - \varpi_2\| \\ \quad \forall \varpi_1, \varpi_2 \in \mathbb{S}^m, \text{a.e. in } \Omega. \\ \text{(b) } \exists\, m_{\mathcal{L}} > 0,\ \text{such that } \mathcal{L}\varpi \cdot \varpi \geq m_{\mathcal{L}}\|\varpi\|^2 \quad \forall \varpi \in S^m, \text{a.e. in } \Omega. \end{cases} \tag{2.16}$$

$$\begin{cases} \mathcal{M} = (m_{ijlk}) : \Omega \times S^m \longrightarrow S^m,\ m_{ijlk} = m_{lkij} = m_{ijlk} \in L^\infty(\Omega), \text{and} \\ \quad \exists\, L_{\mathcal{M}} > 0,\ \text{such that } \|\mathcal{M}(.,\varpi_1) - \mathcal{M}(.,\varpi_2)\| \leq L_{\mathcal{M}}\|\varpi_1 - \varpi_2\| \\ \forall \varpi_1, \varpi_2 \in S^m,\ \text{a.e. in } \Omega. \end{cases} \tag{2.17}$$

$$\begin{cases} \text{(a) } \mathcal{S} : \Omega \times \mathbb{S}^m \times \mathbb{S}^m \times \mathbb{R}^m \to \mathbb{S}^m,\ \text{and } \exists\, L_{\mathcal{S}} > 0,\ \text{such that} \\ \quad \|\mathcal{S}(.,\varpi_1,\varrho_1,\iota_1) - \mathcal{S}(.,\varpi_2,\varrho_2,\iota_2)\| \leq L_{\mathcal{S}}(\|\varpi_1 - \varpi_2\| + \|\varrho_1 - \varrho_2\| + \|\iota_1 - \iota_2\|) \\ \quad \forall \varpi_1, \varpi_2, \varrho_1, \varrho_2 \in \mathbb{S}^m, \text{ and } \iota_1, \iota_2 \in \mathbb{R}^n \text{ a.e. in } \Omega. \\ \text{(b) the function } \mathcal{S}(.,\varpi,\varrho,\iota) \text{ is measurable on } \Omega, \forall \varpi, \varrho \in \mathbb{S}^m, \text{ and } \iota \in \mathbb{R}^n. \\ \quad \text{and } \mathcal{S}(.,0,0,0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{2.18}$$

$$\begin{cases} \text{(a) } \boldsymbol{B} = (B_{ij}) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m, B_{ij} = B_{ji} \in L^\infty(\Omega). \\ \text{(b) } \exists\, m_{\boldsymbol{B}} > 0, \text{ such that } \boldsymbol{B}\varrho.\varrho \geq m_{\boldsymbol{B}}|\varrho|^2 \\ \quad \forall \varrho = (\varrho_i) \in \mathbb{R}^m, \text{ a.e. in } \Omega. \end{cases} \tag{2.19}$$

$$\begin{cases} \text{(a) } \mathcal{E} = (e_{dkl}) : \Omega \times \mathbb{S}^m \to \mathbb{R}^m. \\ \text{(b) } e_{dlk} = e_{dkl} \in L^\infty(\Omega), \quad 1 \leq d, l, k \leq m. \end{cases} \tag{2.20}$$

The function $F : \Omega \times \mathbb{S}^m \times \mathbb{S}^m \times \mathbb{R}^n \to \mathbb{S}^m$ satisfies

$$\begin{cases} \text{(a) } \exists\, L_F > 0 \text{ such that } \|F(., \rho_1, \varrho_1, \iota_1) - F(., \rho_2, \varrho_2, \iota_2)\| \\ \quad \leq L_F \left(\|\rho_1 - \rho_2\| + \|\varrho_1 - \varrho_2\| + \|\iota_1 - \iota_2\|\right), \quad \forall \rho_1, \rho_2, \varrho_1, \varrho_2 \in \mathbb{S}^m, \\ \quad \text{and } \iota_1, \iota_2 \in \mathbb{R}^n \text{ a.e. in } \Omega. \\ \text{(b) the function } F(., \rho, \varrho, \iota) \text{ is measurable on } \Omega, \ \forall \rho, \varrho \in \mathbb{S}^m, \text{ and } \iota \in \mathbb{R}^n. \\ \quad \text{and } F(., 0, 0, 0) \in L^2(\Omega)^m. \end{cases} \tag{2.21}$$

The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) } \exists\, L_\psi > 0 \text{ such that } \|\psi(., \varrho_1) - \psi(., \varrho_2)\| \leq L_\psi \|\varrho_1 - \varrho_2\|, \\ \quad \forall \varrho_1, \varrho_2 \in \mathbb{R}, \text{a.e. in } \Gamma_3. \\ \text{(b) } \exists\, M_\psi > 0 \text{ such that } \|\psi(., \varrho)\| \leq M_\psi \quad \forall \varrho \in \mathbb{R}, \text{ a.e. in } \Gamma_3. \\ \text{(c) the function } \psi(., \varrho) \text{ is measurable on } \Gamma_3, \forall \varrho \in \mathbb{R}, \\ \quad \text{and } \psi(., \varrho) = 0, \forall \varrho \leq 0. \end{cases} \tag{2.22}$$

The function $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, satisfies

$$\begin{cases} \text{(a) } \exists\, L_r > 0 \text{ such that } \|p_r(., \varrho_1) - p_r(., \varrho_2)\| \leq L_r \|\varrho_1 - \varrho_2\|, \\ \quad \forall \varrho_1, \varrho_2 \in \mathbb{R}, \text{ a.e. in } \Gamma_3. \\ \text{(b) the mapping } p_r(., \varrho) \text{ is measurable on } \Gamma_3, \text{and is zero } \forall \varrho \leq 0. \end{cases} \tag{2.23}$$

We assume that

$$\mathbf{f}_0 \in W^{1,p}(I; L^2(\Omega)^m), \quad \mathbf{f}_2 \in W^{1,p}(I; L^2(\Gamma_2)^m), \tag{2.24}$$

$$q_0 \in W^{1,p}(I; L^2(\Omega)), \quad q_2 \in W^{1,p}(I; L^2(\Gamma_b)), \tag{2.25}$$

and

$$g \in L^2(\Gamma_3), \quad g \geq 0 \ \text{ a.e. on } \Gamma_3, \tag{2.26}$$

$$\phi_0 \in L^2(\Gamma_3), \quad \mathbf{u}_0 \in \mathcal{V}, \quad \chi_0 \in Y. \tag{2.27}$$

The variational formulation of the mechanical problem described by equations (1.1)–(1.15) is derived by applying Green's identities together with standard notation. For every $w \in \mathcal{V}$, $\zeta \in \mathcal{Q}$, and for almost every $t \in I$, the problem can be expressed as follows

Problem $\mathcal{PV}^1$. Find a displacement field $u : I \to \mathcal{V}$, a stress field $\sigma : I \to \mathcal{H}_1$, an electric potential $\phi : I \to \mathcal{Q}$, a wear $\omega : I \to L^2(\Gamma_3)$ and an internal state variable $\chi : I \to Y$ such that

$$\sigma = \mathcal{L}\varepsilon(\dot{u}(t)) + \mathcal{M}\varepsilon(\mathbf{u}(t)) - \mathcal{E}^* E(\phi)$$
$$+ \int_0^t \mathcal{S}\big(\sigma(r) - \mathcal{L}\varepsilon(\dot{u}(r)) + \mathcal{E}^* E(\phi), \varepsilon(u(r)), \chi(r)\big) dr, \tag{2.28}$$

$$\dot{\chi}(t) = F\big(\sigma(t) - \mathcal{L}\varepsilon(\dot{u}(t)) + \mathcal{E}^* E(\phi(t)), \varepsilon(u(t)), \chi(t)\big), \tag{2.29}$$

$$\dot{\omega} = -\kappa_\omega \alpha^* \sigma_\nu, \tag{2.30}$$

$$(\sigma(t), \varepsilon(w - \dot{u}(t))_{\mathcal{H}} + \jmath(u, w, \omega(t)) - \jmath(u(t), \dot{u}(t), \omega) \geq (\mathbf{f}(t), w - \dot{u}(t))_{\mathcal{V}}, \tag{2.31}$$

$$(\boldsymbol{B}\nabla\phi(t), \nabla\zeta)_H - (\mathcal{E}\varepsilon(u(t)), \nabla\zeta)_H + (h(u(t), \phi(t)), \zeta) = (q(t), \zeta)_{\mathcal{Q}}, \tag{2.32}$$

$$u(0) = u_0, \chi(0) = \chi_0, \omega(0) = \omega_0, \tag{2.33}$$

where $\jmath : \mathcal{V} \times \mathcal{V} \times L^2(\Gamma_3) \to \mathbb{R}, \mathbf{f} : I \to \mathcal{V}$, $q : I \to \mathcal{Q}$, and $h : \mathcal{V} \times \mathcal{Q} \to \mathcal{Q}$ are

respectively, defined by

$$\jmath(u,w,\omega) = \int_{\Gamma_3} p_\nu(u_\nu - g - \omega)w_\nu \, da + \int_{\Gamma_3} p_\tau(u_\tau - g - \omega)\|w_\tau - w^*\| \, da,$$
$$(\mathbf{f}(t), w)_{\mathcal{V}} = \int_\Omega f_0(t)\cdot w \, dx + \int_{\Gamma_2} f_2(t)\cdot w \, dx,$$
$$(q(t),\zeta)_{\mathcal{Q}} = \int_\Omega q_0(t)\zeta \, dx + \int_{\Gamma_b} q_2(t)\zeta \, dx, \tag{2.34}$$
$$(h(u,\varphi),\zeta)_{\mathcal{Q}} = \int_{\Gamma_3} \psi(u_\nu - g)\Phi_l(\phi - \phi_0)\zeta \, da. \tag{2.35}$$

For all $d \in \mathcal{V}$ and all $\omega \in L^2(\Gamma_3)$, the functional $\jmath(d,\cdot,\omega)$ is proper, convex, and lower semicontinuous on $\mathcal{V}$.

We note that the definitions of $\mathbf{f}$, h and q are based on Riesz representation theorem. Furthermore, conditions (2.24) and (2.25) imply that

$$\mathbf{f} \in W^{1,p}(I;\mathcal{V}), \qquad q \in W^{1,p}(I;\mathcal{Q}). \tag{2.36}$$

2.2 Existence and uniqueness of a solution

This section establishes the existence and uniqueness of a weak solution to Problem $\mathcal{PV}^1$ under suitable regularity and smallness conditions.

The main result is stated below.

Theorem 2.1. *Assume that conditions (2.16)–(2.27) are satisfied, and that the following smallness assumption holds* $M_\psi < \frac{m_{\mathbf{B}}}{c_d^2}$. *Then, there exists a unique solution* $\{u,\sigma,\varphi,\chi,\omega\}$ *to Problem* $\mathcal{PV}^1$, *which also satisfies the following regularity properties*

$$u \in W^{2,p}(I;\mathcal{V}) \cap \mathcal{C}^1(I;\mathcal{V}), \tag{2.37}$$
$$\sigma \in W^{1,p}(I;\mathcal{H}), \quad \mathrm{Div}\sigma \in W^{1,p}(I;H), \tag{2.38}$$
$$\phi \in W^{1,p}(I;\mathcal{Q}), \tag{2.39}$$
$$\chi \in W^{1,p}(I;Y), \quad \omega \in \mathcal{C}^1(I;L^2(\Gamma_3)). \tag{2.40}$$

Theorem (2.1) was proven through a sequence of steps based on results from ordinary differential equations, evolutionary quasivariational inequality and fixed-

point principles.

Let us now examine some further implications of the regularity results. From the constitutive relation (1.2), together with assumptions (2.19), (2.20), and the regularity properties (2.37) and (2.39), we deduce that

$$D \in W^{1,p}(I; H).$$

Moreover, using the definition introduced in (2.32), we obtain the identity

$$\operatorname{div} D(t) = q_0(t), \quad \text{for all } t \in I.$$

Then, from assumption (2.25), it follows that

$$\operatorname{div} D \in W^{1,p}(I; L^2(\Omega)),$$

and hence,

$$D \in W^{1,p}(I; \mathcal{Q}_1).$$

In what follows, we adopt the assumptions of Theorem 2.1 and denote by C a generic positive constant, depending only on the problem data, which may vary from line to line.

Let $\eta \in C(I; \mathcal{H})$ be given. In the first step, we consider the problem described below, for all $w \in \mathcal{V}$, $\zeta \in \mathcal{Q}$ and a.e $t \in I$.

Problem $\mathcal{PV}^1_{1\eta}$. Find $u_\eta \in \mathcal{C}^1(I; \mathcal{V})$ such that

$$\begin{aligned}(\mathcal{L}\varepsilon(\dot{u}_\eta(t)), \varepsilon(w) - \varepsilon(\dot{u}_\eta(t)))_{\mathcal{H}} + (\mathcal{M}\varepsilon(u_\eta(t)), \varepsilon(w) - \varepsilon(\dot{u}_\eta(t)))_{\mathcal{H}} \\ +(\eta(t), \varepsilon(w) - \varepsilon(\dot{u}_\eta(t)))_{\mathcal{H}} + \jmath(u_\eta(t), w) - \jmath(u_\eta(t), \dot{u}_\eta(t)) \\ \geq (\mathbf{f}(t), w - \dot{u}_\eta(t))_V, \qquad (2.41)\end{aligned}$$

$$u_\eta(0) = u_0. \qquad (2.42)$$

Under the stated assumptions, the following holds for $\mathcal{PV}^1_{1\eta}$.

Lemma 2.2 ([25]). *The following statements hold.*

1. *Problem $\mathcal{PV}^1_{1\eta}$ admits a unique solution $u_\eta \in \mathcal{C}^1(I;\mathcal{V})$.*

2. *Let u_{η_1} and u_{η_2} be the solutions of $\mathcal{PV}^1_{1\eta}$ corresponding to η_1 and η_2, respectively. Then there exists a constant $C>0$ such that*

$$\|\dot{u}_{\eta_1}(t)-\dot{u}_{\eta_2}(t)\|_{\mathcal{V}} \leq C\big(\|\eta_1(t)-\eta_2(t)\|_{\mathcal{H}}+\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathcal{V}}\big). \tag{2.43}$$

3. *If $\eta \in W^{1,p}(I;\mathcal{H})$ for $p\in[1,\infty]$, then the solution $u_\eta \in W^{2,p}(I;\mathcal{V})$.*

The proof of this Lemma relies on an abstract theory that guarantees solution existence and uniqueness, previously established in [25].

Let $u_\eta \in \mathcal{C}^1(I,\mathcal{V})$ be the solution obtained in Lemma 2.2. We then consider the following variational formulation for the electric potential, which holds for all $\zeta \in \mathcal{Q}$.

Problem $\mathcal{PV}^1_{2\eta}$. Find $\phi_\eta \in W^{1,p}(I;\mathcal{Q})$ such that

$$(\boldsymbol{B}\nabla\phi_\eta(t),\nabla\zeta)_H-(\mathcal{E}\varepsilon(u_\eta(t)),\nabla\zeta)_H+(h(u_\eta(t),\phi_\eta(t)),\zeta)=(q(t),\zeta)_{\mathcal{Q}}. \tag{2.44}$$

The well-posedness of the problem $\mathcal{PV}^1_{2\eta}$ follows.

Lemma 2.3. *There exists a unique solution $\phi_\eta \in W^{1,p}(I;\mathcal{Q})$ which satisfies (2.44). Moreover, if ϕ_{η_1} and ϕ_{η_2} are the solutions of (2.44) corresponding to η_1, $\eta_2 \in \mathcal{C}(I;\mathcal{H})$ then, there exists $C>0$, such that*

$$\|\phi_{\eta_1}(t)-\phi_{\eta_2}(t)\|_{\mathcal{Q}} \leq C\,\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathcal{V}}, \quad \forall t\in I. \tag{2.45}$$

Proof. Let $t\in I$. Via the Riesz representation theorem, for all $\zeta\in\mathcal{Q}$ we define the operator $\Psi_\eta(t):\mathcal{Q}\to\mathcal{Q}$ by

$$(\Psi_\eta(t)\phi_\eta,\zeta)_{\mathcal{Q}}=(\boldsymbol{B}\nabla\phi_\eta(t),\nabla\zeta)_{\mathcal{Q}}-(\mathcal{E}\varepsilon(u_\eta(t)),\nabla\zeta)_H+(h(u_\eta(t),\phi_\eta(t)),\zeta)_{\mathcal{Q}}. \tag{2.46}$$

For any $\phi_{\eta_1}, \phi_{\eta_2} \in \mathcal{Q}$, assumptions (2.19) and (2.35) imply

$$(\Psi_\eta(t)\phi_{\eta_1} - \Psi_\eta(t)\phi_{\eta_2}, \phi_{\eta_1} - \phi_{\eta_2})_{\mathcal{Q}} \geq m\|\phi_{\eta_1} - \phi_{\eta_2}\|^2_{\mathcal{Q}} + \int_{\Gamma_3} \psi(u_\nu - g)\left(\Phi_l(\phi_{\eta_1} - \phi_0) - \Phi_l(\phi_{\eta_2} - \phi_0)\right)(\phi_{\eta_1} - \phi_{\eta_2})\, da, \tag{2.47}$$

and, by (2.22) together with the monotonicity of Φ_l, we obtain

$$(\Psi_\eta(t)\phi_{\eta_1} - \Psi_\eta(t)\phi_{\eta_2}, \phi_{\eta_1} - \phi_{\eta_2})_{\mathcal{Q}} \geq m\|\phi_{\eta_1} - \phi_{\eta_2}\|^2_{\mathcal{Q}}. \tag{2.48}$$

On the other hand, using again (2.19), (2.20), (2.22), and (2.35), we have

$$(\Psi_\eta(t)\phi_{\eta_1} - \Psi_\eta(t)\phi_{\eta_2}, \zeta)_{\mathcal{Q}} \leq C\|\phi_{\eta_1} - \phi_{\eta_2}\|_{\mathcal{Q}}\|\zeta\|_{\mathcal{Q}} + \int_{\Gamma_C} M_\psi|\phi_{\eta_1} - \phi_{\eta_2}||\zeta|\, da, \quad \forall \zeta \in \mathcal{Q}. \tag{2.49}$$

It then follows from (2.49) and (A.7) that

$$(\Psi_\eta(t)\phi_{\eta_1} - \Psi_\eta(t)\phi_{\eta_2}, \zeta)_{\mathcal{Q}} \leq (C + M_\psi c_d^2)\, \|\phi_{\eta_1} - \phi_{\eta_2}\|_{\mathcal{Q}}\|\zeta\|_{\mathcal{Q}}, \tag{2.50}$$

thus,

$$\|\Psi_\eta(t)\phi_{\eta_1} - \Psi_\eta(t)\phi_{\eta_2}\|_{\mathcal{Q}} \leq C\|\phi_{\eta_1} - \phi_{\eta_2}\|_{\mathcal{Q}}, \tag{2.51}$$

where C is a positive constant depending on the problem data.

The strong monotonicity and Lipschitz continuity of the operator $\Psi_\eta(t)$ on $\mathcal{Q}$, established by inequalities (2.48) and (2.51), guarantee the existence of a unique element $\phi_\eta(t)$ in $\mathcal{Q}$ satisfying the equation

$$\Psi_\eta(t)\varphi_\eta(t) = q(t). \tag{2.52}$$

From relations (2.46) and (2.52), it follows that the element $\phi_\eta(t) \in \mathcal{Q}$ is the sole solution to the nonlinear variational equation (2.44).

To prove that $\phi_\eta(t) \in W^{1,p}(I; \mathcal{Q})$, take any $t_1, t_2 \in I$ and using (2.20),(2.19),(2.35)

and (2.44) we find

$$m_{\mathbf{B}}\|\phi_\eta(t_1)-\phi_\eta(t_2)\|_{\mathcal{Q}}^2 \le C\|u_\eta(t_1)-u_\eta(t_2)\|_{\mathcal{V}}\|\phi_\eta(t_1)-\phi_\eta(t_2)\|_{\mathcal{Q}} + \|q(t_1)-q(t_2)\|_{\mathcal{Q}}\|\phi_\eta(t_1)-\phi_\eta(t_2)\|_{\mathcal{Q}} + R(u_1,u_2;\phi_{\eta 1},\phi_{\eta 2}), \tag{2.53}$$

where

$$R(u_1,u_2;\phi_{\eta 1},\phi_{\eta 2}) = \int_{\Gamma_3} |\psi(u(t_1)-g)\Phi_l(\phi_\eta(t_1)-\phi_0)-\psi(u(t_2)-g)\Phi_l(\phi_\eta(t_2)-\phi_0)||\phi_\eta(t_1)-\phi_\eta(t_2)|\,da.$$

In view of the boundedness of the functions ψ and Φ, their Lipschitz continuity, and the estimate (A.7), we obtain the following inequality

$$\begin{aligned} R(u_1,u_2;\phi_{\eta 1},\phi_{\eta 2}) &\le M_\psi \int_{\Gamma_3} |\phi_\eta(t_1)-\phi_\eta(t_2)|\,da + L_\psi l \int_{\Gamma_3} |u_\eta(t_1)-u_\eta(t_2)||\phi_\eta(t_1)-\phi_\eta(t_2)|\,da \\ &\le M_\psi c_d^2\|\phi_\eta(t_1)-\phi_\eta(t_2)\|_{\mathcal{Q}}^2 + L_\psi l c_d c_s\|u_\eta(t_1)-u_\eta(t_2)\|_{\mathcal{V}}^2\|\phi_\eta(t_1)-\phi_\eta(t_2)\|_{\mathcal{Q}}^2 \end{aligned} \tag{2.54}$$

By substituting this inequality into (2.53), we obtain

$$m_{\mathbf{B}}\|\phi_\eta(t_1)-\phi_\eta(t_2)\|_{\mathcal{Q}} \le (C + L_\psi l c_d c_s)\|u_\eta(t_1)-u_\eta(t_2)\|_{\mathcal{V}} + \|q(t_1)-q(t_2)\|_{\mathcal{Q}} + M_\psi c_d^2\|\phi_\eta(t_1)-\phi_\eta(t_2)\|_{\mathcal{Q}}. \tag{2.55}$$

From Hypothesis $M_\psi < \frac{m_{\mathbf{B}}}{c_d^2}$, it follows that

$$\|\phi_\eta(t_1)-\phi_\eta(t_2)\|_{\mathcal{Q}} \le C\left(\|u_\eta(t_1)-u_\eta(t_2)\|_{\mathcal{V}} + \|q(t_1)-q(t_2)\|_{\mathcal{Q}}\right). \tag{2.56}$$

By the regularity of u and q, we can deduce that $\phi_\eta \in W^{1,p}(I;\mathcal{Q})$.

Let $\eta_1,\eta_2 \in \mathcal{C}(I;\mathcal{H})$. Using (2.44) and arguments similar to those employed in the proof of (2.57), we obtain

$$m_{\mathbf{B}}\|\phi_{\eta_1}(t)-\phi_{\eta_2}(t)\|_{\mathcal{Q}} \le (C + L_\psi l c_d c_s)\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathcal{V}} + M_\psi c_d^2\|\phi_{\eta_1}(t)-\phi_{\eta_2}(t)\|_{\mathcal{Q}}. \tag{2.57}$$

Based on this inequality and assumption $M_\psi < \frac{m_{\mathbf{B}}}{c_d^2}$, we obtain (2.45), thus complet-

ing the proof. □

In the second step, we let $\mu \in L^p(I;\boldsymbol{Y})$ be given, and we define $\chi_\mu \in W^{1,p}(I;\boldsymbol{Y})$ by

$$\chi_\mu(t) = \chi_0 + \int_0^t \mu(r)dr. \tag{2.58}$$

We use the solution (u_η, φ_η), obtained in Lemmas 2.2 and 2.3, to formulate the following Cauchy problem for the stress field.

Problem $\mathcal{P}_{\eta\mu}$. Find $\sigma_{\eta\mu} \in W^{1,p}(I,\mathcal{H})$ such that

$$\sigma_{\eta\mu}(t) = \mathcal{M}\varepsilon(u_\eta(t)) + \int_0^t \mathcal{S}\Big(\sigma_{\eta\mu}(r), \varepsilon(u_\eta(r)), \chi_\mu(r)\Big)\, dr, \quad \text{a.e. } t \in I. \tag{2.59}$$

Lemma 2.4. *Problem $\mathcal{P}_{\eta\mu}$ has a unique solution which satisfies (2.38). Moreover, if u_{η_i} and $\sigma_{\eta_i\mu_i}$ are two solutions of problems $\mathcal{PV}_{\eta_i}$ and $\mathcal{PV}_{\eta_i\mu_i}$, respectively, for $i = 1, 2$, then there exists $C > 0$ such that*

$$\begin{aligned}\|\sigma_{\eta_1\mu_1}(t) - \sigma_{\eta_2\mu_2}(t)\|_{\mathcal{H}}^2 \le C\Big(&\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_{\mathcal{V}}^2 + \int_0^t \|u_{\eta_1}(r) - u_{\eta_2}(r)\|_{\mathcal{V}}^2\, dr \\ &+ \int_0^t \|\chi_{\mu_1}(r) - \chi_{\mu_2}(r)\|_{\boldsymbol{Y}}^2\, dr\Big).\end{aligned} \tag{2.60}$$

Proof. Let $\mathcal{T}_{\eta\mu} : W^{1,p}(I;\mathcal{H}) \to W^{1,p}(I;\mathcal{H})$ be the mapping given by

$$\mathcal{T}_{\eta\mu}\sigma(t) = \mathcal{M}\varepsilon(u_\eta(t)) + \int_0^t \mathcal{S}\big(\sigma(r), \varepsilon(u_\eta(r)), \chi_\mu(r)\big)\, dr, \tag{2.61}$$

for all $\sigma \in W^{1,p}(I;\mathcal{H})$ and $t \in I$. For $\sigma_1, \sigma_2 \in W^{1,p}(I;\mathcal{H})$, using (2.61) and assumption (2.18), we obtain, for all $t \in I$,

$$\|\mathcal{T}_{\eta\mu}\sigma_1(t) - \mathcal{T}_{\eta\mu}\sigma_2(t)\|_{\mathcal{H}} \le L_{\mathcal{S}} \int_0^t \|\sigma_1(r) - \sigma_2(r)\|_{\mathcal{H}}\, dr.$$

By performing n successive integrations of the obtained inequality, we conclude that the operator $\mathcal{T}_{\eta\mu}^n$ is a contraction, and consequently the operator $\mathcal{T}_{\eta\mu}$ is also a contraction in the space $W^{1,p}(I;\mathcal{H})$. Hence, by Banach's fixed-point theorem, the operator admits a unique fixed point, which in turn represents the unique solution to Problem $\mathcal{P}_{\eta\mu}$.

Now, for $i = 1, 2$, we consider $(\eta_i, \mu_i) \in W^{1,p}(I; \mathcal{H} \times \mathbf{Y})$ and denote $u_{\eta_i} = u_i$, $\sigma_{\eta_i,\mu_i} = \sigma_i$, $\chi_{\mu_i} = \chi_i$. We have

$$\sigma_i(t) = \mathcal{M}\varepsilon(u_i(t)) + \int_0^t \mathcal{S}\Big(\sigma_i(r), \varepsilon(u_i(r)), \chi_i(r)\Big)dr, \quad \text{a.e. } t \in I.$$

Using the properties (2.17) and (2.18) of $\mathcal{M}$ and $\mathcal{S}$, we conclude the following

$$\|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}}^2 \leq C\Big(\|u_1(t) - u_2(t)\|_{\mathcal{V}}^2 + \int_0^t \|\sigma_1(r) - \sigma_2(r)\|_{\mathcal{H}}^2 dr + \int_0^t \|u_1(r) - u_2(r)\|_{\mathcal{V}}^2 dr + \int_0^t \|\chi_1(r) - \chi_2(r)\|_{\mathbf{Y}}^2 dr\Big)$$

Applying Grönwall's lemma to the derived inequality yields result (2.60), thereby completing the proof of Lemma 3.4. □

In the third step, we consider the following initial value Problem, using the displacement field u_η obtained from Lemma 2.2.

Problem $\mathcal{P}_{\eta\omega}$ Find $\omega_\eta \in \mathcal{C}^1(I; L^2(\Gamma_3))$ such that

$$\dot{\omega}_\eta = \kappa_\omega \alpha^* p_\nu(u_{\eta\nu} - \omega_\eta - g), \qquad \omega_\eta(0) = \omega_0. \tag{2.62}$$

Lemma 2.5. *The Problem $\mathcal{P}_{\eta\omega}$ has a unique solution which satisfies $\omega_\eta \in \mathcal{C}^1(I; L^2(\Gamma_3))$.*

Proof. We consider the operator $\pounds_\eta : \mathcal{C}^1(I; L^2(\Gamma_3)) \longrightarrow \mathcal{C}^1(I; L^2(\Gamma_3))$ defined by

$$\pounds_\eta\omega(t) = \int_0^t \kappa_\omega \alpha^* p_\nu(u_{\eta\nu}(r) - \omega_\eta(r) - g)\, dr + \omega_0, \qquad \forall t \in I. \tag{2.63}$$

Let $\omega_1, \omega_2 \in \mathcal{C}^1(I; L^2(\Gamma_3))$ and $t \in I$. From (2.63) and assumption (2.23), we deduce that

$$\|\pounds_\eta\omega_1(t) - \pounds_\eta\omega_2(t)\|_{L^2(\Gamma_3)}^2 \leq C\int_0^t \|\omega_1(r) - \omega_2(r)\|_{L^2(\Gamma_3)}^2\, dr.$$

By repeating the previous inequality p times, we get

$$\|\pounds_\eta^p\omega_1 - \pounds_\eta^p\omega_2\|_{\mathcal{C}(0,T;L^2(\Gamma_3))}^2 \leq \frac{(CT)^p}{p!}\|\omega_1 - \omega_2\|_{\mathcal{C}(0,T;L^2(\Gamma_3))}^2.$$

Based on the above inequality, it is clear that when the value of p is selected large enough, the operator $\pounds_\eta^p$ becomes a contraction in the Banach space $\mathcal{C}(I; L^2(\Gamma_3))$. According to Banach's fixed point theorem, the operator $\pounds_\eta$ has a unique fixed point $\omega_\eta \in \mathcal{C}(I; L^2(\Gamma_3))$. From assumption (2.23) and the regularity mentioned in (2.37), we conclude that Lemma (2.5) is valid. □

Finally, by employing the properties of the operators $\mathcal{S}$ and $\mathcal{E}$, along with the function χ and the results established above, we consider the operator

$$\Lambda : W^{1,p}(I; \mathcal{H} \times \mathbf{Y}) \to W^{1,p}(I; \mathcal{H} \times \mathbf{Y}),$$

defined by

$$\Lambda(\eta, \mu)(t) = \big(\Lambda^1(\eta, \mu)(t), \Lambda^2(\eta, \mu)(t)\big) \in \mathcal{H} \times \mathbf{Y},$$

with

$$\begin{aligned}(\Lambda^1(\eta, \mu)(t), w)_{\mathcal{H}\times\mathcal{V}} =& \Big(\int_0^t \mathcal{S}\Big(\sigma_{\eta\mu}(r), \varepsilon(u_\eta(r)), \chi_\mu(r)\Big)\, dr, \varepsilon(w)\Big)_{\mathcal{H}} \\ &+ (\mathcal{E}^*\nabla\phi_\eta(t), \varepsilon(w))_{\mathcal{H}}, \qquad \forall w \in \mathcal{V}, \end{aligned} \tag{2.64}$$

$$\Lambda^2(\eta, \mu)(t) = \chi\big(\sigma_{\eta\mu}(t), \varepsilon(u_\eta(t)), \chi_\mu(t)\big), \tag{2.65}$$

where u_η, φ_η and $\sigma_{\eta\mu}$, represent the displacement field, the potential field, and the stress field obtained in Lemmas (2.2), (2.3) and (2.4), respectively, while χ_μ represents the internal state variable (2.58).

Lemma 2.6. *The operator Λ has a fixed point $(\eta^*, \mu^*) \in W^{1,p}(I; \mathcal{H} \times \boldsymbol{Y})$ such that $\Lambda(\eta^*, \mu^*) = (\eta^*, \mu^*)$.*

Proof. Let $t \in I$ and let (η_1, μ_1), $(\eta_2, \mu_2) \in W^{1,p}(I; \mathcal{H} \times \mathbf{Y})$. We use the notation $u_{\eta_i} = u_i$, $\dot{u}_{\eta_i} = \dot{u}_i$, $\phi_{\eta_i} = \phi_i$, $\sigma_{\eta_i\mu_i} = \sigma_i$, and $\chi_{\mu_i} = \chi_i$ for $i = 1, 2$.

To begin, we apply relation (A.5) together with assumptions (3.14), (2.20), and

(2.23), which yield

$$\|\Lambda^1(\eta_1,\mu_1)(t)-\Lambda^1(\eta_2,\mu_2)(t)\|^2_{\mathcal{H}} \leq \|\mathcal{E}^*\nabla\phi_1(t)-\mathcal{E}^*\nabla\phi_2(t)\|^2_H + \int_0^t \|\mathcal{S}(\sigma_1(r),\varepsilon(u_1(r)),\chi_1(r))-\mathcal{S}(\sigma_2(r),\varepsilon(u_2(r)),\chi_2(r))\|^2_{\mathcal{H}}\,dr \leq C\Big(\|\phi_1(t)-\phi_2(t)\|^2_{\mathcal{Q}} + \int_0^t \|\sigma_1(r)-\sigma_2(r)\|^2_{\mathcal{H}}\,dr + \int_0^t \|\chi_1(r)-\chi_2(r)\|^2_{\mathbf{Y}}\,dr + \int_0^t \|u_1(r)-u_2(r)\|^2_{\mathcal{V}}\,dr\Big).$$

We use estimates (3.57), (2.60) to obtain

$$\|\Lambda^1(\eta_1,\mu_1)(t)-\Lambda^1(\eta_2,\mu_2)(t)\|^2_{\mathcal{H}} \leq C\Big(\|u_1(t)-u_2(t)\|^2_{\mathcal{V}} + \int_0^t \|\chi_1(r)-\chi_2(r)\|^2_{\mathbf{Y}}\,dr + \int_0^t \|u_1(r)-u_2(r)\|^2_{\mathcal{V}}\,dr\Big).$$

By similar arguments, from (2.60), (2.64), and (3.15) we obtain

$$\|\Lambda^2(\eta_1,\mu_1)(t)-\Lambda^2(\eta_2,\mu_2)(t)\|^2_{\mathbf{Y}} \leq C\Big(\|u_1(t)-u_2(t)\|^2_{\mathcal{V}} + \int_0^t \|u_1(r)-u_2(r)\|^2_{\mathcal{V}}\,dr + \|\chi_1(t)-\chi_2(t)\|^2_{\mathbf{Y}} + \int_0^t \|\chi_1(r)-\chi_2(r)\|^2_{\mathbf{Y}}\,dr\Big).$$

Therefore,

$$\|\Lambda(\eta_1,\mu_1)(t)-\Lambda(\eta_2,\mu_2)(t)\|^2_{\mathcal{H}\times\mathbf{Y}} \leq C\Big(\|u_1(t)-u_2(t)\|^2_{\mathcal{V}} + \int_0^t \|u_1(r)-u_2(r)\|^2_{\mathcal{V}}\,dr + \|\chi_1(t)-\chi_2(t)\|^2_{\mathbf{Y}} + \int_0^t \|\chi_1(r)-\chi_2(r)\|^2_{\mathbf{Y}}\,dr\Big), \quad \text{a.e. } t\in I. \tag{2.66}$$

Furthermore, given the integral representation,

$$u_i(t) = u_0 + \int_0^t \dot{u}_i(r)dr,$$

it follows that for a.e. $t \in I$, the following estimate holds

$$\|u_1(t) - u_2(t)\|_{\mathcal{V}} \leq \int_0^t \|\dot{u}_1(r) - \dot{u}_2(r)\|_{\mathcal{V}} dr, \tag{2.67}$$

Substituting this inequality into relation (3.39) yields the following result

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_{\mathcal{V}} \leq C \left(\|\eta_1(t) - \eta_2(t)\|_{\mathcal{H}} + \int_0^t |\dot{u}_1(r) - \dot{u}_2(r)|_{\mathcal{V}} dr \right),$$

By means of a Gronwall-type argument, we obtain

$$\int_0^t \|\dot{u}_1(r) - \dot{u}_2(r)\|_{\mathcal{V}} dr \leq C \int_0^t \|\eta_1(r) - \eta_2(r)\|_{\mathcal{H}} dr,$$

Combining this with estimate (2.67) yields

$$\|u_1(t) - u_2(t)\|_{\mathcal{V}}^2 \leq C \int_0^t \|\eta_1(r) - \eta_2(r)\|_{\mathcal{H}}^2 dr, \quad \text{a.e. } t \in (0, T). \tag{2.68}$$

The definition (2.58) yields

$$\|\chi_1(t) - \chi_2(t)\|_{\mathbf{Y}}^2 \leq C \int_0^t \|\mu_1(r) - \mu_2(r)\|_{\mathbf{Y}}^2 dr. \tag{2.69}$$

From the previous inequality and estimates (2.2), (2.68) , and (2.69), it now follows that

$$\|\Lambda(\eta_1, \mu_1)(t) - \Lambda(\eta_2, \mu_2)(t)\|_{\mathcal{H} \times \mathbf{Y}}^2 \leq C \int_0^t \|(\eta_1, \mu_1)(r) - (\eta_2, \mu_2)(r)\|_{\mathcal{H} \times \mathbf{Y}}^2 \, dr.$$

Repeating this inequality n times yields

$$\begin{aligned} &\|\Lambda^n(\eta_1, \mu_1) - \Lambda^n(\eta_2, \mu_2)\|_{W^{1,p}(I; \mathcal{H} \times \mathbf{Y})}^2 \\ &\leq \frac{C^n T^n}{n!} \|(\eta_1, \mu_1) - (\eta_2, \mu_2)\|_{W^{1,p}(I; \mathcal{H} \times \mathbf{Y})}^2. \end{aligned}$$

Thus, for n sufficiently large, Λ^n is a contraction on $W^{1,p}(I; \mathcal{H} \times \mathbf{Y})$. Therefore, Λ has a unique fixed point in this Banach space. □

Now, we have all the ingredients to prove Theorem (3.1).

Existence. Let $(\eta^*, \mu^*) \in W^{1,p}(I; \mathcal{H} \times \mathbf{Y})$ be the fixed point of Λ defined by (3.83)–(2.64), denote

$$u^* = u_{\eta^*},\ \phi^* = \phi_{\eta^*},\ \sigma^* = \sigma_{\eta^*\mu^*},\ \omega^* = \omega_{\eta^*} \quad \text{and } \chi^* = \chi_{\mu^*}.$$

Now, let $(u^*, \phi^*, \sigma^*, \omega^*)$ be the respective solutions to problems $\mathcal{PV}^1_{1\eta^*}$, $\mathcal{PV}^1_{2\eta^*}$, $\mathcal{P}_{\eta^*\mu^*}$ and $\mathcal{P}_{\eta^*\omega^*}$. Using relations (3.37), (2.44), (2.42), (3.78) and (2.62), and taking into account (2.58), we deduce that the tuple $(u^*, \phi^*, \sigma^*, \omega^*, \chi^*)$ is a solution to Problem $\mathcal{PV}^1$ and satisfies the regularity conditions described in (2.37)–(2.40).

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ, defined by (3.83)–(2.65), together with the unique solvability of the problems $\mathcal{PV}^1_{1\eta}$, $\mathcal{PV}^1_{2\eta}$, $\mathcal{P}_{\eta\mu}$, and $\mathcal{P}_{\eta\omega}$. This concludes the proof.

Chapter 3

Resolution of Thermomechanical Problems with Fractional Time Derivative

In this chapter, we investigate a frictional contact problem by replacing the classical first-order time derivative by a fractional-order derivative in both the heat conduction law and the constitutive relation of the material. The frictional behavior at the contact boundary is described by a Clarke-type subdifferential law, allowing for the treatment of non-smooth and possibly nonmonotone effects. The proposed model captures the intrinsic long-memory effects of materials exhibiting coupled thermal, viscoelastic, and plastic responses. We formulate the associated variational problem and establish the existence and uniqueness of a weak solution by means of a Galerkin approximation scheme combined with Banach's fixed-point theorem and fundamental results from the theory of hemivariational inequalities.

3.1 Mechanical Problem Formulation and Assumptions

The material under investigation is assumed to exhibit thermo-elasto-viscoplastic behavior and occupies a bounded domain $\Omega \subset \mathbb{R}^m$, where $m \in \{2, 3\}$, over the time

interval $I = [0, T]$, with $T > 0$. The boundary Γ of Ω is assumed to be Lipschitz continuous and is partitioned into three pairwise disjoint, relatively open subsets Γ_1, Γ_2, and Γ_3, such that $\text{meas}(\Gamma_1) > 0$. The outward unit normal vector to Γ is denoted by ν.

The body is subjected to a given volume force of density f_0 and to a volumetric heat source of intensity q_0. On Γ_1, the displacement vanishes (clamped boundary condition), while on Γ_2 a prescribed surface traction of density f_2 is applied. The subset Γ_3 represents the potential contact surface with a deformable foundation. Moreover, the temperature is assumed to vanish on a portion $\Gamma_1 \cup \Gamma_2$.

In this study, a Caputo fractional derivative is incorporated into both the constitutive stress law and the thermal evolution equation in order to capture the memory and hereditary effects characteristic of thermo-viscoplastic materials. This fractional framework provides a more accurate representation of the time-dependent behavior and energy dissipation mechanisms within the material.

Based on the above physical setting and boundary conditions, we now formulate the corresponding mathematical problem.

Problem $\mathcal{P}^2$. Find a displacement field $u : \Sigma \to \mathbb{R}^m$, a stress field $\sigma : \Sigma \longrightarrow \mathbb{S}^m$ and a temperature $\theta : \Sigma \to \mathbb{R}$ such that

$$\sigma = \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right) + \mathcal{M}\boldsymbol{\varepsilon}(u) + \int_0^t \mathcal{S}\big(\sigma - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right), \boldsymbol{\varepsilon}(u), \theta\big) dr \quad \text{in } \Sigma, \tag{3.1}$$

$$\operatorname{Div}\sigma + \mathbf{f}_0 = 0 \quad \text{in } \Sigma, \tag{3.2}$$

$${}_0^C D_t^\alpha \theta - \operatorname{div}\mathcal{K}\nabla\theta = \psi\big(\sigma - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right), \boldsymbol{\varepsilon}(u), \theta\big) + q_t \quad \text{in } \Sigma, \tag{3.3}$$

$$u = 0 \quad \text{on } \Xi_1, \tag{3.4}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Xi_2, \tag{3.5}$$

$$\theta = 0 \quad \text{on } \Xi_0, \tag{3.6}$$

$$-\sigma_\nu \in \partial j_\nu\left({}_0^C D_t^\alpha u_\nu\right) \quad \text{on } \Xi_3, \tag{3.7}$$

$$-\sigma_\tau \in \partial j_\tau\left({}_0^C D_t^\alpha u_\tau\right) \quad \text{on } \Xi_3, \tag{3.8}$$

$$\mathcal{K}\frac{\partial\theta}{\partial\nu} = \kappa_e(\theta - \theta_F) \quad \text{on } \Xi, \tag{3.9}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{3.10}$$

$$\theta(0) = \theta_0 \quad \text{in } \Omega. \tag{3.11}$$

In the present model, relation (3.1) provides the constitutive law for thermo-elasto-viscoplastic materials, incorporating Caputo-type time-fractional derivatives to capture memory effects. Mechanical equilibrium is enforced by (3.2), representing the stress balance, while thermal behavior is governed by (3.3), a time-fractional evolution equation for temperature.

The boundary conditions are specified as follows: mechanical constraints are imposed through (3.4) for displacements and (3.5) for traction forces, whereas thermal conditions are given by (3.6) and (3.9). A distinctive feature of the model is the multivalued normal compliance condition (3.7), which incorporates a fractional derivative to describe the contact behavior. Inclusion (3.8) characterizes the frictional interaction on the contact surface, where ∂j_τ denotes the Clarke subdifferential of the function j_τ as defined in [27]. Finally, the initial state of the system is specified by u_0 and θ_0, representing the initial displacement and temperature fields,

respectively.

Before establishing the variational formulation of the problem, it is essential to state the assumptions on the physical setting and on the constitutive behavior of the material. These assumptions ensure the well-posedness of the model and simplify the subsequent mathematical analysis. In the analysis of problem $\mathcal{P}^1$, the following assumptions are introduced.

The operators $\mathcal{L}, \mathcal{M}$, and $\mathcal{S}$ satisfy the following conditions

$$\begin{cases} (\mathbf{a})\ \mathcal{L} = (l_{ijlk}) : \Omega \times S^m \longrightarrow S^m,\ l_{ijlk} = l_{lkij} = l_{ijlk} \in L^\infty(\Omega), \text{and} \\ \quad \exists\, M_{\mathcal{L}} > 0,\ \text{such that}\ \|\mathcal{L}(.,\varpi_1) - \mathcal{L}(.,\varpi_2)\| \leq M_{\mathcal{L}}\|\varpi_1 - \varpi_2\|, \\ \quad \forall \varpi_1, \varpi_2 \in \mathbb{S}^m, \text{a.e. in } \Omega. \\ (\mathbf{b})\ \exists\, m_{\mathcal{L}} > 0,\ \text{such that}\ \mathcal{L}\varpi \cdot \varpi \geq m_{\mathcal{L}}\|\varpi\|^2,\ \forall \varpi \in \mathbb{S}^m, \text{a.e. in } \Omega. \end{cases} \tag{3.12}$$

$$\begin{cases} \mathcal{M} = (m_{ijlk}) : \Omega \times S^m \longrightarrow S^m,\ m_{ijlk} = m_{lkij} = m_{ijlk} \in L^\infty(\Omega), \text{and} \\ \quad \exists\, M_{\mathcal{M}} > 0,\ \text{such that}\ \ \|\mathcal{M}(.,\varpi_1) - \mathcal{M}(.,\varpi_2)\| \leq M_{\mathcal{M}}\|\varpi_1 - \varpi_2\|, \\ \quad \forall \varpi_1, \varpi_2 \in \mathbb{S}^m,\ \text{a.e. in } \Omega. \end{cases} \tag{3.13}$$

$$\begin{cases} (\mathbf{a})\ \mathcal{S} : \Omega \times \mathbb{S}^m \times \mathbb{S}^m \times \mathbb{R} \to \mathbb{S}^m,\ \text{and}\ \exists\, M_{\mathcal{S}} > 0, \text{such that} \\ \quad \|\mathcal{S}(.,\varpi_1,\varrho_1,\iota_1) - \mathcal{S}(.,\varpi_2,\varrho_2,\iota_2)\| \leq M_{\mathcal{S}}(\|\varpi_1 - \varpi_2\| + \|\varrho_1 - \varrho_2\| + \|\iota_1 - \iota_2\|) \\ \quad \forall \varpi_1, \varpi_2, \varrho_1, \varrho_2 \in \mathbb{S}^m, \text{and}\ \iota_1, \iota_2 \in \mathbb{R}\ \text{a.e. in } \Omega. \\ (\mathbf{b})\ \text{the function } \mathcal{S}(.,\varpi,\varrho,\iota) \text{ is measurable on } \Omega,\ \forall \varpi, \varrho \in \mathbb{S}^m, \text{ and } \iota \in \mathbb{R} \\ \quad \text{and } \mathcal{S}(.,0,0,0) \in \mathcal{H}. \end{cases} \tag{3.14}$$

The function $\psi : \Omega \times \mathbb{S}^m \times \mathbb{S}^m \times \mathbb{R} \to \mathbb{S}^m$ satisfies

$$\begin{cases} (\mathbf{a})\ \exists M_\psi > 0 \text{ such that } \|\psi(.,\varpi_1,\varrho_1,\iota_1) - \psi(.,\varpi_2,\varrho_2,\iota_2)\| \\ \quad \leq M_\psi(\|\varpi_1 - \varpi_2\| + \|\varrho_1 - \varrho_2\| + \|\iota_1 - \iota_2\|), \quad \forall \varpi_1, \varpi_2, \varrho_1, \varrho_2 \in \mathbb{S}^m, \\ \quad \text{and } \iota_1, \iota_2 \in \mathbb{R} \text{ a.e. in } \Omega. \\ (\mathbf{b})\ \text{the function } \psi(.,\varpi,\varrho,\iota) \text{ is measurable on } \Omega,\ \forall \varpi, \varrho \in \mathbb{S}^d, \text{ and } \iota \in \mathbb{R}^m, \\ \quad \text{and } \psi(.,0,0,0) \in L^2(\Omega). \end{cases} \tag{3.15}$$

The normal potential $j_\nu : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} (\mathbf{a})\ \exists\, M_\nu > 0 \text{ such that } j_\nu^0(.,\varpi_1,\varpi_2-\varpi_1) + j_\nu^0(.,\varpi_2,\varpi_2-\varpi_1) \\ \quad \leq M_\nu \|\varpi_1-\varpi_2\|_{\mathbb{R}}^2, \quad \forall \varpi_1,\varpi_2 \in \mathbb{R}, \text{a.e. in } \Gamma_3. \\ (\mathbf{b})\ \text{The function } j_\nu(z,\cdot) \text{ is Lipschitz continuous on } \mathbb{R} \text{ for a.e. } z \in \Gamma_3. \\ (\mathbf{c})\ \exists\, m_\nu > 0 \text{ such that } |\partial j_\nu(.,\varpi)| \leq m_\nu, \quad \forall \varpi \in \mathbb{R} \text{ a.e. in } \Gamma_3. \\ (\mathbf{d})\ \text{ for a.e. } z \in \Gamma_3, \text{ the function } j_\nu(z,.) \text{ exhibits regularity .} \\ (\mathbf{e})\ \text{ the function } j_\nu(.,\varpi) \text{ is measurable on } \Gamma_3, \ \forall \varpi \in \mathbb{R} \text{ and} \\ \quad \exists\, r \in L^2(\Gamma_3;\mathbb{R}) \text{ such that } j_\nu(.,r(.)) \in L^1(\Gamma_3). \end{cases} \tag{3.16}$$

The potential function $j_\tau : \Gamma_3 \times \mathbb{R}^m \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} (\mathbf{a})\ \exists\, M_\tau > 0 \text{ such that } j_\tau^0(.,\varpi_1,\varpi_2-\varpi_1) + j_\tau^0(.,\varpi_2,\varpi_2-\varpi_1) \\ \quad \leq M_\tau \|\varpi_1-\varpi_2\|_{\mathbb{R}^m}^2, \quad \forall \varpi_1,\varpi_2 \in \mathbb{R}^m, \text{a.e. in } \Gamma_3. \\ (\mathbf{b})\ \text{The function } j_\nu(z,\cdot) \text{ is Lipschitz continuous on } \mathbb{R}^m \text{ for a.e. } z \in \Gamma_3. \\ (\mathbf{c})\ \exists\, m_\tau > 0 \text{ such that } |\partial j_\tau(.,\varpi)| \leq m_\tau, \quad \forall \varpi \in \mathbb{R}^m \text{ a.e. in } \Gamma_3. \\ (\mathbf{d})\ \text{ for a.e. } z \in \Gamma_3, \text{ the function } j_\tau(z,.) \text{ exhibits regularity.} \\ (\mathbf{e})\ \text{the function } j_\tau(.,\varpi) \text{ is measurable on } \Gamma_3, \ \ \forall \varpi \in \mathbb{R}^m \text{ and} \\ \quad \exists\, r \in L^2(\Gamma_3;\mathbb{R}^m) \text{ such that } j_\tau(.,r(.)) \in L^1(\Gamma_3). \end{cases} \tag{3.17}$$

The thermal tensors are assumed to satisfy the following conditions

$$\begin{cases} \mathcal{K} = k_{lk}, \quad k_{lk} = k_{kl} \in L^\infty(\Omega). \\ k_{lk}\zeta_l\zeta_l \geq m_{\mathcal{K}}\, \zeta_l\zeta_l \quad \text{ where, } m_{\mathcal{K}} > 0 \quad \forall \zeta_l \in \mathbb{R}^m. \end{cases} \tag{3.18}$$

Furthermore, we assume the following additional hypotheses. The forces, the traction and the heat source satisfy the following regularities

$$\mathbf{f}_0 \in \mathcal{C}(I;L^2(\Omega)), \quad q_t \in \mathcal{C}(I;L^2(\Omega)), \quad \mathbf{f}_2 \in \mathcal{C}(I;L^2(\Gamma_2)^m). \tag{3.19}$$

The initial conditions, the thermal potential and the coefficient of heat exchange

satisfy

$$\mathbf{u}_0 \in \mathcal{V}, \quad \theta_F \in L^2(I; L^2(\Gamma_3)), \quad \theta_0 \in \mathcal{W}, \quad \text{and } \kappa_e \in L^\infty(\Gamma_3, \mathbb{R}^+). \tag{3.20}$$

3.2 Variational Formulation

By the Riesz representation theorem, we define the mappings $\mathbf{f} : I \to \mathcal{V}$ and $g : I \to \mathcal{W}^*$ as follows

$$(\mathbf{f}(t), w)_{\mathcal{V}} = \int_\Omega \mathbf{f}_0(t) w \, dx + \int_{\Gamma_2} \mathbf{f}_2(t) w \, da, \tag{3.21}$$

$$(g(t), \xi)_{\mathcal{W}^* \times \mathcal{W}} = \int_\Omega q_t(t) \xi dx + \int_{\Gamma_3} \kappa_e \theta_F(t) \xi da. \tag{3.22}$$

Next, we define the mapping $K : \mathcal{W} \to \mathcal{W}^*$ by

$$(K\theta, \zeta)_{\mathcal{W}^* \times \mathcal{W}} = \int_\Omega \mathcal{K} \nabla\theta \cdot \nabla\zeta \, dx + \int_{\Gamma_3} \kappa_e \theta \, \zeta \, da. \tag{3.23}$$

The operator K satisfies

$$\|K\theta_1 - K\theta_2\|_{\mathcal{W}^*} \leq M_k \|\theta_1 - \theta_2\|_{\mathcal{W}}, \quad \forall \theta_1, \theta_2 \in \mathcal{W}, \text{ with } M_k > 0. \tag{3.24}$$

Now that the fundamental properties of the model have been established, we begin by constructing the variational formulation associated with the contact problem described by equations (3.1)–(3.11). Let (u, σ) be a sufficiently smooth pair defined on Σ and satisfying the system (3.1)–(3.11). For an arbitrary test function $w \in \mathcal{V}$, we multiply the equilibrium equation (3.2) by w. Then, by applying Green's identity together with the definition of the divergence operator, we obtain

$$(\sigma(t), \varepsilon(w))_{\mathcal{H}} = (f_0(t), w)_H + \int_\Gamma \sigma(t)\nu \cdot w da, \quad \text{for a.e. } t \in I. \tag{3.25}$$

Noting that

$$\int_\Gamma \sigma(t)\nu \cdot w da = \int_{\Gamma_1} \sigma(t)\nu \cdot w da + \int_{\Gamma_2} \sigma(t)\nu \cdot w da + \int_{\Gamma_3} \sigma(t)\nu \cdot w da, \tag{3.26}$$

and using the boundary conditions (3.4) and (3.5), we obtain

$$\int_{\Gamma} \sigma(t)\nu \cdot \vartheta da = (f_2(t), \vartheta)_{L^2(\Gamma_2;\mathbb{R}^m)} + \int_{\Gamma_3} \sigma(t)\nu \cdot \vartheta da, \quad \text{for a.e. } t \in I. \tag{3.27}$$

Using the decomposition formula (A.9) together with equation (3.21), we obtain

$$(\sigma(t), \varepsilon(w))_{\mathcal{H}} = (f(t), w)_{\mathcal{V}} + \int_{\Gamma_3} (\sigma_\nu(t) w_\nu + \sigma_\tau(t) \cdot w_\tau)\, da, \quad \text{for a.e. } t \in I. \tag{3.28}$$

Moreover, by combining (3.7) and (3.8) with the definition of the subdifferential, we obtain

$$-\sigma_\nu(t) w_\nu \leq j_\nu^0 \left({}_0^C D_t^\alpha u_\nu; w_\nu\right), \qquad -\sigma_\tau(t) \cdot w_\tau \leq j_\tau^0 \left({}_0^C D_t^\alpha u_\tau; w_\tau\right) \quad \text{on } \Xi_3. \tag{3.29}$$

Substituting (3.1) and (3.29) into (3.28), we obtain the following inequality

$$\begin{aligned} &\left(\mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right) + \mathcal{M}\boldsymbol{\varepsilon}(u), \boldsymbol{\varepsilon}(w)\right)_{\mathcal{H}} + \left(\int_0^t \mathcal{S}\big(\sigma - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right), \boldsymbol{\varepsilon}(u), \theta\big) dr, \boldsymbol{\varepsilon}(w)\right)_{\mathcal{H}} \\ &+ \int_{\Gamma_3} j_\nu^0 \left({}_0^C D_t^\alpha u_\nu(t); w_\nu\right) + j_\tau^0 \left({}_0^C D_t^\alpha u_\tau(t); w_\tau\right) da \geq (\mathbf{f}(t), w)_{\mathcal{V}} \quad \text{for a.e. } t \in I. \end{aligned} \tag{3.30}$$

By proceeding as in the previous case, and now considering the heat equation under the assumptions stated above, we derive the variational formulation of the mechanical problem given in (3.1)–(3.11). For all $\vartheta \in \mathcal{V}$ and $\zeta \in \mathcal{W}$, the variational formulation reads as follows

Problem $\mathcal{PV}^2$. Find a displacement field $u : I \to \mathcal{V}$, a stress field $\boldsymbol{\sigma} : I \to \mathcal{H}_1$ and a temperature $\theta : I \to \mathcal{W}$, such that

$$\sigma = \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right) + \mathcal{M}\boldsymbol{\varepsilon}(u) + \int_0^t \mathcal{S}\big(\sigma - \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u\right), \boldsymbol{\varepsilon}(u), \theta\big) dr, \tag{3.31}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(w))_{\mathcal{H}} + \int_{\Gamma_3} j_\nu^0 \left({}_0^C D_t^\alpha u_\nu(t); w_\nu\right) + j_\tau^0 \left({}_0^C D_t^\alpha u_\tau(t); w_\tau\right) da \geq (\mathbf{f}(t), w)_{\mathcal{V}}, \tag{3.32}$$

$$\begin{aligned} \left({}_0^C D_t^\alpha \theta(t) + K\theta(t), \zeta\right)_{\mathcal{W}^* \times \mathcal{W}} &= \big(\psi\big(\boldsymbol{\sigma} - \mathcal{L}\varepsilon({}_0^C D_t^\alpha u(t)), \boldsymbol{\varepsilon}(u(t)), \theta(t)\big), \zeta\big) \\ &\quad + (g(t), \zeta)_{\mathcal{W}^* \times \mathcal{W}}, \end{aligned} \tag{3.33}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0. \tag{3.34}$$

The existence and uniqueness of the solution to the variational Problem $\mathcal{PV}^2$ will be established in the next section.

3.3 Existence and uniqueness of a solution

We now state the main existence and uniqueness result.

Theorem 3.1. *Assume that the hypotheses* (3.12)–(3.20) *are satisfied, and that*

$$m_{\mathcal{L}} > c_s^2(M_\nu + M_\tau).$$

Then, the variational problem $\mathcal{PV}^2$ *admits a unique solution* (u, σ, θ)*, which has the following regularity:*

$$u \in W^{1,2}(I; \mathcal{V}), \quad \sigma \in W^{1,2}(I; \mathcal{H}_1), \quad \theta \in W^{1,2}(I; \mathcal{W}). \tag{3.35}$$

Under the hypotheses stated in (3.12)–(3.20), the system (3.1)–(3.11) admits a unique weak solution (u, σ, θ) satisfying the regularity properties specified in (3.35).

The proof of Theorem 3.1 is carried out in several stages, using a systematic construction and various analytical tools, including variational inequality techniques, properties of the Caputo fractional derivative, the Galerkin approximation method, and Banach's fixed-point theorem.

Throughout this section, we operate within the framework provided by Theorem 3.1. The symbol C denotes a generic positive constant, which may vary from line to line and depends solely on the problem's given data.

We begin the analysis by selecting an arbitrary element $\eta \in \mathcal{C}(I; \mathcal{H})$. Using the Riesz representation theorem, we define a functional via

$$(f_\eta(t), w)_{\mathcal{V}} = (f(t) - \eta(t), w)\,, \qquad \forall w \in \mathcal{V}. \tag{3.36}$$

This construction naturally leads to the formulation of an associated auxiliary mechanical problem.

Problem $\mathcal{PV}^2_{1\eta}$. Find $u_\eta \in W^{1,p}(I;\mathcal{V})$ such that

$$\begin{aligned}&\left(\mathcal{L}\varepsilon\left({}^C_0D^\alpha_t u_\eta(t)\right)+\mathcal{M}\boldsymbol{\varepsilon}(u_\eta(t)),\boldsymbol{\varepsilon}(w)\right)_{\mathcal{H}}\\ &+\int_{\Gamma_3} j^0_\nu\left({}^C_0D^\alpha_t u_{\nu\eta}(t);w_\nu\right)+j^0_\tau\left({}^C_0D^\alpha_t u_{\tau\eta}(t);w_\tau\right)da\geq(\mathbf{f}_\eta(t),w)_{\mathcal{V}},\quad\forall w\in\mathcal{V}\end{aligned}\tag{3.37}$$

$$u_\eta(0)=u_0.\tag{3.38}$$

We establish the following result in relation to $\mathcal{PV}^2_{1\eta}$

Lemma 3.2 ([37]). *The Problem $\mathcal{PV}^2_{1\eta}$ admits at least one solution. Moreover, if u_{η_1} and u_{η_2} are two solutions of $\mathcal{PV}^2_{1\eta}$ corresponding to the data $\eta_1,\eta_2\in\mathcal{C}(I;\mathcal{H})$, then there exists $C>0$ such that*

$$\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathcal{V}}\leq C\,\|\eta_1(t)-\eta_2(t)\|_{\mathcal{H}},\quad\forall\,t\in I.\tag{3.39}$$

Proof. We set ${}^C_0D^\alpha_t u_\eta=\vartheta_\eta$. According to Proposition A.11, we then have

$$u_\eta(t)={}_0I^\alpha_t\vartheta_\eta(t)+u_0,\qquad\text{for a.e. } t\in I.$$

Consequently, $\mathcal{PV}^2_{1\eta}$ may be reformulated in the following form.

Problem $\mathcal{PV}^2_{2\eta}$. Find $\vartheta_\eta\in L^1(I;\mathcal{V})$ such that

$$\begin{aligned}&\Big(\mathcal{L}\varepsilon(\vartheta_\eta(t))+\mathcal{M}\Big(\boldsymbol{\varepsilon}\big({}_0I^\alpha_t\vartheta_\eta(t)+u_0\big)\Big),\boldsymbol{\varepsilon}(w)\Big)_{\mathcal{H}}\\ &+\int_{\Gamma_3} j^0_\nu\left(\vartheta_{\nu\eta}(t);w_\nu\right)+j^0_\tau\left(\vartheta_{\tau\eta}(t);w_\tau\right)da\geq\left(f_\eta(t),w\right)_{\mathcal{V}},\quad\forall w\in\mathcal{V},\end{aligned}$$

$$u_\eta(0)=u_0.\tag{3.40}$$

This equivalence implies that the function $u_\eta\in W^{1,2}(I;\mathcal{V})$ is a solution to the problem $\mathcal{PV}^2_{1\eta}$ if and only if there exists a corresponding function $\vartheta_\eta\in L^1(I;\mathcal{V})$ that satisfies the conditions of Problem $\mathcal{PV}^2_{2\eta}$. In other words, solving $\mathcal{PV}^2_{1\eta}$ is completely equivalent to finding ϑ_η that fulfills $\mathcal{PV}^2_{2\eta}$; both formulations describe the same physical and mathematical phenomena, but in different functional settings: one in terms of the displacement u_η and the other in terms of the auxiliary variable ϑ_η.

We consider the functional J defined on the space $L^2(\Gamma_3, \mathbb{R}^m)$ by

$$J(\vartheta) = \int_{\Gamma_3} (j_\nu(., \vartheta_\nu) + j_\tau(., \vartheta_\tau))da \tag{3.41}$$

Based on Assumptions (3.16) and (3.17), and by applying Corollary 4.15 and Lemma 3.39, in [27], we conclude that the function J satisfies a local Lipschitz condition, and that the following equality holds

$$J^0(\vartheta, \varpi) = \int_{\Gamma_3} (j^0_\nu(., \vartheta_\nu; \varpi_\nu) + j^0_\tau(., \vartheta_\tau; \varpi_\tau))da, \quad \forall \vartheta, \varpi \in L^2(\Gamma_3, \mathbb{R}^m). \tag{3.42}$$

Now, let us introduce the following auxiliary Problem, which will be used to reformulate the main variational formulation in a more convenient form.

In other words, we aim to determine ϑ_η such that, for every $w \in \mathcal{V}$, the following condition holds

$$\begin{aligned} &\Big(\mathcal{L}\varepsilon(\vartheta_\eta(t)) + \mathcal{M}\Big(\boldsymbol{\varepsilon}\big({}_0I_t^\alpha \vartheta_\eta(t) + u_0\big)\Big), \boldsymbol{\varepsilon}(w)\Big)_{\mathcal{H}} \\ &\quad + J^0(\gamma\vartheta_\eta(t); \gamma w) \geq (f_\eta(t), w)_{\mathcal{V}}\,, \end{aligned} \tag{3.43}$$

where $\gamma : \mathcal{V} \to L^2(\Gamma_3, \mathbb{R}^m)$ denotes the trace operator, which is compact. It follows from Assumption (3.12) that the operator $\mathcal{L}$ belongs to $\mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ and is coercive, while Assumption (3.13) ensures that $\mathcal{M} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$. Moreover, based on hypotheses (3.19) and (3.21), together with the regularity of η, we have $f_\eta \in \mathcal{C}(I; \mathcal{V}^*)$. Using Assumptions (3.16) and (3.17) and Corollary 4.15 in [27], we obtain

$$\|\partial J(\vartheta)\|_{\mathcal{V}^*} \leq C\,(1 + \|\vartheta\|_{\mathcal{V}})\,, \quad \forall \vartheta \in \mathcal{V}. \tag{3.44}$$

Hence, all conditions of Theorem 16 in [37] are satisfied, and problem (3.43) admits at least one solution $\vartheta_\eta \in L^1(I; \mathcal{V})$. Consequently, ϑ_η provides a solution to the auxiliary problem $\mathcal{PV}^2_{2\eta}$, and from this result, we deduce that $u_\eta \in W^{1,2}(I; \mathcal{V})$ constitutes a solution to Problem $\mathcal{PV}^2_{1\eta}$.

On the other hand, suppose that $\vartheta_{\eta_1}, \vartheta_{\eta_2} \in \mathcal{V}$ are two solutions to Problem $\mathcal{PV}^2_{2\eta}$.

Then, for all $w \in \mathcal{V}$, it follows that

$$\begin{aligned}&\Big(\mathcal{L}\varepsilon(\vartheta_{\eta_1}(t)) + \mathcal{M}\Big(\boldsymbol{\varepsilon}\big({}_0I_t^{\alpha}\vartheta_{\eta_1}(t) + u_0\big)\Big), \boldsymbol{\varepsilon}(w)\Big)_{\mathcal{H}} \\ &+ \int_{\Gamma_3} j_\nu^0\left((\vartheta_{\eta_1})_\nu(t); w_\nu\right) + j_\tau^0\left((\vartheta_{\eta_1})_\tau(t); w_\tau\right)\, da \geq (f_{\eta_1}(t), w)_{\mathcal{V}}\,, \end{aligned} \tag{3.45}$$

$$\begin{aligned}&\Big(\mathcal{L}\varepsilon(\vartheta_{\eta_2}(t)) + \mathcal{M}\Big(\boldsymbol{\varepsilon}\big({}_0I_t^{\alpha}\vartheta_{\eta_2}(t) + u_0\big)\Big), \boldsymbol{\varepsilon}(w)\Big)_{\mathcal{H}} \\ &+ \int_{\Gamma_3} j_\nu^0\left((\vartheta_{\eta_2})_\nu(t); w_\nu\right) + j_\tau^0\left((\vartheta_{\eta_2})_\tau(t); w_\tau\right)\, da \geq (f_{\eta_2}(t), w)_{\mathcal{V}}\,, \end{aligned} \tag{3.46}$$

Take $w = \vartheta_{\eta_2} - \vartheta_{\eta_1}$ in (3.45) and $w = \vartheta_{\eta_1} - \vartheta_{\eta_2}$ in (3.46) to obtain

$$\begin{aligned}&(\mathcal{L}\varepsilon(\vartheta_{\eta_1}(t)), \boldsymbol{\varepsilon}(\vartheta_{\eta_2} - \vartheta_{\eta_1}))_{\mathcal{H}} + \Big(\mathcal{M}\Big(\boldsymbol{\varepsilon}\big({}_0I_t^{\alpha}\vartheta_{\eta_1}(t) + u_0\big)\Big), \boldsymbol{\varepsilon}(\vartheta_{\eta_2} - \vartheta_{\eta_1})\Big)_{\mathcal{H}} \\ &+ \int_{\Gamma_3} j_\nu^0\left(\vartheta_{\nu\eta_1}(t); (\vartheta_{\eta_2} - \vartheta_{\eta_1})_\nu\right)\, da + \int_{\Gamma_3} j_\tau^0\left(\vartheta_{\tau\eta_1}(t); (\vartheta_{\eta_2} - \vartheta_{\eta_1})_\tau\right)\, da \\ &\qquad \geq (f_{\eta_1}(t), \vartheta_{\eta_2} - \vartheta_{\eta_1})_{\mathcal{V}}\,. \end{aligned} \tag{3.47}$$

$$\begin{aligned}&(\mathcal{L}\varepsilon(\vartheta_{\eta_2}(t)), \boldsymbol{\varepsilon}(\vartheta_{\eta_1} - \vartheta_{\eta_2}))_{\mathcal{H}} + \Big(\mathcal{M}\Big(\boldsymbol{\varepsilon}\big({}_0I_t^{\alpha}\vartheta_{\eta_2}(t) + u_0\big)\Big), \boldsymbol{\varepsilon}(\vartheta_{\eta_1} - \vartheta_{\eta_2})\Big)_{\mathcal{H}} \\ &+ \int_{\Gamma_3} j_\nu^0\left(\vartheta_{\nu\eta_2}(t); (\vartheta_{\eta_1} - \vartheta_{\eta_2})_\nu\right)\, da + \int_{\Gamma_3} j_\tau^0\left(\vartheta_{\tau\eta_2}(t); (\vartheta_{\eta_1} - \vartheta_{\eta_2})_\tau\right)\, da \\ &\qquad \geq (f_{\eta_2}(t), \vartheta_{\eta_1} - \vartheta_{\eta_2})_{\mathcal{V}}\,. \end{aligned} \tag{3.48}$$

By combining the two inequalities mentioned above, we get

$$\begin{aligned}&(\mathcal{L}\varepsilon(\vartheta_{\eta_1}(t)) - \mathcal{L}\varepsilon(\vartheta_{\eta_2}(t)), \boldsymbol{\varepsilon}(\vartheta_{\eta_2}(t) - \vartheta_{\eta_1}(t)))_{\mathcal{H}} \\ &\leq \Big(\mathcal{M}\Big(\boldsymbol{\varepsilon}\big({}_0I_t^{\alpha}\vartheta_{\eta_1}(t) + u_0\big)\Big) - \mathcal{M}\Big(\boldsymbol{\varepsilon}\big({}_0I_t^{\alpha}\vartheta_{\eta_2}(t) + u_0\big)\Big), \boldsymbol{\varepsilon}(\vartheta_{\eta_2}(t) - \vartheta_{\eta_1}(t))\Big)_{\mathcal{H}} \\ &\qquad + (\eta_1(t) - \eta_2(t), \vartheta_{\eta_2}(t) - \vartheta_{\eta_1}(t)) + R(\vartheta_{\eta_1}, \vartheta_{\eta_2}; \vartheta_{\eta_1}, \vartheta_{\eta_2}). \end{aligned} \tag{3.49}$$

where

$$R(\vartheta_{\eta_1},\vartheta_{\eta_2};\vartheta_{\eta_1},\vartheta_{\eta_2}) = \int_{\Gamma_3} j_\tau^0\left(\vartheta_{\tau\eta_2}(t);(\vartheta_{\eta_1}-\vartheta_{\eta_2})_\tau\right)\,da + \int_{\Gamma_3} j_\tau^0\left(\vartheta_{\tau\eta_1}(t);(\vartheta_{\eta_2}-\vartheta_{\eta_1})_\tau\right)\,da + \int_{\Gamma_3} j_\nu^0\left(\vartheta_{\nu\eta_2}(t);(\vartheta_{\eta_1}-\vartheta_{\eta_2})_\nu\right)\,da + \int_{\Gamma_3} j_\nu^0\left(\vartheta_{\nu\eta_1}(t);(\vartheta_{\eta_2}-\vartheta_{\eta_1})_\nu\right)\,da. \tag{3.50}$$

Using assumptions (3.16), (3.17) and (A.6), we find

$$|R(\vartheta_{\eta_1},\vartheta_{\eta_2};\vartheta_{\eta_1},\vartheta_{\eta_2})| \le c_s^2(M_\nu+M_\tau)\|\vartheta_{\eta_1}-\vartheta_{\eta_2}\|_{\mathcal{V}}^2. \tag{3.51}$$

From hypothesis (3.12), (3.13) and (3.51), we find

$$\begin{aligned} m_{\mathcal{L}}\|\vartheta_{\eta_1}(t)-\vartheta_{\eta_2}(t)\|_{\mathcal{V}}^2 &\le c_s^2(M_\nu+M_\tau)\|\vartheta_{\eta_1}(t)-\vartheta_{\eta_2}(t)\|_{\mathcal{V}}^2 \\ &+M_{\mathcal{M}}\|{}_0I_t^\alpha\vartheta_{\eta_1}(t)-{}_0I_t^\alpha\vartheta_{\eta_2}(t)\|_{\mathcal{V}}\|\vartheta_{\eta_1}(t)-\vartheta_{\eta_2}(t)\|_{\mathcal{V}} \\ &+\|\vartheta_{\eta_1}(t)-\vartheta_{\eta_2}(t)\|_{\mathcal{V}}\|\eta_1(t)-\eta_2(t)\|_{\mathcal{H}}. \end{aligned} \tag{3.52}$$

Therefore, by applying a generalized fractional Gronwall inequality under the assumption $m_{\mathcal{L}} > c_s^2(M_\nu+M_\tau)$, we obtain

$$\|\vartheta_{\eta_1}(t)-\vartheta_{\eta_2}(t)\|_{\mathcal{V}} \le C\,\|\eta_1(t)-\eta_2(t)\|_{\mathcal{H}}. \tag{3.53}$$

On the other hand, using the relation $u_\eta(t) = {}_0I_t^\alpha\vartheta_\eta(t)+u_0$, we obtain the following estimate

$$\begin{aligned} \|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathcal{V}} &= \|{}_0I_t^\alpha\vartheta_{\eta_1}(t)-{}_0I_t^\alpha\vartheta_{\eta_2}(t)\|_{\mathcal{V}} \\ &\le \frac{t^\alpha}{\alpha\Gamma(\alpha)}\|\vartheta_{\eta_1}(t)-\vartheta_{\eta_2}(t)\|_{\mathcal{V}}. \end{aligned} \tag{3.54}$$

Combining (3.53) and (3.54), we deduce (3.39), which completes the proof of Lemma 3.2.

□

Next, for a given $\mu \in C(\mathrm{I}; \mathcal{W})$, we formulate the following variational problem

Problem $\mathcal{PV}^2_\mu$. Find $\theta_\mu \in W^{1,p}(I; \mathcal{W})$ such that

$$\left({}^C_0 D^\alpha_t \theta_\mu(t) + K\theta_\mu(t), \zeta\right)_{\mathcal{W}'\times\mathcal{W}} = \left(\mu(t) + g(t), \zeta\right)_{\mathcal{W}'\times\mathcal{W}}, \quad \forall \zeta \in \mathcal{W}, \tag{3.55}$$

$$\theta_\mu(0) = \theta_0. \tag{3.56}$$

Lemma 3.3. *For every $\mu \in C(I; \mathcal{W})$, there exists a unique solution $\theta_\mu \in W^{1,2}(I; \mathcal{W})$ satisfying (3.55)–(3.56). Moreover, for any $\mu_1, \mu_2 \in C(I; \mathcal{W})$, the corresponding solutions θ_{μ_1} and θ_{μ_2} of $\mathcal{PV}^2_\mu$ satisfy the estimate*

$$\|\theta_{\mu_1}(t) - \theta_{\mu_2}(t)\|_{\mathcal{W}} \leq C \|\mu_1(t) - \mu_2(t)\|_{\mathcal{W}}, \quad \forall t \in I, \tag{3.57}$$

where $C > 0$ is a positive constant independent of t.

Proof. The proof proceeds by applying the Galerkin method. Let $\{\lambda_j\}_{j=1}^\infty$ be an orthogonal basis of $\mathcal{W}$ that also forms an orthonormal basis of $L^2(\Omega)$ (see [14]). For a fixed positive integer n, we define the Galerkin approximation

$$\theta_{\mu_n}(t) = \sum_{i=1}^{n} z_n^i(t)\, \lambda_i, \quad t \in I, \tag{3.58}$$

where $z_n^i(t)$ are scalar functions to be determined. Let $W_n = \mathrm{span}\{\lambda_1, \ldots, \lambda_n\}$ denote the corresponding finite-dimensional subspace of $\mathcal{W}$. Clearly, $\theta_{\mu_n} \in W_n$, and as $n \to \infty$, $\theta_{\mu_n} \to \theta_\mu$ in $\mathcal{W}$. For each $n \in \mathbb{N}$, we then consider the following finite-dimensional approximate problem

Find $\theta_{\mu_n} : I \longrightarrow W_n$ such that

$$\left({}^C_0 D^\alpha_t \theta_{\mu_n}(t) + K\theta_{\mu_n}(t), \lambda_j\right) = \left(\mu(t) + g(t), \lambda_j\right), \quad \forall \lambda_j \in W_n, \tag{3.59}$$

$$\theta_{\mu_n}(0) = \theta_0. \tag{3.60}$$

From (3.58), we have

$$\begin{aligned}\left({}_0^C D_t^\alpha \theta_{\mu_n}(t), \lambda_j\right) &= {}_0^C D_t^\alpha z_n^i(t),\\ (K\theta_{\mu_n}, \lambda_j) &= K z_n^i(t),\\ (\mu(t) + g(t), \lambda_j) &= \mu_i(t) + g_i(t).\end{aligned}$$

Problems (3.59) and (3.60) can be written as follows

$${}_0^C D_t^\alpha z_n^i(t) = Q(t, z_n^i(t)), \tag{3.61}$$

$$z_n^i(0) = (\theta_0, \lambda_j), \tag{3.62}$$

where $Q(t, z_n^i(t)) = \mu_i(t) + g_i(t) - K z_n^i(t)$.

It follows from assumption (3.24) that there exists a positive constant $C > 0$ such that

$$|Q(t, z_{i_1}(t)) - Q(t, z_{i_2}(t))| \leq C\,|z_{i_1}(t) - z_{i_2}(t)|, \quad \forall t \in I. \tag{3.63}$$

According to Theorem (A.7), one can assert that there exists a unique absolutely continuous function $z_n(t) = \left(z_n^1(t), \ldots, z_n^n(t)\right)$, which satisfies equations (3.61) and (3.62) for almost every $t \in I$.

Estimates: Multiply (3.59) by $z_n^i(t)$, for $i = 1, ..., n$. We find

$$\left({}_0^C D_t^\alpha \theta_{\mu_n}(t) + K\theta_{\mu_n}(t), \theta_{\mu_n}(t)\right) = (\mu(t) + g(t), \theta_{\mu_n}(t)). \tag{3.64}$$

Since $\frac{1}{2}\|\theta\|^2$ is convex, Proposition (A.9) implies that

$${}_0^C D_t^\alpha \frac{1}{2}\|\theta_{\mu_n}(t)\|_{\mathcal{W}}^2 \leq \left({}_0^C D_t^\alpha \theta_{\mu_n}(t), \theta_{\mu_n}(t)\right). \tag{3.65}$$

Applying the inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \qquad \text{for } \epsilon > 0, \tag{3.66}$$

we obtain

$$(\mu(t)+g(t),\theta_{\mu_n}(t)) \leq \frac{1}{4\epsilon}\|\mu(t)+g(t)\|^2_{\mathcal{W}} + \epsilon\|\theta_{\mu_n}(t)\|^2_{\mathcal{W}}. \tag{3.67}$$

From (3.65), (3.67) and using assumption (3.18), we conclude that

$$ {}^{C}_{0}D^{\alpha}_{t}\frac{1}{2}\|\theta_{\mu_n}(t)\|^2_{\mathcal{W}} + C_1\|\theta_{\mu_n}(t)\|^2_{\mathcal{W}} \leq C_2\|\mu(t)+g(t)\|^2_{\mathcal{W}}. \tag{3.68}$$

As well, according to proposition (A.11), we obtain

$$\|\theta_{\mu_n}(t)\|^2_{\mathcal{W}} + \frac{2C_1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}\|\theta_{\mu_n}(s)\|^2_{\mathcal{W}}dr \leq C_3\left(\|\mu(t)+g(t)\|^2_{\mathcal{W}} + \|\theta_0\|^2_{\mathcal{W}}\right). \tag{3.69}$$

Let $\zeta \in \mathcal{W}$, with $\|\zeta\|_{\mathcal{W}} \leq 1$, and write $\zeta = \zeta_1 + \zeta_2$ where $\zeta_1 \in \text{span}\,\{\lambda_i\}_{i=1}^n$ and $(\zeta_2, \lambda_i) = 0$ for $i = 1, ..., n$. Using (3.59), we find

$$\left({}^{C}_{0}D^{\alpha}_{t}\theta_{\mu_n}(t) + K\theta_{\mu_n}(t), \zeta_1\right) = (\mu(t)+g(t), \zeta_1). \tag{3.70}$$

Similarly to (3.68), we conclude that

$$\|{}^{C}_{0}D^{\alpha}_{t}\theta_{\mu_n}(t)\|_{\mathcal{W}^*} \leq C\left(\|\theta_{\mu_n}(t)\|_{\mathcal{W}} + \|\mu(t)+g(t)\|_{\mathcal{W}}\right). \tag{3.71}$$

It follows from (3.69) that one can find a constant $C > 0$ such that

$$\|{}^{C}_{0}D^{\alpha}_{t}\theta_{\mu_n}(t)\|_{L^2(I;\mathcal{W}^*)} \leq C. \tag{3.72}$$

Passage to the limit: From the uniform bounds provided by (3.69) and (3.72), we infer that the sequence $\{\theta_{\mu_n}\}_{n=1}^{\infty}$ is bounded in $L^2(I;\mathcal{W})$, while the sequence of Caputo derivatives $\left\{{}^{C}_{0}D^{\alpha}_{t}\theta_{\mu_n}\right\}_{n=1}^{\infty}$ is bounded in $L^2(I;\mathcal{W}^*)$. Hence, there exists a subsequence, still denoted by $\{\theta_{\mu_n}\}$, and a function $\theta_\mu \in L^2(I;\mathcal{W})$ with ${}^{C}_{0}D^{\alpha}_{t}\theta_\mu \in L^2(I;\mathcal{W}^*)$, such that

$$\theta_{\mu_n} \longrightarrow \theta_\mu \quad \text{strongly in } L^2(I;L^2(\Omega)) \text{ as } n \to \infty. \tag{3.73}$$

Now, let $\zeta \in C_c^{\infty}([0,T)\times\Omega;\mathcal{W})$ be any test function. We can expand ζ as a

series in terms of the basis $\{\lambda_i\}_{i\geq 1}$

$$\zeta = \sum_{i=1}^{\infty} \beta_i \lambda_i,$$

where β_i are the corresponding coefficients.

For a fixed integer $n \geq 1$, we define the truncated approximation

$$\zeta_n = \sum_{i=1}^{n} \beta_i \lambda_i.$$

Since ζ is a smooth function in t that vanishes at T, the same property holds for ζ_n. Moreover, we have the convergence

$$ {}_0^C\tilde{D}_t^{\alpha}\zeta_n \longrightarrow {}_0^C\tilde{D}_t^{\alpha}\zeta \quad \text{in } L^2\big(I;\mathcal{W}\big).$$

We first fix $n_0 \geq 1$, and for any $n_j \geq n_0$, we obtain

$$\big({}_0^C D_t^{\alpha}\theta_{\mu_{nj}}(t), \zeta_{n_0}\big) = \Big(\theta_{\mu_{nj}} - \theta_0, {}_0^C\tilde{D}_t^{\alpha}\zeta_{n_0}\Big) = -(K\theta_{\mu_{nj}}(t), \zeta_{n_0}) + (\mu(t) + g(t), \zeta_{n_0}).$$

Here, the first equality follows from the fractional integration by parts formula, which allows the fractional derivative to be transferred from $\theta_{\mu_{nj}}$ to the test function ζ_{n_0}. The second equality holds because the function ${}_0^C\tilde{D}_t^{\alpha}\zeta_{n_0}$ belongs to the finite-dimensional space $\text{span}\{\lambda_1, \ldots, \lambda_{n_j}\}$.

By Convergence (3.73), taking $j \to \infty$, we have

$$\Big(\theta_{\mu} - \theta_0, {}_0^C\tilde{D}_t^{\alpha}\zeta_{n_0}\Big) = -(K\theta_{\mu}(t), \zeta_{n_0}) + (\mu(t) + g(t), \zeta_{n_0}).$$

Then, by taking $n_0 \to \infty$, and since $\zeta_{n_0} \to \zeta$ strongly in $L^2\big(I;\mathcal{W}\big)$, we conclude that the first part of the Lemma is proved.

For the second part, let $\theta_{\mu_1}, \theta_{\mu_2} \in \mathcal{W}$ be two solutions of the problem $\mathcal{PV}^2_\mu$. From (3.55), we can write

$$\begin{aligned}\left({}^C_0D^\alpha_t\theta_{\mu_1}(t) - {}^C_0D^\alpha_t\theta_{\mu_2}(t), \zeta\right)_{\mathcal{W}^*\times\mathcal{W}} + \left(K\theta_{\mu_1}(t) - K\theta_{\mu_2}(t), \zeta\right)_{\mathcal{W}^*\times\mathcal{W}} \\ = \left(\mu_1(t) - \mu_2(t), \zeta\right)_{\mathcal{W}^*\times\mathcal{W}}.\end{aligned} \tag{3.74}$$

Let ${}^C_0D^\alpha_t\theta_\mu = \varsigma_\mu$. Then, for a.e. $t \in I$, we have

$$\theta_\mu(t) = {}_0I^\alpha_t\varsigma_\mu(t) + \theta_0.$$

By choosing $\zeta = \varsigma_{\mu_1} - \varsigma_{\mu_2}$ as a test function, we deduce that

$$\begin{aligned}\left(\varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t), \varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\right)_{\mathcal{W}^*\times\mathcal{W}} = \left(\mu_1(t) - \mu_2(t), \varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\right)_{\mathcal{W}^*\times\mathcal{W}} \\ - \left(K\left({}_0I^\alpha_t\varsigma_{\mu_1}(t) + \theta_0\right) - K\left({}_0I^\alpha_t\varsigma_{\mu_2}(t) + \theta_0\right), \varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\right)_{\mathcal{W}^*\times\mathcal{W}}.\end{aligned} \tag{3.75}$$

Using the hypothesis (3.24), we find

$$\|\varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\|_{\mathcal{W}} \leq M_k\|{}_0I^\alpha_t\varsigma_{\mu_1}(t) - {}_0I^\alpha_t\varsigma_{\mu_2}(t)\|_{\mathcal{W}} + \|\mu_1(t) - \mu_2(t)\|_{\mathcal{W}}. \tag{3.76}$$

According to Lemma (A.12), we derive the following estimate

$$\|\varsigma_{\mu_1}(t) - \varsigma_{\mu_2}(t)\|_{\mathcal{W}} \leq C\,\|\mu_1(t) - \mu_2(t)\|_{\mathcal{W}}. \tag{3.77}$$

By employing reasoning analogous to that in (3.54), we obtain the estimate (3.57).

□

We now utilize u_η from Lemma 3.2 and θ_μ from Lemma 3.3 to set up the following Cauchy problem for the stress field.

Problem $\mathcal{P}_{\eta\mu}$. Find a stress field $\boldsymbol{\sigma}_{\eta\mu} : I \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta\mu}(t) = \mathcal{M}\varepsilon(u_\eta(t)) + \int_0^t \mathcal{S}\Big(\boldsymbol{\sigma}_{\eta\mu}(r), \varepsilon(u_\eta(r)), \theta_\mu(r)\Big)\,dr, \quad \text{a.e. } t \in I. \tag{3.78}$$

Lemma 3.4. *There exists a unique solution $\boldsymbol{\sigma}_{\eta\mu} \in \mathcal{C}(0,T;\mathcal{H})$ to the problem $\mathcal{P}_{\eta\mu}$. Furthermore, let u_{η_i}, θ_{μ_i}, and $\boldsymbol{\sigma}_{\eta_i\mu_i}$ denote the solutions of the problems $\mathcal{PV}_{1\eta_i}$, $\mathcal{PV}_{\mu_i}$, and $\mathcal{P}_{\eta_i\mu_i}$, respectively, corresponding to $(\eta_i,\mu_i) \in \mathcal{C}(I;\mathcal{H}\times\mathcal{W})$ for $i=1,2$. Then, there exists a positive constant $C>0$ such that*

$$\|\boldsymbol{\sigma}_{\eta_1\mu_1}(t)-\boldsymbol{\sigma}_{\eta_2\mu_2}(t)\|_{\mathcal{H}}^2 \le C\Big(\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathcal{V}}^2+\int_0^t \|u_{\eta_1}(r)-u_{\eta_2}(r)\|_{\mathcal{V}}^2\,dr + \int_0^t \|\theta_{\mu_1}(r)-\theta_{\mu_2}(r)\|_{L^2(\Omega)}^2\,dr\Big). \tag{3.79}$$

Proof. For all $\boldsymbol{\sigma} \in \mathcal{C}(I;\mathcal{H})$, let $\mathcal{T}_{\eta\mu} : \mathcal{C}(I;\mathcal{H}) \to \mathcal{C}(I;\mathcal{H})$ be the mapping defined by

$$\mathcal{T}_{\eta\mu}\boldsymbol{\sigma}(t) = \mathcal{M}\varepsilon(u_\eta(t)) + \int_0^t \mathcal{S}\big(\boldsymbol{\sigma}(r),\boldsymbol{\varepsilon}(u_\eta(r)),\theta_\mu(r)\big)\,dr. \tag{3.80}$$

For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in \mathcal{C}(0,T;\mathcal{H})$, we use (3.14) and (3.80) to obtain

$$\|\mathcal{T}_{\eta\mu}\boldsymbol{\sigma}_1(t)-\mathcal{T}_{\eta\mu}\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le M_{\mathcal{S}}\int_0^t \|\boldsymbol{\sigma}_1(r)-\boldsymbol{\sigma}_2(r)\|_{\mathcal{H}}\,dr. \tag{3.81}$$

Hence, for sufficiently large n, the operator $\mathcal{T}_{\eta\mu}^n$ is a contraction. Therefore, by the Banach fixed-point theorem, there exists a unique $\boldsymbol{\sigma} \in \mathcal{C}(0,T;\mathcal{H})$ such that

$$\mathcal{T}_{\eta\mu}\boldsymbol{\sigma} = \boldsymbol{\sigma},$$

which establishes that $\boldsymbol{\sigma}$ is the unique solution of the problem $\mathcal{P}_{\eta\mu}$.

Consider now (η_1,μ_1), $(\eta_2,\mu_2) \in \mathcal{C}(0,T;\mathcal{H}\times\mathcal{W})$. Using the properties (3.13) and (3.14) of $\mathcal{M}$ and $\mathcal{S}$, we find

$$\|\boldsymbol{\sigma}_{\eta_1\mu_1}(t)-\boldsymbol{\sigma}_{\eta_2\mu_2}(t)\|_{\mathcal{H}}^2 \le C\Big(\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_{\mathcal{V}}^2+\int_0^t \|\boldsymbol{\sigma}_{\eta_1\mu_1}(r)-\boldsymbol{\sigma}_{\eta_2\mu_2}(r)\|_{\mathcal{H}}^2\,dr + \int_0^t \|u_{\eta_1}(r)-u_{\eta_2}(r)\|_{\mathcal{V}}^2\,dr+\int_0^t \|\theta_{\mu_1}(r)-\theta_{\mu_2}(r)\|_{\mathcal{W}}^2\,dr\Big). \tag{3.82}$$

An application of Gronwall's inequality to the above estimate yields (3.79), thus completing the proof of Lemma 3.4. □

In the final step, we define the operator

$$\Lambda : \mathcal{C}(I; \mathcal{H} \times \mathcal{W}) \longrightarrow \mathcal{C}(I; \mathcal{H} \times \mathcal{W})$$

by

$$\Lambda(\eta, \mu)(t) = \big(\Lambda^1(\eta, \mu)(t), \Lambda^2(\eta, \mu)(t)\big), \tag{3.83}$$

with

$$(\Lambda^1(\eta, \mu)(t), \vartheta)_{\mathcal{V}} = \Big(\int_0^t \mathcal{S}\Big(\boldsymbol{\sigma}_{\eta\mu}(r), \boldsymbol{\varepsilon}(u_\eta(r)), \theta_\mu(r)\Big)\, dr, \boldsymbol{\varepsilon}(\vartheta)\Big)_{\mathcal{H}}, \quad \forall\, \vartheta \in \mathcal{V}, \tag{3.84}$$

$$\Lambda^2(\eta, \mu)(t) = \psi\big(\boldsymbol{\sigma}_{\eta\mu}(t), \boldsymbol{\varepsilon}(u_\eta(t)), \theta_\mu(t)\big). \tag{3.85}$$

In this context, u_η, θ_μ, and $\boldsymbol{\sigma}_{\eta\mu}$ denote the displacement, temperature, and stress fields, respectively, as established in Lemmas 3.2, 3.3, and 3.4.

Lemma 3.5. *The operator Λ admits a unique fixed point $(\eta^*, \mu^*) \in \mathcal{C}(I; \mathcal{H} \times \mathcal{W})$ such that*

$$\Lambda(\eta^*, \mu^*) = (\eta^*, \mu^*).$$

Proof. Let (η_1, μ_1), $(\eta_2, \mu_2) \in \mathcal{C}(0, T; \mathcal{H} \times \mathcal{W})$. For convenience of notation, we introduce $u_i := u_{\eta_i}$, $\boldsymbol{\sigma}_i := \boldsymbol{\sigma}_{\eta_i\mu_i}$, and $\theta_i := \theta_{\mu_i}$, for $i = 1, 2$. Using assumption (3.14), we deduce that

$$\begin{aligned} \|\Lambda^1(\eta_1, \mu_1)(t) - \Lambda^1(\eta_2, \mu_2)(t)\|_{\mathcal{H}}^2 \leq C\Big(& \int_0^t \|\boldsymbol{\sigma}_1(r) - \boldsymbol{\sigma}_2(r)\|_{\mathcal{H}}^2\, dr \\ & + \int_0^t \|u_1(r) - u_2(r)\|_{\mathcal{V}}^2\, dr + \int_0^t \|\theta_1(r) - \theta_2(r)\|_{\mathcal{W}}^2\, dr\Big). \end{aligned} \tag{3.86}$$

From estimates (3.79), we find

$$\begin{aligned} \|\Lambda^1(\eta_1, \mu_1)(t) - \Lambda^1(\eta_2, \mu_2)(t)\|_{\mathcal{H}}^2 \leq C\Big(& \|u_1(t) - u_2(t)\|_{\mathcal{V}}^2 \\ & + \int_0^t \|u_1(r) - u_2(r)\|_{\mathcal{V}}^2\, dr + \int_0^t \|\theta_1(r) - \theta_2(r)\|_{\mathcal{W}}^2\, dr\Big). \end{aligned} \tag{3.87}$$

Furthermore, by applying arguments analogous to those used in (3.87), we conclude

that

$$\|\Lambda^2(\eta_1,\mu_1)(t)-\Lambda^2(\eta_2,\mu_2)(t)\|_{\mathcal{H}}^2 \leq C\Big(\|u_1(t)-u_2(t)\|_{\mathcal{V}}^2 + \int_0^t \|u_1(r)-u_2(r)\|_{\mathcal{V}}^2\,dr + \|\theta_1(t)-\theta_2(t)\|_{\mathcal{W}}^2 + \int_0^t \|\theta_1(r)-\theta_2(r)\|_{\mathcal{W}}^2\,dr\Big). \tag{3.88}$$

Consequently,

$$\|\Lambda(\eta_1,\mu_1)(t)-\Lambda(\eta_2,\mu_2)(t)\|_{\mathcal{H}\times\mathcal{W}}^2 \leq C\Big(\|u_1(t)-u_2(t)\|_{\mathcal{V}}^2 + \int_0^t \|u_1(r)-u_2(r)\|_{\mathcal{V}}^2\,dr + \|\theta_1(t)-\theta_2(t)\|_{\mathcal{W}}^2 + \int_0^t \|\theta_1(r)-\theta_2(r)\|_{\mathcal{W}}^2\,dr\Big). \tag{3.89}$$

By substituting (3.39) and (3.57) into (3.89), we obtain

$$\|\Lambda(\eta_1,\mu_1)(t)-\Lambda(\eta_2,\mu_2)(t)\|_{\mathcal{H}\times\mathcal{W}}^2 \leq C\int_0^t \|(\eta_1,\mu_1)(r)-(\eta_2,\mu_2)(r)\|_{\mathcal{H}\times\mathcal{W}}^2\,dr.$$

An n-fold iteration of this inequality yields

$$\|\Lambda^n(\eta_1,\mu_1)-\Lambda^n(\eta_2,\mu_2)\|_{\mathcal{C}(I;\mathcal{H}\times\mathcal{W})}^2 \leq \frac{C^nT^n}{n!}\|(\eta_1,\mu_1)-(\eta_2,\mu_2)\|_{\mathcal{C}(I;\mathcal{H}\times\mathcal{W})}^2.$$

For sufficiently large n, one can verify that the iterate Λ^n is a contraction mapping on the Banach space $\mathcal{C}(I;\mathcal{H}\times\mathcal{W})$. Therefore, an application of the Banach fixed-point theorem ensures the existence and uniqueness of a fixed point of Λ in this space. □

We are now in a position to prove Theorem 3.1, having established all the necessary preliminary results.

Let $(\eta^*,\mu^*) \in \mathcal{C}(I;\mathcal{H}\times\mathcal{W})$ be the fixed point of Λ defined by (3.84), (3.85) and denote:

$$u = u_{\eta^*}, \quad \theta = \theta_{\mu^*}. \tag{3.90}$$

$$\boldsymbol{\sigma} = \mathcal{L}\varepsilon\left({}_0^C D_t^\alpha u(t)\right) + \boldsymbol{\sigma}_{\eta^*\mu^*}. \tag{3.91}$$

We begin by rewriting (3.78) with $\eta=\eta^*$ and $\mu=\mu^*$, and, using (3.90) and (3.91),

we deduce (3.31).

Next, we consider (3.37) and (3.64) with $\eta = \eta^*$ and $\mu = \mu^*$. Employing the identities

$$\Lambda^1(\eta, \mu)(t) = \eta \quad \text{and} \quad \Lambda^2(\eta, \mu)(t) = \mu,$$

together with (3.84), (3.85), and (3.90), we conclude that (3.32) and (3.33) hold. Moreover, (3.34) and the regularity properties stated in (3.35) follow from Lemmas 3.2, 3.3, and 3.4, together with the relations (3.90) and (3.91), and the assumptions on $\mathcal{L}$.

Finally, the uniqueness of the solution is a direct consequence of the uniqueness of the fixed point of the operator Λ, combined with the unique solvability of the auxiliary problems $\mathcal{PV}^2_{1\eta}$, $\mathcal{PV}^2_{\mu}$, and $\mathcal{P}_{\eta\mu}$, which completes the proof of Theorem 3.1.

Conclusion and Future Perspectives

This thesis studied contact problems for an electro-thermal viscoplastic body, including mechanical effects, friction, and wear, within a rigorous mathematical framework that ensures the existence of weak solutions.

Chapter 2 focused on the contact problem for an elastic viscoplastic electrical body with friction, wear, and internal state variables, and established the existence of a weak solution.

Chapter 3 addressed the thermal contact problem using time-fractional derivatives and the Clarke subdifferential, allowing the treatment of energy functionals J that are not necessarily convex or continuous, while establishing the existence and uniqueness of weak solutions. The results show that quasi-variational equations combined with time-fractional derivatives provide a robust framework to model viscoplastic materials under mechanical, electrical, and thermal effects, while guaranteeing stability and numerical feasibility.

Future Perspectives

- Apply this framework to more complex contact problems or bodies with intricate geometries.
- Investigate additional physical laws affecting material behavior, such as friction, wear, and electrical or thermal conduction.
- Develop advanced numerical methods to approximate weak solutions for accurate simulations in engineering applications.

- Study the influence of material properties varying in time or space and compare results with experimental data to improve accuracy.
- Explore alternative mathematical formulations or nonstandard derivatives, including Clarke's derivative or additional fractional derivatives, to extend model flexibility and reduce restrictions on energy functionals.

Appendix A

Functional Setting and Theoretical Preliminaries

This annex presents fundamental definitions and properties of the functional spaces used in the study of mechanical problems. It also introduces some classical existence and uniqueness theorems that are essential for the mathematical framework of the thesis. The considered domain Ω is assumed to be a bounded Lipschitz domain in $\mathbb{R}^m$ (with $m \in \{2, 3\}$), whose boundary is divided into measurable and disjoint parts corresponding to different boundary conditions. The theoretical framework presented here is mainly based on references [8, 22, 27, 34].

A.1 Functional spaces for vector fields

The contact problems investigated in this work require analyzing their behavior and evolution over a prescribed time interval $I \equiv [0, T]$, where $T > 0$ denotes the final time under consideration. This analysis is carried out by employing vector-valued function spaces constructed over the real Banach space $(E, \|\cdot\|_E)$, providing a suitable functional framework for accurately describing and studying the mechanical and mathematical properties of the solutions.

We define $C(I; E)$ to be the space of functions $z : I \to E$ which are continuous

on the closed interval I. This is a Banach space equipped with the norm

$$\|z\|_{C(I;E)} := \max_{s\in I} \|z(s)\|_E.$$

Let $k \geq 0$ be an integer. We define the space

$$C^k(I;E) = \left\{ z \in C(I;E) : \vartheta^{(j)} \in C(I;E),\ j = 1, \ldots, k \right\}.$$

This is a Banach space equipped with the norm

$$\|z\|_{C^k(I;E)} = \sum_{j=0}^{k} \max_{s\in I} \|\vartheta^{(j)}(s)\|_E.$$

Let $[a, b]$ be a bounded interval, and denote by $AC([a, b])$ the class of absolutely continuous functions on $[a, b]$. A classical characterization states that a function h belongs to $AC([a, b])$ precisely when it admits a representation of the form

$$h(s) = c + \int_a^s \phi(r)\, dr, \quad \phi \in L^1(a, b), \tag{1.1.5}$$

for some constant c.

In this case, the integrand ϕ coincides almost everywhere with the derivative of h, that is,

$$h'(s) = \phi(s) \quad \text{a.e. on } [a, b], \qquad c = h(a).$$

A.2 Sobolev Spaces in Contact Modeling

In the context of physical quantities with a tensorial character, we denote by $\mathbb{S}^m$ the set of all symmetric second-order tensors defined on $\mathbb{R}^m$. This notation is fundamental for describing stress and strain distributions in thermo-mechanical and coupled multi-physics problems. Let $\Omega \subset \mathbb{R}^m$ ($m \in \{2, 3\}$) be a bounded domain with a Lipschitz boundary Γ. The boundary is partitioned into three non-overlapping relatively open segments, denoted by Γ_1, Γ_2, and Γ_3. Furthermore, the union $\Gamma_1 \cup \Gamma_2$ is subdivided into two disjoint open subsets, Γ_a and Γ_b, such that $|\Gamma_1| > 0$ and

$|\Gamma_a| > 0$. Within this geometric framework, the basic functional spaces relevant to mathematical analysis are identified.

On $\mathbb{R}^m$ and $\mathbb{S}^m$, the standard inner products and their associated norms are defined by

$$\upsilon \cdot \vartheta = \upsilon_i \cdot \vartheta_i, \quad \|\upsilon\| = \sqrt{\vartheta \cdot \vartheta} \quad \forall \upsilon, \vartheta \in \mathbb{R}^m, \tag{A.1}$$

$$\sigma \cdot \varsigma = \sigma_{ij} \cdot \varsigma_{ij}, \quad \|\varsigma\| = \sqrt{\varsigma \cdot \varsigma} \quad \forall \sigma, \varsigma \in \mathbb{S}^m. \tag{A.2}$$

Based on these definitions, we proceed to introduce the functional spaces associated with vector and tensor fields, which form the foundation of the variational framework used in this study

$$\begin{gathered}
H = \left\{\vartheta = (\vartheta_i) : \vartheta_i \in L^2(\Omega)\right\} = L^2(\Omega)^m, \\
\mathcal{H} = \left\{\varsigma = (\varsigma_{ij}) : \varsigma_{ij} = \varsigma_{ji} \in L^2(\Omega)\right\}, \\
H^1(\Omega)^m = \{\vartheta = (\vartheta_i) \in H : \varepsilon(\vartheta) \in \mathcal{H}\}, \\
\mathcal{H}_1 = \{\varrho \in \mathcal{H} : \operatorname{Div} \varrho \in H\}.
\end{gathered}$$

Here, $\varepsilon : H_1 \to \mathcal{H}$ and $\operatorname{Div} : \mathcal{H}_1 \to H$ denote the linearized deformation and divergence operators, defined by

$$\varepsilon(\vartheta) = (\varepsilon_{ij}(\vartheta)), \quad \varepsilon_{ij}(\vartheta) = \frac{1}{2}(\vartheta_{i,j} + \vartheta_{j,i}), \quad \operatorname{Div} \vartheta = (\tau_{ij,j}).$$

These spaces are Hilbert spaces, endowed with the canonical inner products and the associated norms, given by

$$\begin{gathered}
(\upsilon, \vartheta)_H = \int_\Omega \upsilon_i \vartheta_i \, dx, \quad (\sigma, \varsigma)_{\mathcal{H}} = \int_\Omega \sigma_{ij} \varsigma_{ij} \, dx, \\
(\upsilon, \vartheta)_{H^1} = (\upsilon, \vartheta)_H + (\varepsilon(\upsilon), \varepsilon(\vartheta))_{\mathcal{H}}, \\
(\sigma, \varsigma)_{\mathcal{H}_1} = (\sigma, \varsigma)_{\mathcal{H}} + (\operatorname{Div} \sigma, \operatorname{Div} \varsigma)_H.
\end{gathered}$$

The associated norms corresponding to these inner products are, respectively, denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$, and $\|\cdot\|_{\mathcal{H}_1}$.

Based on the previous framework, we now introduce the functional spaces associated with the electric field, the displacement field, the temperature field, and the internal state variable, respectively

$$\mathcal{V} = \{\vartheta \in H_1 : \vartheta = 0 \text{ on } \Gamma_1\},$$
$$\mathcal{Q} = \left\{\phi \in H^1 : \phi = 0 \text{ on } \Gamma_a\right\},$$
$$\mathcal{Q}_1 = \{\phi \in H, \operatorname{Div}\phi \in L^2(\Omega)\},$$
$$\mathcal{W} = \left\{\theta \in H^1 : \theta = 0 \text{ on } \Gamma_1 \cup \Gamma_2\right\},$$
$$Y = \{\xi = (\xi_i) : \xi_i \in L^2(\Omega), 1 \leq i \leq n\}.$$

On the spaces $\mathcal{V}$, $\mathcal{Q}$ and $\mathcal{W}$, we define the following inner products

$$(\upsilon, \vartheta)_{\mathcal{V}} = (\varepsilon(\upsilon), \varepsilon(\vartheta))_{\mathcal{H}}, \ \forall\, \upsilon, \vartheta \in \mathcal{V}, \tag{A.3}$$
$$(\rho, \varrho)_{\mathcal{Q}} = (\nabla\rho, \nabla\varrho)_H, \quad \forall\, \rho, \varrho \in \mathcal{Q}. \tag{A.4}$$
$$(\theta, \zeta)_{\mathcal{W}} = (\nabla\theta, \nabla\zeta)_H, \quad \forall\, \theta, \zeta \in \mathcal{W}, \tag{A.5}$$

Since $|\Gamma_a| > 0$ and $|\Gamma_1| > 0$, the Korn's and Friedrichs–Poincaré inequalities are satisfied

$$\|\varepsilon(\upsilon)\|_{\mathcal{H}} \geq c_k \|\upsilon\|_{H^1(\Omega)^m}, \quad \forall\, \upsilon \in \mathcal{V},$$

$$\|\nabla\theta\|_H \geq c_F \|\theta\|_{\mathcal{W}}, \quad \forall\, \theta \in \mathcal{W}.$$

From relations (A.2) and (A.3), one deduces that the norms $\|\cdot\|_{H^1(\Omega)^m}$ and $\|\cdot\|_{\mathcal{V}}$ are equivalent on the space $\mathcal{V}$, which implies that they generate the same topology. Similarly, employing relations (A.2) and (A.5), the norms $\|\cdot\|_{H^1(\Omega)^m}$ and $\|\cdot\|_{\mathcal{W}}$ are demonstrated to be equivalent on $\mathcal{W}$.

Moreover, considering the Sobolev trace theorem and the equalities (A.3)–(A.5), there are positive constants c_s and c_d such that

$$\|\vartheta\|_{L^2(\Gamma_3)^m} \leq c_s \|\vartheta\|_{\mathcal{V}}, \quad \forall \vartheta \in \mathcal{V}, \tag{A.6}$$
$$\|\zeta\|_{L^2(\Gamma_3)} \leq c_d \|\zeta\|_{\mathcal{W}}, \quad \forall \zeta \in \mathcal{Q}. \tag{A.7}$$

For each element $\vartheta \in H_1$, we use the same notation ϑ to represent its trace $\gamma\vartheta$ on the boundary Γ. The normal and tangential components of ϑ on Γ are defined as

$$\vartheta_\nu = \vartheta \cdot \nu, \qquad \vartheta\tau = \vartheta - \vartheta\nu\nu. \tag{A.8}$$

Similarly, for a sufficiently smooth tensor field $\delta : \Omega \to \mathbb{S}^m$, we define its normal and tangential components on Γ by

$$\delta_\nu = (\delta\nu) \cdot \nu, \qquad \delta_\tau = \delta\nu - \delta_\nu\nu, \tag{A.9}$$

where ν denotes the unit outward normal vector to Γ.

The variational formulation of the contact problem is obtained by applying Green's theorem as follows

$$(\delta, \varepsilon(\vartheta))_{\mathcal{H}} + (\operatorname{Div}\delta, \vartheta)_H = \int_\Gamma \delta\nu \cdot \vartheta \, da \quad \forall \vartheta \in H_1. \tag{A.10}$$

A.3 Foundations of Nonlinear Analysis

This section introduces the fundamental concepts of nonsmooth analysis, the generalized differentiation of locally Lipschitz functions, and multivalued quasi-monotone operators. In addition, some basic properties of fractional calculus are presented. The main theoretical results required for this thesis are also provided, with particular emphasis on the essential properties of multivalued quasi-monotone operators, which play a crucial role in establishing the existence of stationary and evolutionary inclusions. Throughout this section, E denotes a reflexive Banach space equipped with the norm $\|\cdot\|_E$, unless otherwise specified in special cases.

Definition A.1. *A sequence $\{\vartheta_n\} \subset E$ is said to converge weakly to $\vartheta \in E$ if*

$$(\vartheta_n, z)_{E\times E^*} \to (\vartheta, z)_{E\times E^*} \quad \forall\, z \in E^*.$$

In this case, ϑ is called the weak limit of the sequence $\{\vartheta_n\}$. The notation $\vartheta_n \rightharpoonup \vartheta$ is used to denote weak convergence.

Definition A.2. *We say an operator $M : E \to E^*$ is monotone whenever*

$$(M\vartheta_1 - M\vartheta_2,\, \vartheta_1 - \vartheta_2)_{E^*\times E} \geq 0, \quad \forall\, \vartheta_1, \vartheta_2 \in E.$$

We say an operator $M : E \to E^*$ is pseudomonotone if, for every sequence $\{\vartheta_n\} \subset E$ such that $\vartheta_n \rightharpoonup \vartheta$ weakly in E and

$$\limsup_{n\to\infty} (M\vartheta_n, \vartheta_n - \vartheta) \leq 0,$$

it follows that

$$(M\vartheta, \vartheta - w) \leq \liminf_{n\to\infty} (M\vartheta_n, \vartheta_n - w), \quad \forall w \in E.$$

We say an operator $M : E \to E$ is Lipschitz continuous if there exists a constant L such that

$$\|M\vartheta_1 - M\vartheta_2\|_E \leq L\|\vartheta_1 - \vartheta_2\|_E, \quad \forall\, \vartheta_1, \vartheta_2 \in E.$$

Definition A.3. *Let a multivalued operator $M : E \to 2^{E^*}$ satisfy the following conditions*

(1) For every $\vartheta \in E$, the set $M\vartheta$ is nonempty, closed, and convex in E^*.

(2) The operator M is bounded.

(3) If $\vartheta_n \rightharpoonup \vartheta$ weakly in E and $\vartheta_n^* \rightharpoonup \vartheta^*$ weakly in E^* with $\vartheta_n^* \in M\vartheta_n$, and if

$$\limsup_{n\to\infty} (\vartheta_n^*, \vartheta_n - \vartheta) \leq 0,$$

then $v^* \in M\vartheta$ and $(\vartheta_n^*, \vartheta_n) \to (\vartheta^*, \vartheta)$.

Then the operator M is pseudomonotone.

Theorem A.4. *Let a multivalued operator $M : E \to 2^{E^*}$ be pseudomonotone and coercive. Then M is surjective; that is, for every $h \in E^*$, there exists $\vartheta \in E$ such that $h \in M\vartheta$.*

Definition A.5. *Let $I = [0, T]$ be a finite time interval. For a function $h \in L^1(I; E)$ and for any order $\gamma > 0$, the Riemann–Liouville fractional integral of order γ is defined by*

$$ {}_0I_t^\gamma h(s) = \frac{1}{\Gamma(\gamma)} \int_0^s (s-r)^{\gamma-1} f(r)\, dr, \quad \forall\, s \in I, $$

where $\Gamma(\cdot)$ denotes the Gamma function. For completeness, we set ${}_0I_t^0 = I$, where I denotes the identity operator, that is,

$$ {}_0I_s^0 h(s) = h(s), \quad \text{for a.e. } s \in I. $$

Definition A.6. *Let $I = [0, T]$ be a finite time interval and let $0 < \gamma \leq 1$. For a function $h \in AC(I; E)$, the* Caputo fractional derivative *of order γ is given by*

$$ {}_0^C D_s^\gamma h(s) = {}_0I_s^{1-\gamma}\, h'(s) = \frac{1}{\Gamma(1-\gamma)} \int_0^s (s-r)^{-\gamma}\, h'(r)\, dr, \qquad s \in I, $$

where $\Gamma(\cdot)$ denotes the Gamma function.

Observe that when $\gamma = 1$, the Caputo derivative coincides with the usual first-order derivative, namely,

$$ {}_0^C D_s^1 h(s) = I\, h'(s) = h'(s), \qquad \text{for a.e. } s \in I. $$

Theorem A.7. *Let $[a, b]$ be a bounded interval and consider a function $h : [0, \infty) \times [a, b] \to \mathbb{R}$ which is continuous and locally Lipschitz continuous with respect to its second variable.*

More precisely, for every $A > 0$ and every compact set $K \subset [a, b]$, there exists a constant $L_{A,K} > 0$ such that

$$ \sup_{0 \leq s \leq A} |h(s, \vartheta_1) - h(s, \vartheta_2)| \leq L_{A,K}\, |\vartheta_1 - \vartheta_2|, \qquad \forall\, \vartheta_1, \vartheta_2 \in K. $$

Let $0 < \gamma < 1$ and fix an initial value $\vartheta_0 \in [a, b]$. Then the fractional Cauchy problem

$$ {}^C D_s^\gamma \vartheta(s) = h(s, \vartheta(s)), \qquad \vartheta(0) = \vartheta_0, $$

admits a unique solution $\vartheta(\cdot)$ *on an interval* $[0, T_\beta)$*, where*

$$T_\beta = \sup\Big\{ l > 0 : \ \vartheta \in \mathcal{C}([0,l)),\ \vartheta(s) \in [a,b] \ \forall\ s \in [0,l) \Big\}$$

denotes the maximal existence time of the solution, with $T_\beta \in (0, \infty]$*.*

Furthermore, if the maximal time is finite, i.e. $T_\beta < \infty$*, then the solution approaches the boundary of the interval* $[a,b]$ *in the sense that*

$$\liminf_{s \to T_\beta^-} \vartheta(s) = a \qquad or \qquad \limsup_{s \to T_\beta^-} \vartheta(s) = b.$$

Definition A.8. *Consider the fractional differential equation*

$${}^C D_s^\gamma \vartheta(s) = h(s, \vartheta(s)), \quad s \in I,$$

with the initial condition

$$\vartheta(0) =_0,$$

where $0 < \gamma < 1$ *and* ${}^C D_s^\gamma$ *denotes the Caputo fractional derivative.*

A function $\vartheta \in L^1(I; E)$ *is said to be a weak solution if, for every test function* $\varphi \in C_c^\infty(I)$*, the following relation holds*

$$\int_0^T \vartheta(s)\, ({}^C D_s^\gamma \varphi(s))\, ds = \int_0^T h(s, \vartheta(s))\, \varphi(s)\, ds + \vartheta_0\, ({}_0 I_s^{1-\gamma} \varphi)(0),$$

where ${}_0 I_s^{1-\gamma}$ *denotes the Riemann–Liouville fractional integral of order* $1 - \gamma$*.*

Remark. *An equivalent formulation of the weak solution is given by the integral equation*

$$\vartheta(s) = \vartheta_0 + \frac{1}{\Gamma(\gamma)} \int_0^s (s-r)^{\gamma-1} h(r, \vartheta(r))\, dr, \quad s \in I.$$

This integral form is often used as a practical definition of the weak solution in the framework of fractional calculus.

Proposition A.9. *Let* $\gamma \in (0,1)$*. If* $\vartheta : [0,T) \to E$ *satisfies* $\vartheta \in C^1((0,T); E) \cap$

$C^0([0,T);E)$, and the mapping $\vartheta \mapsto Z(\vartheta) \in \mathbb{R}$ is a C^1 convex functional on E, then

$$ {}^{C}D_s^{\gamma}\vartheta(s) = \frac{1}{\Gamma(1-\gamma)}\left(\frac{\vartheta(s)-\vartheta(0)}{s^{\gamma}} + \gamma\int_0^s \frac{\vartheta(s)-\vartheta(r)}{(s-r)^{\gamma+1}}\,dr\right), $$

and

$$ {}^{C}D_s^{\gamma} Z(\vartheta(s)) \leq \left({}^{C}D_s^{\gamma}\vartheta(s), D_F Z(\vartheta(s))\right), $$

where $D_F Z(\cdot) : E \to E^*$ denotes the Fréchet differential.

Definition A.10. *Let $J : E \to \mathbb{R}$ be a locally Lipschitz functional defined on the Banach space E. The* Clarke generalized directional derivative *of J at a point $\varrho \in E$ in the direction $\theta \in E$ is given by*

$$ J^0(\varrho;\theta) = \limsup_{\substack{\lambda \downarrow 0 \\ \rho \to \varrho}} \frac{J(\rho+\lambda\theta) - J(\rho)}{\lambda}. $$

This quantity represents the maximal rate of increase of J near ϱ along θ.

The Clarke subdifferential of J at ϱ is the set of all continuous linear functionals $\eta \in E^*$ satisfying

$$ J^0(\varrho;\theta) \geq \langle \eta, \theta \rangle \qquad \text{for every } \theta \in E. $$

We denote this set by

$$ \partial J(\varrho) = \left\{ \eta \in E^* \;\middle|\; J^0(\varrho;\theta) \geq \langle \eta, \theta\rangle,\ \forall \theta \in E \right\}. $$

Proposition A.11. *Let E be a Banach space and let $\alpha, \beta > 0$. Then the following properties hold:*

1) *If $z \in L^1(I;E)$, then for a.e. $s \in I$,*

$$ {}_0I_s^{\alpha}\,{}_0I_s^{\beta} z(s) = {}_0I_s^{\alpha+\beta} z(s). $$

2) *For every $z \in AC(I;E)$ and $\alpha \in (0,1]$, we have*

$$ {}_0I_s^{\alpha}\,{}_0^{C}D_s^{\alpha} z(s) = z(s) - z(0) \qquad \text{for a.e. } s \in I. $$

3) If $z \in L^1(I;E)$, then

$$ {}_0^C D_s^\alpha \, {}_0 I_s^\alpha z(s) = z(s) \qquad \text{for a.e. } s \in I. $$

Lemma A.12. *Let b and h be non-decreasing functions on $I = [0,T]$, and suppose that $h(s) \le c$ for all $s \in I$ with some constant $c > 0$. Let $0 < \alpha \le 1$. If $z \in L_+^\infty(0,T)$ satisfies*

$$ z(s) \le b(s) + h(s) \int_0^s (s-r)^{\alpha-1} z(r)\, dr, \qquad s \in I, $$

then

$$ z(s) \le b(s)\, E_\alpha(h(s)\,\Gamma(\alpha)\, s^\alpha)\,, \qquad s \in I, $$

where E_α denotes the Mittag–Leffler function

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. $$

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Abstract: In this thesis, we study mechanical contact problems while taking frictional effects into account. The physical model is based on a nonlinear constitutive law describing viscoelastic–viscoplastic materials, incorporating electrical or thermal effects within the formulation. The contact is modeled using various conditions, including normal compliance with wear phenomena, as well as multivalued normal compliance, which allows for a more general description of contact situations.

The study aims to establish the existence of a weak solution for the proposed problems and to prove its uniqueness under suitable assumptions. These results were achieved using analytical techniques relying on tools from ordinary differential equation theory, evolutionary hemivariational inequality, and fixed-point principles, providing an appropriate framework for the analysis of such nonlinear systems.

Keywords: Electro-Elasto-Viscoplastic, Thermo-Elastic-Viscoplastic, Wear, Internal state variables, Time-fractional derivative.

Résumé: Dans cette thèse, nous étudions des problèmes de contact mécanique en tenant compte des effets de frottement. Le modèle physique repose sur une loi constitutive non linéaire décrivant des matériaux viscoélastiques–viscoplastiques, intégrant des effets électriques ou thermiques dans la formulation. Le contact est modélisé à l'aide de différentes conditions, notamment la conformité normale accompagnée de phénomènes d'usure, ainsi que la conformité normale multivaluée, permettant une description plus générale des situations de contact.

L'étude vise à établir l'existence d'une solution faible pour les problèmes proposés et à démontrer son unicité sous des hypothèses appropriées. Ces résultats ont été obtenus en utilisant des techniques analytiques basées sur des outils issus de la théorie des équations différentielles ordinaires, des inégalités hémivariationnelles évolutives et des principes de point fixe, fournissant un cadre approprié pour l'analyse de tels systèmes non linéaires.

Mots-clés: Électro-Élasto-Viscoplastique, Thermo-Élasto-Viscoplastique, Usure, Variable interne d'état, Dérivée fractionnaire temporelle.

ملخص: في هذه الرسالة ندرس مسائل تلامس ميكانيكي مع الأخذ بعين الاعتبار تأثيرات الاحتكاك. يعتمد النموذج الفيزيائي على قانون تأسيسي غير خطي يصف سلوك مواد مرنة–لزجة–لدنة مع دمج التأثيرات الكهربائية أو الحرارية ضمن الصياغة. ويُنمذج التلامس بواسطة شروط مختلفة، من بينها الامتثال الطبيعي المصحوب بعمليات التآكل، إضافة إلى الامتثال الطبيعي متعدد القيم الذي يسمح بوصف أكثر عمومية لظروف التلامس.

تهدف الدراسة إلى إثبات وجود حل ضعيف للمشاكل المطروحة وإثبات وحدانيته تحت فرضيات مناسبة. وقد تم تحقيق هذه النتائج باستخدام تقنيات تحليلية تعتمد على أدوات من نظرية المعادلات التفاضلية العادية، والتفاوتات التطورية شبه المتغيرة، إضافة إلى مبادئ النقطة الثابتة التي توفر إطارًا ملائمًا لدراسة مثل هذه الأنظمة غير الخطية.

الكلمات المفتاحية: كهر-مرن-لزج-لدن، حراري-مرن-لزج-لدن، تآكل، متغير حالة داخلي، مشتق زمني كسري.