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Efficient techniques to solve nonlinear fractional-order integro-differential equations

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بسم الله الرحمن الرحيم

شكر وعرفان

في بداية هذه الرسالة، أود أن أشكر الله الذي ساعدني ومنحني الصبر والشجاعة طوال هذه السنوات الطويلة من الدراسة.

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والديّ العزيزين الذين كانا دوما سندي ودعمي الأكبر تعبيرا عن شكري وامتناني لكل ما قدماه لي من حب وتضحية حفظهما الله.

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I dedicate this humble work to:

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Introduction

In the past few years, fractional calculus, which is a generalization of integer order calculus or integration of integer order to arbitrary order (real or complex order), plays a fundamental and significant role in mathematical modeling and solving the models obtained in diverse fields of science and technology, such as biological models, fluid dynamics, viscoelasticity, chemistry, diffusion, economics, and many other applications. Nowadays, fractional integro-differential equations have been widely applied as an important and crucial tool for describing many different types of complex mechanical and physical behaviors, sciences and engineering, anomalous diffusion, vibration, control theory, electricity, electrochemistry, epidemiology and others, since most of the models in various research areas and engineering applications are nonlinear, obtaining their solutions plays an important role.

In the present thesis, we will conduct a comprehensive study on nonlinear fractional integro-differential equations, especially nonlinear fractional Volterra integro-differential equations in the form

$$D^{\alpha}x(t) = f(t) + \int_0^t K(t,s)F(x(s))ds,\ t \geq 0,$$

with the initial condition

$$x(0) = x_0 \in \mathbb{R}.$$

where $0 < \alpha \leq 1$ is a parameter that indicates the fractional order of the derivative, $x(t)$ is an unknown real function of the variable t, f, K are known real functions and F is nonlinear function. When $\alpha = 1$ this equation reduces to the classical nonlinear Volterra integro-differential equation. The nonlinear fractional Volterra integro-differential equation has been examined using different strategies. Jiang and Tian **[21]** utilized reproducing

kernel strategy to solve the problem considering the fractional derivative in Caputo sense. Wang and Zhu [**37**] presented a numerical strategy known as the Euler wavelet strategy using the Caputo fractional derivative to solve the nonlinear fractional integro-differential Volterra equation. Ghosh and Mohapatra [**17**] determined the solution of nonlinear fractional integro-differential Volterra equation using finite difference strategies with Caputo fractional derivative.

Fractional derivative operators are more generalized than the integer-order derivative operators that appear in classical calculus. Fractional calculus has gained increasing popularity over the past few years. The fundamental notions of fractional derivative are introduced by Caputo and Riemann-Liouville, which involve the singular kernel

$$K(t,s) = (t-s)^{-\alpha},\ 0 < \alpha < 1.$$

However, Caputo and Fabrizio noted in [**12**] that the modeling of many physical phenomena cannot be adequately modeled by the Caputo and Riemann-Liouville fractional derivatives. In order to resolve this problem, Caputo and Fabrizio introduced a new definition of the fractional derivative with a non-singular kernel

$$K(t,s) = \exp\left(-\frac{\alpha\,(t-s)}{1-\alpha}\right),\ 0 < \alpha \leq 1.$$

Currently, the Caputo-Fabrizio derivative is one of the most popular definitions of fractional derivatives and is employed to resolve many mathematical problems in engineering sciences. It has been successfully employed in the study of the general form of Walter's-B fluid model [**4**], a new dynamical model of hepatitis E [**25**], a new fractional differential model for COVID-19 transmission [**9**], mathematical modeling of human liver [**8**] and others. The Caputo-Fabrizio derivative has also been used to resolve the fractional SIS epidemic model [**23**].

The motivation of this thesis is to propose a novel analytical strategy for the solutions of nonlinear fractional Volterra integro-differential equations using the Caputo-Fabrizio derivative. The proposed strategy utilizes the Khalouta transform and the Caputo-Fabrizio fractional derivative along with the Adomian polynomials, to construct the approximate analytical solution of the nonlinear fractional Volterra integro-differential equationn. Three numerical experiments are presented to illustrate the proposed strategy. The change in

the solution under the influence of fractional parameter is observed through numerical and graphical observations.

The present thesis is structured as follows:

Chapter 1, entitled "Preliminary concepts" provides the fundamental concepts of the theory of fractional calculus including the definitions and properties of fractional integrals and derivatives, especially the Riemann-Liouville and Caputo, as well as the existence and uniqueness of the solution of fractional-order differential equations and some applications of fractional calculus in certain scientific fields.

Chapter 2, entitled "Generalities on classical integro-differential equations" introduces the definitions and classifcations of integro-differential equations. It also focuses on the existence and uniqueness of solution to various types of integro-differential equations.

Chapter 3, entitled "Application of decomposition strategies for standard nonlinear integro-differential equations" presents four decomposition strategies for finding approximate analytical solutions to standard integro-differential equations. Numerical examples are provided to demonstrate the accuracy and reliability of the presented strategies.

Chapter 4, entitled "Adaptation of decomposition strategies to resolve Caputo fractional integro-differential equations" focuses on adapting the decomposition strategies introduced in Chapter 3 to resolve fractional integro-differential equations under the Caputo fractional derivative. The accuracy and reliability of these strategies are demonstrated by solving various types of such equations.

Chapter 5, entitled "Investigation of solutions to nonlinear fractional Volterra integro-differential equations under the Caputo-Fabrizio fractional derivative" introduces the mathematical formulation of KhDS to resolve nonlinear fractional integro-differential equations, and more precisely nonlinear fractional Volterra integro-differential equations under the Caputo-Fabrizio fractional derivative. It also presents the convergence of the proposed strategy in the sense of Banach space and numerical simulations of some applications of the proposed equations.

Finally, the conclusions of this study are presented in the last portion entitled "Conclusions and future research".

Chapter 1

Preliminary concepts

In the present chapter, we introduce the concepts necessary for a good understanding of this thesis, where this part includes a brief reminder of the basic elements of the theory of fractional calculus, as well as the question of the existence and uniqueness of the solution of fractional-order differential equations and conclude this chapter with some applications of fractional calculus in certain scientific fields.

1.1 Fractional calculus

The concept of fractional calculus is a generalization of ordinary derivation and integration to an arbitrary order. Non-integer derivatives are widely applied in many fields, such as probability, viscoelasticity, electronics, economics, mechanics and biology, and others. A particular interest in fractional derivations is related to the mechanical modeling of rubbers and erasers. In short, all types of materials that retain the memory of their previous deformations, especially those of a viscoelastic nature. In fact, the fractional derivative arises automatically. In this portion, we will introduce the fractional integration operator, as well as the two most commonly used definitions for fractional derivatives (the Riemann-Liouville and Caputo definitions), and present the main properties of these concepts.

1.1.1 L^P Spaces

Definition 1.1.1 ***[10]*** *Let $[a,b]$ $(-\infty \leq a \leq b \leq \infty)$ a finite or infinite interval of $\mathbb{R}$ and $p \in \mathbb{R}$ with $1 \leq p < \infty$. We indicate by $L^p([a,b])$ the space of equivalence classes of $p-$integrable power functions on $[a,b]$ with values in $\mathbb{C}$*

$$L^p([a,b]) = \{x : [a,b] \to \mathbb{C},\ x \text{ measurable and } \|x\|_{L^p} < \infty\},$$

with

$$\|x\|_{L^p([a,b])} = \left(\int_a^b |x(t)|^p\, dt\right)^{\frac{1}{p}}.$$

The space $L^p([a,b])$ equipped with the norm $\|.\|_{L^p}$ is a Banach space.

If $p = 2$, then $L^2([a,b])$ is the space of equivalence classes of square-integrable measurable functions on $[a,b]$.

The space $L^2([a,b])$ provided by the norm

$$\|x\|_{L^2([a,b])} = \left(\int_a^b |x(t)|^2\, dt\right)^{\frac{1}{2}},$$

is a Hilbert space.

1.1.2 Absolutely continuous functions

Definition 1.1.2 ***[10]*** *Let $[a,b]$ $(-\infty \leq a \leq b \leq \infty)$ a finite or infinite interval of $\mathbb{R}$. We indicate by $AC([a,b])$ the space of absolutely continuous functions on $[a,b]$ consisting of functions x which are primitives of Lebesgue-integrable functions, i.e.*

$$x \in AC([a,b]) \Longleftrightarrow \exists \varphi \in L^1([a,b]) \text{ such that } x(t) = x(a) + \int_a^t \varphi(s)ds,\ \text{for all } t \in [a,b].$$

Definition 1.1.3 ***[10]*** *We indicate by $AC^n([a,b])$, $n \in \mathbb{N}^*$, the space of functions x defined on $[a,b]$ with values in $\mathbb{C}$ which have continuous derivatives on $[a,b]$ up to order $n-1$ and such that $x^{(n-1)} \in AC([a,b])$, i.e.*

$$AC^n([a,b]) = \left\{x : [a,b] \longrightarrow \mathbb{C} : x^{(k)} \in C([a,b]), k = 0, ..., n-1, x^{(n-1)} \in AC([a,b])\right\}.$$

Remark 1.1.1 *In $n = 1$, we have*

$$AC^1([a,b]) = AC([a,b]).$$

A characterization of the functions of the space $AC^n([a,b])$ is given by the following lemma.

Lemma 1.1.1 *A function $x \in AC^n([a,b]), n \in \mathbb{N}^*$ if and only if it is given by the following relation*

$$x(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}x^{(n)}(s)ds + \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^k.$$

1.1.3 Gamma and Beta functions

In this portion, we introduce the Gamma and Beta functions, which will be used later. These two functions play an essential role in the theory of fractional calculus.

Definition 1.1.4 *[30] The gamma function is a function that naturally extends the factorial to real numbers and even to complex numbers. For $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, we define the gamma function by*

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt. \tag{1.1.1}$$

By integrating by parts in (1.1.1), we show that

$$\Gamma(z+1) = z\Gamma(z), \mathrm{Re}(z) > 0. \tag{1.1.2}$$

Relation (1.1.2) makes it possible to establish that

$$\Gamma(n+1) = n!, \forall n \in \mathbb{N}.$$

Definition 1.1.5 *[30] The Beta function is a type of Euler integral defined for numbers complexes z and w by*

$$\beta(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0. \tag{1.1.3}$$

The Beta function is related to the gamma function by the following relationship

$$\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0. \tag{1.1.4}$$

It follows from (1.1.4) that

$$\beta(z,w) = \beta(w,z), \mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0.$$

1.1.4 Riemann-Liouville fractional integral and derivative

According to the Riemann-Liouville approach to fractional calculus, the concept of a fractional integral of order $\alpha, (\alpha > 0)$ generalizes the famous formula of integrals repeated $n-$times

$$\begin{aligned} I_a^n x(t) &= \int_a^t ds_1 \int_a^{s_1} ds_2 ... \int_a^{s_{n-1}} x(s_n) ds_n \\ &= \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} x(s) ds, n \in \mathbb{N}^*, \end{aligned}$$

which reduces the calculation of the nth primitive of a continuous function x on an interval $[a, b]$ to a single convolution-type integral.

Let us indicate by $D^n, n \in \mathbb{N}$, the derivation operator of order n, then we have

$$D^n I_a^n = I_d, I_a^n D^n \neq I_d,$$

where I_d is the identity operator.

Definition 1.1.6 *[30] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x \in L^1([a, b])$ is given by*

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds, t > a. \tag{1.1.5}$$

Definition 1.1.7 *[26] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function x defined on an interval $[a, b]$, is given by*

$$\begin{aligned} D_a^\alpha x(t) &= D^n I_a^{n-\alpha} x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} x(s) ds, t > a, \end{aligned} \tag{1.1.6}$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

In particular, if $\alpha = 0$, then

$$D_a^0 x(t) = I_a^0 x(t) = x(t).$$

If $\alpha = n \in \mathbb{N}$, then

$$D_a^n x(t) = x^{(n)}(t).$$

In addition, if $0 < \alpha < 1$, then $n = 1$, hence

$$D_a^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} x(s) ds, t > a.$$

Remark 1.1.2 *The definition of fractional-order derivation is based on that of fractional-order integration, a fractional-order derivation has a global character, unlike an integer derivation. It turns out that the fractional-order derivative of a function requires knowledge of $x(t)$ over the entire interval $]a, t[$, where, as in the integer case, only local knowledge of x around t is necessary. This property allows fractional-order systems to be interpreted as long-memory systems, and thus the entire set of systems to be interpreted as short-memory systems.*

Example 1.1.1 *Let $\alpha > 0, \gamma > -1$, and $x(t) = (t-a)^\gamma$, then*

$$\begin{aligned} (1)\ I_a^\alpha x(t) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha}, \\ (2)\ D_a^\alpha x(t) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha}. \end{aligned}$$

Indeed,

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(s-a)^\gamma ds. \tag{1.1.7}$$

Performing the following variable change

$$s - a = \theta(t-a) \Longrightarrow ds = (t-a)d\theta, \tag{1.1.8}$$

then, (1.1.7) becomes

$$\begin{aligned} I_a^\alpha x(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t ((t-a) - \theta(t-a))^{\alpha-1}(\theta(t-a))^\gamma (t-a)d\theta \\ &= \frac{(t-a)^{\alpha+\gamma}}{\Gamma(\alpha)} \int_0^1 \theta^\gamma (1-\theta)^{\alpha-1} d\theta. \end{aligned}$$

Using (1.1.3) and the property (1.1.4) we arrive at

$$\begin{aligned} I_a^\alpha x(t) &= \frac{\beta(\gamma+1,\alpha)}{\Gamma(\alpha)}(t-a)^{\alpha+\gamma} \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\alpha+\gamma}. \end{aligned}$$

Similarly, we can prove (2).

In particular, if $\gamma = 0$ *and* $\alpha > 0$, *then the Riemann-Liouville fractional derivative of a constant is generally non-zero. In fact, we have*

$$D_a^\alpha C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \alpha > 0.$$

It is easy to establish the following result.

Proposition 1.1.1 *Let* $\alpha > 0$ *and* $n = [\alpha] + 1$, *then*

$$D_a^\alpha x(t) = 0 \Longleftrightarrow x(t) = \sum_{j=1}^{n} C_j (t-a)^{\alpha-j}, \forall C_1, ...C_n \in \mathbb{R}.$$

In particular, if $0 < \alpha < 1$, *then*

$$D_a^\alpha x(t) = 0 \Longleftrightarrow x(t) = C(t-a)^{\alpha-1}, \forall C \in \mathbb{R}.$$

Let us now state some properties of the operators D_a^α and I_a^α.

Lemma 1.1.2 ***[30]*** *The Riemann-Liouville fractional integral operator* $I_a^\alpha, \alpha > 0$ *is linear and bounded by the space* $L^p([a,b])$ $(1 \leq p \leq \infty)$ *in itself, i.e.* $\exists k > 0$ *such that:*

$$\left\| I_a^\alpha x \right\|_p \leq k \left\| x \right\|_p \forall x \in L^p([a,b]), \text{ where } k = \frac{(b-a)^\alpha}{\alpha \left| \Gamma(\alpha) \right|}.$$

The following result characterizes the conditions necessary for the existence of the Riemann-Liouville fractional derivative D_a^α.

Lemma 1.1.3 ***[30]*** *Let* $\alpha > 0$ *and* $n = [\alpha] + 1$. *If* $x \in AC^n([a,b])$, *then the Riemann-Liouville fractional derivative* $D_a^\alpha x$ *exists almost everywhere on* $[a,b]$ *and is represented in the form*

$$D_a^\alpha x(t) = \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-a} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds.$$

An important property of the Riemann-Liouville fractional integral operator I_a^α called the property of semigroup, is given by the following lemma.

Lemma 1.1.4 *If* $\alpha > 0$ *and* $\beta > 0$, *then the relation*

$$I_a^\alpha I_a^\beta x(t) = I_a^{\alpha+\beta} x(t), \tag{1.1.9}$$

is satisfied almost everywhere on $[a,b]$ *for all* $x \in L^p([a,b]), 1 \leq p < \infty$.

Proof. Let $x \in L^p([a,b])$, then according to Fubini's theorem, we have

$$\begin{aligned} I_a^\alpha I_a^\beta x(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} I_a^\beta x(s) ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-s)^{\alpha-1} ds \int_a^s (s-\xi)^{\beta-1} x(\xi) d\xi \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t x(\xi) d\xi \int_\tau^t (t-s)^{\beta-1} ds \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-\xi)^{\alpha+\beta-1} x(\xi) d\xi \\ &= I_a^{\alpha+\beta} x(t). \end{aligned}$$

Hence the desired result. ■

One of the important properties that connect the Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral is the following.

Lemma 1.1.5 *For $\alpha > 0$ and $x \in L^1([a,b])$, we have*

$$D_a^\alpha I_a^\alpha x(t) = x(t), \ \text{a.e. on } [a,b]. \tag{1.1.10}$$

Proof. According to (1.1.6) and the property (1.1.9), we have for $n = [\alpha] + 1$

$$D_a^\alpha I_a^\alpha x(t) = D^n I_a^{n-\alpha} I_a^\alpha x(t) = D^n I_a^n x(t) = x(t), \ \text{a.e. on } [a,b].$$

This establishes the result. ■

Remark 1.1.3 *Property (1.1.10) means that the Riemann-Liouville fractional derivative operator is the left inverse of the Riemann-Liouville fractional integral operator of the same order.*

1.1.5 Caputo fractional derivative

The Riemann-Liouville fractional derivative has some drawbacks for modeling real phenomena. The problems studied require a definition of fractional derivatives that allows the use of physically interpretable initial conditions, including $x(0), x'(0), x''(0)$, etc. These failures led to an alternative definition of fractional derivatives meeting these requirements in the late

1960s, introduced by Caputo [**11**]. In fact, Caputo and Mainardi [**13**] used this definition in their work on viscoelasticity.

In this portion, we give the definition of the fractional derivative in the sense of Caputo as well as some essential properties.

Let $[a, b]$ be a finite interval of $\mathbb{R}$ and let I_a^α and D_a^α be the Riemann-Liouville fractional integral and derivative operators given by (1.1.5) and (1.1.6).

Definition 1.1.8 ***[26]*** *The Caputo fractional derivative of order $\alpha > 0$ of a function x defined on $[a, b]$ is given by*

$$ {}^C D_a^\alpha x(t) = D_a^\alpha \left(x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(t-a)^k \right), \tag{1.1.11} $$

where

$$ n = [\alpha] + 1, \ \text{if } \alpha \notin \mathbb{N}, \ \text{and } n = \alpha \ \text{if } \alpha \in \mathbb{N}^*. \tag{1.1.12} $$

If $\alpha = 0$, then

$$ {}^C D_a^0 x(t) = x(t). $$

In particular, when $0 < \alpha < 1$, the relation (1.1.11) takes the form

$$ {}^C D_a^\alpha x(t) = D_a^\alpha \left(|x(t) - x(a)| \right). $$

Thus, if $\alpha \notin \mathbb{N}$ and x is a function for which the Caputo fractional derivative (1.1.11) and the Riemann-Liouville fractional derivative (1.1.6) exist, then they are related to each other by the relation

$$ {}^C D_a^\alpha x(t) = D_a^\alpha x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-a}, n = [\alpha] + 1. \tag{1.1.13} $$

In particular, if $0 < \alpha < 1$, then we have

$$ {}^C D_a^\alpha x(t) = D_a^\alpha x(t) - \frac{x(a)}{\Gamma(\alpha-1)}(t-a)^{-a}. $$

According to relation (1.1.13), if $\alpha \notin \mathbb{N}$, the Caputo fractional derivative (1.1.11) is identical to the Riemann-Liouville fractional derivative (1.1.6) if the function x and all its derivatives up to order $n-1$ ($n = [\alpha] + 1$) vanish at the point a, i.e.

$$ {}^C D_a^\alpha x(t) = D_a^\alpha x(t) \Longleftrightarrow x(a) = x'(a) = ... = x^{(n-1)}(a) = 0. $$

If $\alpha = n \in \mathbb{N}$ and the usual derivative $x^{(n)}(t)$ exists, then ${}^{C}D_{a}^{\alpha}x(t)$ coincides with $x^{(n)}(t)$, i.e.

$$ {}^{C}D_{a}^{\alpha}x(t) = x^{(n)}(t). \tag{1.1.14}$$

The Caputo fractional derivative (1.1.11) is defined for functions $x(t)$ for which the Riemann-Louville derivative (1.1.6) exists, in particular for functions $x(t) \in AC^{n}([a,b])$ we have the following theorem.

Theorem 1.1.1 *[26] Let $\alpha > 0$, and n be given by (1.1.12). If $x \in AC^{n}([a,b])$, then the Caputo fractional derivative ${}^{C}D_{a}^{\alpha}x(t)$ exists almost everywhere on $[a,b]$.*

i) If $\alpha \notin \mathbb{N}$, then ${}^{C}D_{a}^{\alpha}x(t)$ is given by

$$\begin{aligned} {}^{C}D_{a}^{\alpha}x(t) &= \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-s)^{n-\alpha-1}x^{(n)}(s)ds, \\ &= I^{n-\alpha}D^{n}x(t). \end{aligned} \tag{1.1.15}$$

In particular, when $0 < \alpha < 1$ and $x \in AC^{1}([a,b])$, then

$$\begin{aligned} {}^{C}D_{a}^{\alpha}x(t) &= \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t}(t-s)^{-\alpha}x^{'}(s)ds, \\ &= I^{1-\alpha}x^{'}(t). \end{aligned} \tag{1.1.16}$$

ii) If $\alpha = n$, then

$$ {}^{C}D_{a}^{n}x(t) = x^{(n)}(t).$$

Proof. According to the definition of the Caputo fractional derivative (1.1.11), we have

$$\begin{aligned} {}^{C}D_{a}^{\alpha}x(t) &= D_{a}^{\alpha}\left(x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^{k}\right) \\ &= D^{n}I_{a}^{n-\alpha}\left(x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^{k}\right). \end{aligned}$$

We put

$$D(t) = I_{a}^{n-\alpha}\left(x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(t-a)^{k}\right).$$

According to (1.1.5), we have

$$D(t) = \int_{a}^{t}\frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left(x(s) - \sum_{k=0}^{n-1}\frac{x^{(k)}(a)}{k!}(s-a)^{k}\right)ds.$$

Integrating by parts, we will have

$$\begin{aligned} D(t) &= \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(s-a)^k \right) ds \\ &= -\frac{(t-s)^{n-\alpha}}{\Gamma(n-\alpha+1)} \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(s-a)^k \right)_{s=a}^{s=t} \\ &\quad + \frac{(t-s)^{n-\alpha}}{\Gamma(n-\alpha+1)} \int_a^t (t-s)^{n-\alpha} D \left(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(s-a)^k \right) ds \\ &= I_a^{n-\alpha+1} D \left(x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(t-a)^k \right). \end{aligned}$$

By repeating this process n-times, we find

$$\begin{aligned} D(t) &= I_a^{n-\alpha+n} D^n \left(x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(t-a)^k \right) \\ &= I_a^n I^{n-\alpha} D^n \left(x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(t-a)^k \right). \end{aligned}$$

But

$$\sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{k!}(t-a)^k,$$

is a polynomial of degree $n-1$, therefore

$$D(t) = I_a^n I_a^{n-\alpha} D^n x(t).$$

So,

$$\begin{aligned} {}^C D_a^\alpha x(t) &= D^n D(t) \\ &= D^n I_a^n I_a^{n-\alpha} D^n x(t) \\ &= I_a^{n-\alpha} D^n x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds. \end{aligned}$$

Hence the desired result. ■

Theorem 1.1.2 *[26] Let $\alpha > 0$, n given by (1.1.12) and $x \in C^n([a,b])$. Then the Caputo fractional derivative ${}^C D_a^\alpha x(t)$ is continuous on $[a,b]$.*

i) If $\alpha \notin \mathbb{N}$, *then* ${}^{C}D_{a}^{\alpha}x(t)$ *is given by (1.1.15) and takes the forme (1.1.16) for* $0 < \alpha < 1$.

ii) If $\alpha = n \in \mathbb{N}$, *then*

$$ {}^{C}D_{a}^{n}x(t) = x^{(n)}(t). $$

Example 1.1.2 *Let* $0 < \alpha < 1$ *and* $x(t) = (t-a)^{\gamma}, \gamma > -1$. *Then*

$$ {}^{C}D_{a}^{\alpha}x(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha}. $$

Indeed,

$$ \begin{aligned} {}^{C}D_{a}^{\alpha}x(t) &= I_{a}^{1-\alpha}x'(t) \\ &= \frac{\gamma}{\Gamma(1-\alpha)}\int_{a}^{t}(t-s)^{-\alpha}(s-a)^{\gamma-1}ds. \end{aligned} $$

By using the variable change (1.1.8), we will have

$$ \begin{aligned} {}^{C}D_{a}^{\alpha}x(t) &= \frac{\gamma}{\Gamma(1-\alpha)}(t-a)^{\gamma-\alpha}\int_{0}^{1}\theta^{\gamma-1}(1-\theta)^{-\alpha}d\theta \\ &= \frac{\gamma}{\Gamma(1-\alpha)}(t-a)^{\gamma-\alpha}\beta(\gamma,1-\alpha) \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(t-a)^{\gamma-\alpha}. \end{aligned} $$

In particular, the Caputo fractional derivative of a constant is zero

$$ {}^{C}D_{a}^{\alpha}C = 0, \forall C \in \mathbb{R}. $$

The Caputo fractional derivative, like the Riemann-Liouville fractional derivative, represents the left inverse operation of the fractional integral I_{a}^{α}.

Lemma 1.1.6 *Let* $\alpha > 0$ *and* $x \in L^{\infty}([a,b])$, *then*

$$ {}^{C}D_{a}^{\alpha}I_{a}^{\alpha}x(t) = x(t). $$

Lemma 1.1.7 *Let* $\alpha > 0$ *and* $x \in L^{\infty}([a,b])$, *then*

$$ I_{a}^{\alpha C}D_{a}^{\alpha}x(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^{k}, \tag{1.1.17} $$

for certain constants $x^{(k)}(0) \in \mathbb{R}$, $k \in \{0,1,...,n-1\}, n = [\alpha]+1$.

We defined the Caputo fractional derivative of a function x on a finite interval $[a,b]$ by (1.1.13), and we saw in Theorem 1.1.1 that it can be represented by (1.1.15) and (1.1.16) provided that $x \in AC^n([a,b])$. In fact, formula (1.1.15) allows us to define the Caputo fractional derivative on the semi-axis $\mathbb{R}^+$. Indeed, the Caputo fractional derivative of a function $x \in AC^n([0,\infty))$ on the semi-axis $\mathbb{R}^+$ is given by

$$ {}^{C}D^{\alpha}_{0^+}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds, t>0. $$

In particular, if $0<\alpha<1$ and $x \in AC^1([a,\infty))$, then

$$ {}^{C}D^{\alpha}_{0^+}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}x'(s)ds, t>0. $$

1.2 Caputo-type fractional differential equations

Definition 1.2.1 *Let $\alpha>0, \alpha \notin \mathbb{N}$, $n=[\alpha]+1$ and $f:\Omega \subset \mathbb{R}^2 \to \mathbb{R}$, then*

$$ {}^{C}D^{\alpha}x(t) = f(t,x(t)), $$

is called a fractional differential equation (FDE) of the Caputo type. The initial conditions for this type of equation, we use

$$ x^{(k)}(0) = x_k, k = 0,1,2,...,n-1, $$

where Ω is an open set of $\mathbb{R}^2$.

1.2.1 Existence and uniqueness of the solution for FDE

In this portion, we address the question of the existence and uniqueness of the solution of a Cauchy problem for fractional differential equations (we use the derivative in the sense of Caputo) and we have the problem in the following form

$$ \begin{cases} {}^{C}D^{\alpha}x(t) = f(t,x(t)), \\ x(0) = x_0, \end{cases} \tag{1.2.1} $$

where $t \in [0,T], 0<\alpha \leq 1, x_0 \in \mathbb{R}$, and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Theorem 1.2.1 *[2] (Banach's fixed point theorem)*

Let B be a Banach space and a contracting operator $\mathbb{P} : B \longrightarrow B$. Then $\mathbb{P}$ admits a unique fixed point. i.e $\exists x \in B$ such that $\mathbb{P}(x) = x$.

Theorem 1.2.2 *[2] Let $\mathbb{P}$ be an operator defined on a Banach space B, such that $\mathbb{P}^n$ is contracting on $B, \forall \ \ n \geq 1$. Then $\mathbb{P}$ has a unique fixed point.*

Lemma 1.2.1 *Let $0 < \alpha \leq 1$ and let $h : [0, T] \to \mathbb{R}$ a continuous function. A function x is a solution of the Cauchy problem*

$$\begin{cases} {}^{C}D^{\alpha}x(t) = h(t), t \in [0, T], 0 < \alpha \leq 1, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.2.2}$$

if and only if it is the solution of the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. \tag{1.2.3}$$

Proof. We apply the operator I^{α} to Eq. (1.2.2) we find

$$\begin{aligned} I^{\alpha C}D^{\alpha}x(t) &= I^{\alpha}h(t) \\ &\Longrightarrow \ x(t) + C_0 = I^{\alpha}h(t) \\ &\Longrightarrow \ x(t) = I^{\alpha}h(t) - C_0. \end{aligned}$$

The initial condition gives

$$\begin{aligned} x(0) &= I^{\alpha}h(0) - C_0 = -C_0 \\ &\Longrightarrow \ C_0 = -x_0. \end{aligned}$$

So

$$\begin{aligned} x(t) &= I^{\alpha}h(t) - (-x_0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + x_0. \end{aligned}$$

Conversely we have

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &= I^{\alpha}h(t) + x_0. \end{aligned}$$

We apply ${}^{C}D^{\alpha}$ to the integral equation (1.2.3)

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= {}^{C}D^{\alpha}I^{\alpha}h(t) + {}^{C}D^{\alpha}x_0 \\ &= h(t). \end{aligned}$$

All that remains is to verify that $x(0) = C_0$

$$\begin{aligned} x(0) &= I^{\alpha}h(0) + x_0 = 0 + x_0 \\ &= x_0. \end{aligned}$$

Then there is a solution to the problem (1.2.2).

Hence the desired result. ■

Theorem 1.2.3 *Let $0 < \alpha \leq 1$ and $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and verifies the following Lipschitz condition*

$$|f(t,x) - f(t,y)| \leq k\,|x-y|\,, \forall t \in [0,T] \text{ and } x, y \in \mathbb{R}.$$

If

$$\frac{kT^{\alpha}}{\Gamma(\alpha+1)} < 1,$$

then there exists a unique solution to the Cauchy problem (1.2.1).

Proof. We use Banach's fixed point theorem 1.2.1.

We transform the problem (1.2.1) into a fixed point problem (Lemma 1.2.1), by considering the operator

$$\begin{aligned} \mathbb{P} : C([0,T],\mathbb{R}) &\longrightarrow C([0,T],\mathbb{R}) \\ x &\longrightarrow \mathbb{P}x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s))ds, \end{aligned}$$

where $C([0,T],\mathbb{R})$ is the Banach space of continuous functions x defined from $[0,T]$ in $\mathbb{R}$, equipped with the norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)|\,.$$

It is clear that the fixed points of operator $\mathbb{P}$ are the solutions of problem (1.2.1). $\mathbb{P}$ is well defined, in fact: if $x(t) \in C([0,T],\mathbb{R})$, then $\mathbb{P}x(t) \in C([0,T],\mathbb{R})$.

To show that $\mathbb{P}$ admits a fixed point, it suffices to show that F is a contraction, in fact if $x_1, x_2 \in C([0,T],\mathbb{R}), t \in [0,T]$, using the Lipschitiz condition we obtain

$$\begin{aligned}
|\mathbb{P}x_1(t) - \mathbb{P}x_2(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(f(s, x_1(s)) - f(s, x_2(s)) \right) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t |f(s, x_1(s)) - f(s, x_2(s))| \, (t-s)^{\alpha-1} ds \\
&\leq \frac{k}{\Gamma(\alpha)} \int_0^t |x_1(s) - x_2(s)| \, (t-s)^{\alpha-1} ds \\
&\leq \frac{k}{\Gamma(\alpha)} |x_1(s) - x_2(s)| \int_0^t (t-s)^{\alpha-1} ds \\
&\leq \frac{kT^\alpha}{\alpha\Gamma(\alpha)} |x_1(s) - x_2(s)| .
\end{aligned}$$

So,

$$\|\mathbb{P}x_1 - \mathbb{P}x_2\| \leq \frac{kT^\alpha}{\Gamma(\alpha+1)} \|x_1 - x_2\| .$$

By virtue of $\dfrac{kT^\alpha}{\Gamma(\alpha+1)} < 1$, we can deduce that $\mathbb{P}$ is a contraction and according to Banach's theorem, $\mathbb{P}$ admits a unique fixed point which is a solution of the Cauchy problem (1.2.1).

Hence the desired result. ■

1.3 Applications of fractional systems

Fractional systems are increasingly appearing in various research fields. However, the growing interest in these systems lies in their applications in fundamental and applied sciences. It can be noted that, for most of the fields presented below, fractional operators are used to take into account memory effects. Let us mention the works **[32],[36],** which bring together various applications of fractional calculus.

In automatic:

In automatic model, it was only in the early 1990s that the CRONE regulator (non-integer order robust control) was proposed by A. Oustaloup **[35]**. By exploiting the advantages of the properties of fractional order systems, this regulator ensures the robustness of the control in a given frequency band. The success of this approach was enormous, several

variants of this order emerged (1st, 2nd and 3rd generations). Since then, fractional order has attracted the attention of many researchers. In 1999, I. Podlubny **[31]** proposed the $I^{\alpha}D^{\beta}$ regulator, which includes a fractional integration of order α and a fractional derivative of order β, thus expanding the application of fractional calculus to control theory.

In physics:

One of the most remarkable applications of fractional calculus in physics is in the context of classical mechanics. F. Riewe **[34]** showed that the Lagrangian containing the time derivatives of fractional orders leads to an equation of motion with non-conservative forces such as friction. This result is remarkable because frictional forces and nonconservative forces are essential for the treatment of usual macroscopic variations, and consequently, for the most common methodological advances in classical mechanics. F. Riewe generalized the usual calculus of variations to the Lagrangian dependent on fractional derivatives **[33]** in order to treat the usual nonconservative forces. Furthermore, several approaches have been developed to generalize the principle of least action and the Euler-Lagrange equation to the case of fractional derivatives **[3]**,**[5]**.

In continuous media mechanics:

The deformation of continuous media (solid or fluid) is often described using two tensors, one for deformations indicate ε_{ij}, and the other for stresses σ_{ij}. Some materials, such as polymers (glue, rubber, etc.), exhibit behavior intermediate between viscous and elastic, described as viscoelastic. These systems can be modeled using the following relationship between the two tensors:

$$\sigma_{ij} = E\varepsilon_{ij}(t) + \eta D^{\alpha}\varepsilon_{ij}(t), 0 < \alpha < 1.$$

This law is justified by R.L. Bagley and P.J. Torvik in **[6]**,**[7]** (for $\alpha = \frac{1}{2}$). In **[29]**, the introduction of fractional derivatives in the case of polymers is motivated by the following analysis: due to the length of the fibers, the applied deformations take time to communicate step by step (the length of the wound fibers being much greater than the geometric distance). They gradually dampen and induce memory effects (the state at time t will depend on the

previous states). If the stress decreases when $t^{-(1+\alpha)}$, it can induce a fractional derivative of order α. This operator thus makes it possible to give a simple macroscopic description (requiring only a few parameters) of complex microscopic phenomena.

In acoustics:

For some wind musical instruments, visco-thermal losses can be modeled effectively using time-fractional derivatives [**20**].

Chapter 2

Generalities on classical integro-differential equations

Integro-differential equations appear in many scientific applications, especially whene we convert initail value problems or boundary value problems into integral equations. Integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions can appear at any order. In this present chapter, we provide the definition and classifcation of integro-differential equations. Moreover, we focus on the existence and uniqueness of the solution of different types of integro-differential equations.

2.1 Integro-differential equations

Definition 2.1.1 *[27] An integro-differential equation (IDE) is an equation composed of two operators, integral and differential, that involve the unknown function $x(t)$, with the general form of a nonlinear integro-differential equation of order n is*

$$x^{(n)}(t) = F\left(t, x(t), x^{'}(t), ...x^{(n-1)}(t), \gamma \int_{\Omega} K(t, s, x(s), ..., x^{(n-1)}(s))ds\right), \tag{2.1.1}$$

or in an equivalent form

$$G(x(t), x^{'}(t), ..., x^{(n)}(t)) = f(t) + \gamma \int_{\Omega} K(t, s, x(s), ..., x^{(n-1)}(s))ds,$$

with K is a known function, called the kernel of the integro-differential equation, γ is a real constant, and Ω is an open subset of $\mathbb{R}$. Since this equation contains the differential it is necessary to have the following initial conditions

$$x(t_0) = C_0, x^{'}(t_0) = C_1, x^{''}(t_0) = C_2, ..., x^{(n-1)}(t_0) = C_{n-1},$$

with $t_0 \in \Omega$ and $C_i \in \mathbb{R}, \forall i \in \{1, 2, ..., n-1\}$ are positive natural numbers.

Remark 2.1.1 *An integro-differential equation of order n is of the form*

$$\sum_{i=0}^{n} p_i(t)x^{(i)}(t) = f(t) + \gamma \int_{\Omega} K(t,s) \sum_{j=0}^{m} b_j(s)x^{(j)}(s)ds. \tag{2.1.2}$$

We can write

$$\sum_{i=0}^{n} p_i(t)x^{(i)}(t) = M_t(x).0 \leq i \leq n,$$

and

$$\sum_{j=0}^{m} b_j(s)x^{(j)}(s) = L_s(x).0 \leq j \leq m,$$

with the function $b_j(t)$, $p_i(t)$, $f(t)$ are known functions, and Ω is an open subset of $\mathbb{R}$.

2.2 Classification of integro-differential equations

In the classification of integro-differential equations we will follow the same category as previously used.

2.2.1 Fredholm integro-differential equation

If the two endpoints of the integral are constants, then the integro-differential equation is said to be of the Fredholm type with the general form

$$(F): \begin{cases} x^{(n)}(t) = f(t) + \gamma \int_a^b K(t,s)F(x(s))ds, \ t \in [a,b], \\ x(a) = C_0, x'(a) = C_1, ..., x^{(n-1)}(a) = C_{n-1}, \end{cases}$$

where F is a nonlinear function of the unknown function x.

Example 2.2.1

$$\begin{cases} x^{'}(t) = t^2 - \frac{1}{3}t + \gamma \int_0^1 (t-s)^2 x^2(s)ds, \\ x(0) \in \mathbb{R}. \end{cases}$$

2.2.2 Volterra integro-differential equation

If one of the two endpionts of the interval varies with respect to t, then this integro-differential equation is of the Volterra type with the general form

$$(V): \begin{cases} x^{(n)}(t) = f(t) + \gamma \int_a^t K(t,s)F(x(s))ds, t \in [a,b] \\ x(a) = C_0, x'(a) = C_1, ..., x^{(n-1)}(a) = C_{n-1}, \end{cases}$$

with F is a nonlinear function of the unknown function x and its standard form is

$$\int_a^t K_1(t,s)F(x(s))ds + \int_a^t K_2(t,s)x^{(i)}(s)ds = f(t).$$

Example 2.2.2

$$\begin{cases} x^{'}(t) = -\frac{1}{3} - t + \gamma \int_0^t s^2 e^{x(s)}ds, \\ x(0) \in \mathbb{R}. \end{cases}$$

and

$$\begin{cases} \int_0^t (t-s)x^2(s)ds + \int_0^t (t-s)x^{'}(s)ds = \frac{7}{8} + \frac{1}{4}t^2 - \cos(t) + \frac{1}{8}\cos(2t), \\ x(0) = 0. \end{cases}$$

2.2.3 Fredholm-Volterra integro-differential equation

If both types of integrals Fredholm and Volterra exist, then this integro-differential equation is of Fredholm-Volterra type of order k and its general form is

$$(F\text{–}V): \begin{cases} \sum_{j=0}^{k} \xi_j(t)\, x^{(j)}(t) = f(t) + \lambda_1 \int_a^b K_1(t,s)\, G_1(x(s))ds + \lambda_2 \int_a^t K_2(t,s)\, G_2(x(s))ds, t \neq 0, \\ x^{(r)}(a) = C_r \in \mathbb{R}, r = 0, 1, 2, ..., k-1, \end{cases}$$

where $x^{(j)}(t)$ is the jth derivative of the unknown function $x(t)$ to be determined, $K_i(t,s), i = 1, 2$ are the kernels of the equation, $\xi_j(t)$ is an analytic function, G_1 and G_2 are nonlinear functions of x, and $a, b, \lambda_1, \lambda_2$ and b_r are finite real constants.

Example 2.2.3

$$\begin{cases} x^{'}(t) = t^4 + 24t + 3 - \int_0^1 sx^3(s)ds - \int_0^t (t-s)x^2(s)ds, t \neq 0, \\ x(0) = C_1 \in \mathbb{R}. \end{cases}$$

2.3 Properties of integro-differential equation

2.3.1 First and second type

If the differential part of an integro-differential equation is zero, the equation is said to be of the first kind (type).

Example 2.3.1

$$\begin{cases} 0 = 1 - t^2 + \gamma \int_0^t x^2(s)ds, t \neq 0, \\ x(0) \in \mathbb{R}. \end{cases}$$

If the differential part of an integro-differential equation is not zero, the equation is said to be of the second kind (type).

Example 2.3.2

$$\begin{cases} x'(t) = 1 - t^2 + \gamma \int_0^t x^2(s)ds, t \neq 0, \\ x(0) \in \mathbb{R}. \end{cases}$$

2.3.2 Homogeneity and linearity

Integro-differential equations of the second kind are classified as homogeneous or non-homogeneous, If the function $f(t)$ in the second kind of integro-differential equations is identically zero, the equation is said to be homogeneous. Otherwise, it is said to be non-homogeneous. Note that this property is valid only for equations of the second kind.

To clarify this concept by considering the following equation:

$$(1): \begin{cases} x'(t) = 1 + \int_0^t se^{x(s)}ds, t \neq 0, \\ x(0) = 1. \end{cases}$$

$$(2): \begin{cases} x'(t) = \frac{1}{12}\gamma \int_0^1 x^2(s)\, ds, \\ x(0) = 1. \end{cases}$$

$$(3): \begin{cases} x''(t) = \int_a^b e^{x(s)}ds, \\ x(a) = \alpha_1, x'(a) = \alpha_2. \end{cases}$$

$$(4): \begin{cases} x''(t) = \int_a^b \log(x(s))ds + \int_a^t (t-s)^2 x^2(s)ds, t \neq 0, \\ x(a) = \beta_1, x'(a) = \beta_2. \end{cases}$$

The Eqs. (1) and (4) are non-homogeneous, while Eqs. (2) and (3) are homogeneous.

An integro-differential equation is said to be nonlinear if it is of type (2.1.1) and linear if it is of type (2.1.2).

2.3.3 Singularity

If one of the two endpoints or both endepoint are infinite, or if the kernel of the equation is not finite in the neighborhood of at least one point within any given interval, then the equation is said to be singular.

Example 2.3.3

$$\begin{cases} x''(t) = f(t) + \int_a^t \frac{\cos(x(s))}{(t-s)^\alpha} ds, 0 < \alpha < 1, \\ x(a) = \delta_1, x'(a) = \delta_2. \end{cases}$$

2.3.4 Order of an integro-differential equation

The order of an integro-differential equation is the highest order of the derivative in the equation.

2.4 Existence and uniqueness theorems

In this portion, we present some results related to the existence and uniqueness theorems of the solution for different types of integro-differential equations.

First, let us note that in most cases, we consider the space $C(I)$ of continuous functions on the interval $I = [a, b]$ in $\mathbb{R}$ equipped with the norm

$$\|x\|_\infty = \max_{t \in I.} |x|, \forall x \in C(I).$$

2.4.1 Existence and uniqueness of the solution of the Fredholm integro-differential equation

Here, we use the contraction mapping theorem to prove the existence and uniqueness of the solution of the following Fredholm integro-differential equation

$$\begin{cases} x'(t) = f(t) + \int_a^b K(t, s, x(s)) ds, t \in [a, b], \\ x(a) = x_0 \in \mathbb{R}, \end{cases} \tag{2.3.1}$$

where $f : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Theorem 2.4.1 *Let the operator* $\mathbb{P}$ *such that*

$$\mathbb{P} : C([a,b]) \to C([a,b]).$$

If K *is Lipschitz in the third variable, then for all real* $L > 0$ *and* $n \geq 1$*, we have*

$$\left\|\mathbb{P}^n(x) - \mathbb{P}^n(y)\right\|_\infty \leq \frac{L^n (b-a)^{2n}}{n!} \left\|x - y\right\|_\infty, \tag{2.3.2}$$

and Eq. (2.3.1) admits a unique solution.

Proof. By integration (2.3.1) between a and t, we obtain

$$x(t) = x_0 + \int_a^t f(\tau)\, d\tau + \int_a^t \int_a^b K(\tau, s, x(s))\, ds d\tau, \ t \in [a,b].$$

We define the operator $\mathbb{P}$ from $C([a,b])$ in $C([a,b])$ as follows

$$\mathbb{P}x(t) = x_0 + \int_a^t f(\tau)\, d\tau + \int_a^t \int_a^b K(\tau, s, x(s))\, ds d\tau, \ t \in [a,b].$$

We use proof by induction to show (2.3.2) for $n = 1$, we have

$$\begin{aligned} |\mathbb{P}x(t) - \mathbb{P}y(t)| &= \left| \int_a^t \left(\int_a^b (K(\tau, s, x(s)) - K(\tau, s, y(s)))\, ds \right) d\tau \right| \\ &\leq \int_a^t \left(\int_a^b |K(\tau, s, x(s)) - K(\tau, s, y(s))|\, ds \right) d\tau. \end{aligned}$$

Since K is Lipschitz in the third variable with the Lipschitz constant $L > 0$, for all $t \in [a,b]$, we have

$$\begin{aligned} \int_a^t \left(\int_a^b |K(\tau, s, x(s)) - K(\tau, s, y(s))|\, ds \right) d\tau &\leq \int_a^t \left(\int_a^b L\, |x(s) - y(s)|\, ds \right) d\tau \\ &\leq \int_a^x \left(\int_a^b L \left\|x - y\right\|_\infty ds \right) d\tau \\ &= L(b-a)(t-a) \left\|x - y\right\|_\infty \\ &\leq L(b-a)^2 \left\|x - y\right\|_\infty, \text{ because } t \leq b. \end{aligned}$$

That is to say (2.3.2) is true for $n = 1$.

Suppose the inequality (2.3.2) is true for $n-1$ and let show that it also true for n , $\forall t \in [a,b]$, we have

$$\begin{aligned}
\left|\mathbb{P}^n x(t) - \mathbb{P}^n y(t)\right| &= \left|\int_a^t \left(\int_a^b \left(K\left(\tau, s, \mathbb{P}^{n-1}x(s)\right) - K\left(\tau, s, \mathbb{P}^{n-1}y(s)\right)\right) ds\right) d\tau\right| \\
&\leq \int_a^t \left(\int_a^b L\left|\mathbb{P}^{n-1}x(s) - \mathbb{P}^{q-1}y(s)\right| ds\right) d\tau \\
&\leq \int_a^t \left(\int_a^b L.L^{n-1}\left\|x-y\right\|_\infty \frac{(b-a)^{n-1}}{(n-1)!}(s-a)^{n-1} ds\right) d\tau \\
&= \frac{L^n(b-a)^n(t-a)^n}{n!}\left\|x-y\right\|_\infty \\
&\leq \frac{L^n(b-a)^{2n}}{n!}\left\|x-y\right\|_\infty, \text{ because } t \leq b.
\end{aligned}$$

This completes the proof of (2.3.2).

Thus, if the value of n is a large enough $\dfrac{L^n(b-a)^{2n}}{n!} < 1$, then $\mathbb{P}^n$ is a contraction operator from $C([a,b])$ in itself and by the contaction mapping theorem $\mathbb{P}^n$ admits a unique fixed point.

From Banach fixed point theorem 1.2.1, we deduce that

$$\mathbb{P}^n(\mathbb{P}(x)) = \mathbb{P}(\mathbb{P}^n(x)) = \mathbb{P}(x).$$

So, $\mathbb{P}(x)$ is a fixed point of $\mathbb{P}^n$, and by the the uniqueness of the fixed point of $\mathbb{P}^n$, we have $\mathbb{P}(x) = x$.

Since the fixed point of $\mathbb{P}$ are also fixed point of $\mathbb{P}^n$, we conclud that x is the unique fixed point of $\mathbb{P}$.

Thus, Eq. (2.3.1) admits a unique solution.

Hence the desired result. ■

2.4.2 Existence and uniqueness of the solution of the Volterra integro-differential equation

Let us consider the Velterra integro-differential equation with the following initial condition

$$\begin{cases} x'(t) = f(t) + \int_a^t K(t,s,x(s))\, ds, t \in [a,b], \\ x(a) = x_0 \in \mathbb{R} \end{cases} \tag{2.3.3}$$

where $f : [a,b] \to \mathbb{R}$, $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$, $x : [a,b] \to \mathbb{R}$ and $x' : [a,b] \to \mathbb{R}$ are continuous functions.

Here, we use the contraction mapping theorem to prove the existence and uniqueness of the solution of (2.3.3).

Let $C_\beta(I,\mathbb{R})$ be the Banach space of all continuous function from $I = [a,b]$ in $\mathbb{R}$, equipped with the following norm

$$\|x\|_\infty^\beta = \sup_{t\in I} \frac{|x(t)|}{e^{\beta(t-a)}} \text{ with } \beta > 0.$$

Theorem 2.4.2 *If the kernel K is Lipschitz with respect to the third variable, that is to say, $\exists\ L > 0$, $\forall t, s \in I$ and $x, y \in \mathbb{R}$*

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq L|x(s) - y(s)|,$$

and if $\frac{L}{\beta^2} < 1$, then Eq. (2.3.3) admits a unique solution $x \in C_\beta(I,\mathbb{R})$.

Proof. Let $x \in C_\beta(I,\mathbb{R})$, by integrating Eq. (2.3.3) from a to t, we obtain

$$x(t) = x_0 + \int_a^t f(\tau)\,d\tau + \int_a^t \int_a^s K(\tau,s,x(s))\,dsd\tau,\ t \in I.$$

We define the operator $\mathbb{P}$ on the space $C_\beta(I,\mathbb{R})$, as follows

$$\mathbb{P}x(t) = x_0 + \int_a^t f(\tau)\,d\tau + \int_a^t \int_a^s K(\tau,s,x(s))\,dsd\tau,\ t \in I.$$

It is clear that the operator $\mathbb{P}(C_\beta(I,\mathbb{R})) \subset C_\beta(I,\mathbb{R})$.

Now, we must verify that the operator $\mathbb{P}$ is a contraction. Let $x, y \in C_\beta(I,\mathbb{R})$, we have

$$\begin{aligned}
|\mathbb{P}x(t) - \mathbb{P}y(t)| &\leq \int_a^t \int_a^s |K(\tau,s,x(s)) - K(\tau,s,y(s))|\,dsd\tau \\
&\leq \int_a^t \int_a^s L|x(s) - y(s)|\,dsd\tau \\
&\leq L\int_a^t \int_a^s |x(s) - y(s)|\,e^{-\beta(s-a)}e^{\beta(s-a)}dsd\tau \\
&\leq L\|x - y\|_\infty^\beta \left(\frac{e^{\beta(t-a)} - 1}{\beta^2} - \frac{t-a}{\beta}\right) \\
&\leq L\left(\frac{e^{\beta(t-a)}}{\beta^2}\right)\|x - y\|_\infty^\beta.
\end{aligned}$$

Thus, we obtain

$$\left\|\mathbb{P}x\left(t\right)-\mathbb{P}y\left(t\right)\right\|_{\infty}^{\beta}\leq\frac{L}{\beta^{2}}\left\|x-y\right\|_{\infty}^{\beta},\forall x,y\in C_{\beta}\left(I,\mathbb{R}\right)\text{ and }\beta>0.$$

Since $\frac{L}{\beta^2}<1$, so $\mathbb{P}$ is a contraction. Therefore, by Banach's fixed point theorem 1.2.1, $\mathbb{P}$ has a unique fixed point in $C_{\beta}\left(I,\mathbb{R}\right)$, which is the solution of Eq. (2.3.3).

Hence the desired result. ■

2.4.3 Existence and uniqueness of the solution of the Fredholm-Volterra integro-differential equation

In this portion, we will give the results of the existence and uniqueness of the solution of the Fredholm-Volterra integro-differential equation of the form

$$(F-V):\begin{cases}\sum\limits_{j=0}^{k}\xi_{j}\left(t\right)x^{(j)}\left(t\right)=f(t)+\lambda_{1}\int_{a}^{b}K_{1}\left(t,s\right)G_{1}\left(x(s)\right)ds+\lambda_{2}\int_{a}^{t}K_{2}\left(t,s\right)G_{2}\left(x(s)\right)ds,\\ x^{(r)}(a)=C_{r}\in\mathbb{R},r=0,1,2,...,k-1.\end{cases}\tag{2.3.4}$$

We can write Eq. (2.3.4) in the form

$$\begin{aligned}x(t) &= L^{-1}\left(\frac{f(t)}{\xi_{k}\left(t\right)}\right)+\sum_{r=0}^{k-1}\frac{\left(t-a\right)^{r}}{r!}C_{r}+\lambda_{1}L^{-1}\left(\int_{a}^{b}\frac{K_{1}\left(t,s\right)}{\xi_{k}\left(t\right)}G_{1}\left(x(s)\right)ds\right)\\ &\quad+\lambda_{2}L^{-1}\left(\int_{a}^{t}\frac{K_{2}\left(t,s\right)}{\xi_{k}\left(t\right)}G_{2}\left(x(s)\right)ds\right)-\sum_{r=0}^{k-1}L^{-1}\left(\frac{\xi_{j}\left(t\right)}{\xi_{k}\left(t\right)}x^{(j)}\left(t\right)\right)\end{aligned}$$

such as

$$L^{-1}\left(\int_{a}^{b}\frac{K_{1}\left(t,s\right)}{\xi_{k}\left(t\right)}G_{1}\left(x(s)\right)ds\right)=\int_{a}^{b}\frac{\left(t-s\right)^{k}}{\left(k!\right)\,\xi_{k}\left(t\right)}K_{1}\left(t,s\right)G_{1}\left(x(s)\right)ds,$$

$$L^{-1}\left(\int_{a}^{t}\frac{K_{2}\left(t,s\right)}{\xi_{k}\left(t\right)}G_{2}\left(x(s)\right)ds\right)=\int_{a}^{t}\frac{\left(t-s\right)^{k}}{\left(k!\right)\,\xi_{k}\left(t\right)}K_{2}\left(t,s\right)G_{2}\left(x(s)\right)ds,$$

$$\sum_{r=0}^{k-1}L^{-1}\left(\frac{\xi_{j}\left(t\right)}{\xi_{k}\left(t\right)}x^{(j)}\left(t\right)\right)=\sum_{j=0}^{k-1}\int_{a}^{t}\frac{\left(t-s\right)^{k-1}\xi_{j}\left(t\right)}{\left(k-1\right)!\,\xi_{k}\left(t\right)}x^{(j)}\left(s\right)ds.$$

We fix

$$\Psi\left(t\right)=L^{-1}\left(\frac{f(t)}{\xi_{k}\left(t\right)}\right)+\sum_{r=0}^{k-1}\frac{\left(t-a\right)^{r}}{r!}C_{r}.$$

Before starting and proving the main results, we introduce the following hypotheses:

(H1): There exist constants L_1, L_2 and $\gamma_j > 0$, $j = 0, 1, 2, ..., k$, such that for all $x_1, x_2 \in C(J, \mathbb{R})$, we have

$$\begin{aligned} |G_1(x_1) - G_1(x_2)| &\leq L_1 |x_1 - x_2|, \\ |G_2(x_1) - G_2(x_2)| &\leq L_2 |x_1 - x_2|, \end{aligned}$$

and

$$\left|D^j(x_1) - D^j(x_2)\right| \leq \gamma_j |x_1 - x_2|.$$

(H2): We assume that the nonlinear terms $G_1(x(t)), G_2(x(t))$ and $x^{(j)}(t) = \sum_{j=0}^{k-1} \gamma_j$, $j = 0, 1, 2, ..., k-1$, are Lipschitz continuous.

We assume that for all $a \leq s \leq t \leq b$ and $j = 0, 1, 2, ..., k$, we have

$$\left|\lambda_1 \frac{(t-s)^k}{(k!)\ \xi_k(t)} K_1(t,s)\right| \leq \theta_1, \left|\lambda_2 \frac{(t-s)^k}{(k!)\xi_k(t)} K_2(t,s)\right| \leq \theta_2,$$

$$\left|\frac{(t-s)^{k-1} \xi_j(t)}{(k-1)!\ \xi_k(t)}\right| \leq \theta_3.$$

(H3): There exist three functions θ_2^*, θ_3^* and $\gamma^* \in C(D, \mathbb{R}^+)$ the set of all positive continuous functions on $D = \{(t,s) \in \mathbb{R} \times \mathbb{R} : a \leq s \leq t \leq b\}$ such that

$$\theta_2^* = \max_{[a,b]} |\theta_2|, \ \theta_3^* = \max_{[a,b]} |\theta_3|, \ \gamma^* = \max_{[a,b]} \left|\gamma_j\right|.$$

(H4): $\Psi(x)$ is a bounded function for all t in $J = [a, b]$.

Theorem 2.4.3 *Assume that **(H1)-(H4)** are satisfied.*

If

$$0 < \psi = (L_1\theta_1 + L_2\theta_2^* + k\gamma^*\theta_3^*)(b-a) < 1,$$

then there exists a unique solution $x(t) \in C(J)$ to the Fredholm-Volterra integro-differential equation (2.3.4).

Proof. Let x_1 et x_2 be two different solutions of Eq. (2.3.4), then

$$\begin{aligned}
|x_1 - x_2| &= \left| \lambda_1 \int_a^b \frac{(t-s)^k K_1(t,s)}{(k!)\ \xi_k(t)} \left(G_1(x_1) - G_1(x_2)\right) ds \right. \\
&\quad + \lambda_2 \int_a^t \frac{(t-s)^k K_2(t,s)}{(k!)\ \xi_k(t)} \left(G_2(x_1) - G_2(x_2)\right) ds \\
&\quad \left. - \sum_{j=0}^{k-1} \int_a^t \frac{(t-s)^{k-1} \xi_j(t)}{(k-1)!\ \xi_k(t)} \left(x_1^{(j)}(s) - x_2^{(j)}(s)\right) ds \right| \\
&\leq \lambda_1 \int_a^b \left| \frac{(t-s)^k K_1(t,s)}{(k!)\ \xi_k(t)} \right| \left|G_1(x_1) - G_1(x_2)\right| ds \\
&\quad + \lambda_2 \int_a^t \left| \frac{(t-s)^k K_2(t,s)}{(k!)\ \xi_k(t)} \right| \left|G_2(x_1) - G_2(x_2)\right| ds \\
&\quad + \sum_{j=0}^{k-1} \int_a^t \left| \frac{(t-s)^{k-1} \xi_j(t)}{(k-1)!\ \xi_k(t)} \right| \left|x_1^{(j)}(s) - x_2^{(j)}(s)\right| ds \\
&\leq \left(L_1\theta_1 + L_2\theta_2^* + k\gamma^*\theta_3^*\right)(b-a)\left|x_1 - x_2\right|.
\end{aligned}$$

We put

$$\psi = \left(L_1\theta_1 + L_2\theta_2^* + k\gamma^*\theta_3^*\right)(b-a),$$

and we have

$$(1-\psi)\left|x_1 - x_2\right| \leq 0.$$

Since $0 < \psi < 1$, therefore

$$|x_1 - x_2| = 0.$$

Consequently

$$x_1 = x_2.$$

Hence the desired result. ■

Chapter 3

Application of decomposition strategies for standard nonlinear integro-differential equations

In the present chapter, we present four decomposition strategies for finding approximate analytical solutions to standard integro-differential equations. Several numerical applications are provided to display the precision and credibility of the proposed strategies.

3.1 Homotopy analysis strategy (HAS)

In 1992, Chinese mathematician Shijun Liao [**28**] presented an innovative and powerful analytical method for solving nonlinear problems, known as the homotopy analysis strategy (HAS). This strategy has proven to be a very effective and adaptive tool for finding analytical solutions to a wide range of nonlinear problems. One of the most significant advantages of the HAS is that it provides an easy and simple way to adjust and control the convergence of the series solution. It offers numerous important benefits. First, it is applicable even when the given nonlinear problem does not contain any parameters, whether they are small or large. Secondly, it allows for easy adjustment and control of the convergence region and approximation rate of the series solution when needed. Third, by choosing different sets of basis functions, the HAS can be used to efficiently approximate nonlinear problems.

3.1.1 The HAS description

In this portion, we explain the HAS procedure. For this, let us take the nonlinear differential equation

$$\mathcal{N}[x(t)] = f(t), \tag{3.1.1}$$

where $x(t)$ is an unknown function, $\mathcal{N}$ is a nonlinear operator, and $f(t)$ is a known function.

In accordance with the HAS, we construct the so-called zeroth-order deformation equation as

$$(1-q)\mathcal{L}[\phi(t,q) - x_0(t)] = qhH(t)[\mathcal{N}[\phi(t,q)] - f(t)], \tag{3.1.2}$$

where $q \in [0,1]$ is an embedding parameter, $h \neq 0$ is an auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, $H(t) \neq 0$ is an auxiliary function, $x_0(t)$ is an initial guess of $x(t)$, $\phi(t,q)$ is an unknown function.

It is clear that when the embedding parameter $q = 0$ and $q = 1$, Eq. (3.1.2) becomes

$$\phi(t,0) = x_0(t), \ \mathcal{N}[\phi(t,1)] = f(t),$$

respectively.

Thus, as q increases from 0 to 1, the solution $\phi(t,q)$ various from the initial guess $x_0(t)$ to the exact solution $x(t)$. Expanding $\phi(t,q)$ in the Taylor series with respect to the embedding parameter q, we obtain

$$\phi(t,q) = x_0(t) + \sum_{n=1}^{+\infty} q^n x_n(t), \tag{3.1.3}$$

where

$$x_n(t) = \frac{1}{n!} \left. \frac{\partial^n \phi(t,q)}{\partial q^n} \right|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are correctly chosen, the series (3.1.3) converges at $q = 1$, then we have

$$x(t) = x_0(t) + \sum_{n=1}^{+\infty} x_n(t),$$

which must be one of the solutions of the original nonlinear differental equation (3.1.1).

Let us define the vector

$$\overrightarrow{x_n}(t) = \{x_0(t), x_1(t), x_2(t), ..., x_n(t)\}.$$

Differentiating the zeroth-order deformation equation (3.1.2) $n-$times with respect to the embedding parameter q, and then dividing them by $n!$ and finally setting $q = 0$, we get the following n^{th}-order deformation equation

$$\mathcal{L}\left[x_n(t) - \chi_n x_{n-1}(t)\right] = hH(t)\mathcal{R}_n\left[\overrightarrow{x}_{n-1}(t)\right], \tag{3.1.4}$$

where

$$\mathcal{R}_n\left[\overrightarrow{x}_{n-1}(t)\right] = \frac{1}{(n-1)!}\left.\frac{\partial^{n-1}\left[\mathcal{N}\left[\phi(t,q)\right] - f(t)\right]}{\partial q^{n-1}}\right|_{q=0},$$

and

$$\chi_n = \begin{cases} 0, n \leq 1, \\ 1, n > 1. \end{cases}$$

Taking the inverse operator $\mathcal{L}^{-1}$ on both sides of Eq. (3.1.4), we get

$$x_n(t) = \chi_n x_{n-1}(t) + \mathcal{L}^{-1}\left[hH(t)\mathcal{R}_n\left[\overrightarrow{x}_{n-1}(t)\right]\right]. \tag{3.1.5}$$

The nth deformation equation (3.1.5) is a linear that can be easily solved.

Consequently, the approximate solution can be written in this way

$$x(t) = \sum_{n=0}^{N} x_n(t).$$

When $N \to \infty$, we obtain an accurate approximation solution to Eq. (3.1.1) expressed as

$$x(t) = \sum_{n=0}^{+\infty} x_n(t).$$

Example 3.1.1 *Let us consider the nonlinear Volterra integro-differential equation*

$$x'(t) = -1 + \int_0^t x^2(s)ds, \tag{3.1.6}$$

within the initial condition

$$x(0) = 0. \tag{3.1.7}$$

There is no exact solution of Eq (3.1.6).

To resolve Eq. (3.1.6) using HAS, we rewrite it in the operator form

$$\mathcal{N}\left[\phi(t,q)\right] = f(t),$$

where

$$\begin{aligned} \mathcal{N}\left[\phi(t,q)\right] &= \frac{\partial}{\partial t}\phi(t,q) - \int_0^t \phi^2(s,q)ds, \\ f(t) &= -1. \end{aligned}$$

With hypothesis $H(t) = 1$, *we construct the zeroth-order deformation equation as*

$$(1-q)\mathcal{L}\left[\phi(t,q) - x_0(t)\right] = qh\left[\mathcal{N}\left[\phi(t,q)+1\right]\right].$$

When $q = 0$ *and* $q = 1$, *we get*

$$\begin{cases} \phi(t,0) = x_0(t), \\ \mathcal{N}\phi(t,1) = -1. \end{cases}$$

Thus, we obtain the nth*-order deformation equation*

$$\mathcal{L}\left[x_n(t) - \chi_n x_{n-1}(t)\right] = h\mathcal{R}_n\left[\overrightarrow{x}_{n-1}(t)\right], \tag{3.1.8}$$

where

$$\mathcal{R}_n\left[\overrightarrow{x}_{n-1}(t)\right] = \frac{\partial}{\partial t}x_{n-1}(t) - \int_0^t \sum_{j=0}^{n-1} x_j(s)x_{n-1-j}(s)ds + 1.$$

Taking the inverse operator $\mathcal{L}^{-1}$ *on both sides of Eq. (3.1.8), we get the* n^{th} *deformation equation as in*

$$x_n(t) = \chi_n x_{n-1}(t) + \mathcal{L}^{-1}\left[h\mathcal{R}_n\left[\overrightarrow{x}_{n-1}(t)\right]\right]. \tag{3.1.9}$$

Since $\mathcal{L}x(t) = \dfrac{dx(t)}{dt}$, *then we can choose the initial guess as follows*

$$x_0(t) = -t. \tag{3.1.10}$$

From Eq. (3.1.9), we have

$$\begin{aligned} x_1(t) &= h\mathcal{L}^{-1}\left[h\mathcal{R}_1\overrightarrow{x}_0(t)\right], \\ x_2(t) &= x_1(t) + h\mathcal{L}^{-1}\left[\mathcal{R}_2\overrightarrow{x}_1(t)\right], \\ x_3(t) &= x_2(t) + h\mathcal{L}^{-1}\left[\mathcal{R}_3\overrightarrow{x}_2(t)\right], \\ &\vdots \end{aligned} \tag{3.1.11}$$

where

$$\begin{aligned}
\mathcal{R}_1 \overrightarrow{x}_0(t) &= \frac{\partial}{\partial t}x_0(t) - \int_0^t x_0^2(s)ds + 1, \\
\mathcal{R}_2 \overrightarrow{x}_1(t) &= \frac{\partial}{\partial t}x_1(t) - \int_0^t 2x_0(s)x_1(s)ds + 1, \\
\mathcal{R}_3 \overrightarrow{x}_2(t) &= \frac{\partial}{\partial t}x_2(t) - \int_0^t \left(2x_0(s)x_2(s) + x_1^2(s)\right) ds + 1, \\
&\vdots
\end{aligned} \tag{3.1.12}$$

Using Eq. (3.1.10) and the iteration formulas (3.1.11) and (3.1.12), we obtain

$$\begin{aligned}
x_0(t) &= -t, \\
x_1(t) &= -\frac{h}{12}t^4, \\
x_2(t) &= -\left(\frac{h}{12} + \frac{h^2}{12}\right)t^4 - \frac{h^2}{252}t^7, \\
x_3(t) &= -\left(\frac{h}{12} + \frac{h^2}{12} - \frac{(1+h)h^2}{12}\right)t^4 - \left(\frac{(1+2h)h^2}{252} + \frac{h^2}{252}\right)t^7 - \frac{h^3}{6048}t^{10}, \\
&\vdots
\end{aligned}$$

and so on.

Similarly, $x_4(t), x_5(t), ...$ are calculated.

Finally, the approximate solution of Eq. (3.1.16) under the initial condition (3.1.17) at $h = -1$ is given by

$$\begin{aligned}
x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t) + ... \\
&= -t + \frac{1}{12}t^4 - \frac{1}{252}t^7 + \frac{1}{6048}t^{10} + \frac{1}{157248}t^{13} + \frac{37}{158505984}t^{16} + ...
\end{aligned}$$

*which is the same result found in **[15].***

3.2 Variational iteration strategy (VIS)

In the late 1990s, Chinese mathematician Ji-Huan He **[28]** improved an efficient, reliable, and accurate scheme for analytically solving linear and nonlinear differential equations. This technique is called the varational iteration strategy (VIS). Contrary to other strategies, VIS

avoids linearization and other restrictive assumptions by directly incorporating the nonlinear terms into the iterative process. The strategy consists of constructing a correction functional by introducing a Lagrange multiplier and then applying the functional derivative to find the value of this multiplier. This correction functional generates successive approximations that rapidly converge to the exact solution of the problem.

3.2.1 The VIS description

In this portion, we clarify the VIS procedure. For this reason, let us take the general nonlinear differential equation

$$\mathcal{L}\left(x(t)\right)+\mathcal{N}\left(x(t)\right)=f(t), t>0, \tag{3.2.1}$$

within the initial conditions

$$x^{(k)}(0)=x_k, k=0,1,...,m-1, \tag{3.2.2}$$

where $\mathcal{L}=\dfrac{d^m}{dt^m}, m\in\mathbb{N}$ indicates a linear differential operator, $\mathcal{N}$ indicates a nonlinear operator, and f indicates a non-homogeneous term.

According to the VIS, the correction functional formula for (3.2.1) can be constructed as follows

$$x_{n+1}(t)=x_n(t)+\int_0^t\lambda(\tau)\left[\mathcal{L}\left(x_n(\tau)\right)+\mathcal{N}\left(x_n(\tau)\right)-f(\tau)\right]d\tau, \tag{3.2.3}$$

where $x_n(t)$ is the nth approximate solution, $\lambda(\tau)$ is a general Lagrange multiplier that can be identified optimally via the variational theory.

We apply restricted variations to the nonlinear term $\mathcal{N}\left(x_n(\tau)\right)$, i.e, $\delta\mathcal{N}\left(x_n(\tau)\right)=0$.

By taking the variation to the correction functional (3.2.3) and making it stationary, i.e, $\delta x_{n+1}(t)=0$, we get

$$\delta x_{n+1}(t)=\delta x_n(t)+\delta\int_0^t\lambda(\tau)\left[\mathcal{L}\left(x_n(\tau)\right)-f(\tau)\right]d\tau=0. \tag{3.2.4}$$

Notice that $\delta\ f(\tau)=0$, then Eq. (3.2.4) reduces to

$$\begin{aligned}\delta x_{n+1}(t) &= \delta x_n(t)+\int_0^t\lambda(\tau)\left[\delta\mathcal{L}\left(x_n(\tau)\right)-\delta f(\tau)\right]d\tau\\ 0 &= \delta x_n(t)+\int_0^t\lambda(\tau)\delta\mathcal{L}\left(x_n(\tau)\right)d\tau.\end{aligned}$$

Integrating the integral by parts and using $\delta x_n(0) = 0$, we obtain the following Lagrange multipliers

$$\begin{aligned} \lambda(\tau) &= -1 \text{ for } m = 1, \\ \lambda(\tau) &= \tau - t \text{ for } m = 2, \end{aligned}$$

and in general, we have

$$\lambda(\tau) = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \text{ for } m \geq 1. \tag{3.2.5}$$

As a result, by substituting the value of $\lambda(\tau)$ found in Eq. (3.2.5) into Eq. (3.2.3), we obtain the following iteration relation

$$x_{n+1}(t) = x_n(t) + \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left[\mathcal{L}(x_n(\tau)) + \mathcal{N}(x_n(\tau)) - f(\tau)\right] d\tau,$$

Now we define the operator $\mathcal{P}(x(t))$ as

$$\mathcal{P}(x(t)) = \int_0^t \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left[\mathcal{L}(x_n(\tau)) + \mathcal{N}(x_n(\tau)) - f(\tau)\right] d\tau, \tag{3.2.6}$$

and we define the components $y_n(t), n = 0, 1, 2, ...$, as follows

$$\begin{cases} y_0(t) = x_0(t), \\ y_1(t) = \mathcal{P}(y_0(t)), \\ y_2(t) = \mathcal{P}(y_0(t) + y_1(t)), \\ \vdots \\ y_{n+1}(t) = \mathcal{P}(y_0(t) + y_1(t) + ... + y_n(t)). \end{cases} . \tag{3.2.7}$$

Then, we have

$$x(t) = \lim_{n \longrightarrow \infty} x_n(t) = \sum_{k=0}^{\infty} y_n(t).$$

Finally, the solution of Eq. (3.2.1) under the the initial conditions (3.2.2) can be deduced using (3.2.6) and (3.2.7) in the form of a series as

$$x(t) = \sum_{n=0}^{\infty} y_n(t).$$

The initial approximation $y_0(t) = x_0(t)$ can be chosen freely if it satisfies the initial conditions of the problem. The success of the technique depends on the correct choice of the

initial approximation $y_0(t)$. In this alternative approach, we choose the initial approximation as follows

$$y_0(t) = \sum_{k=0}^{m-1} \frac{x_k}{k!} t^k.$$

Example 3.2.1 *Let us consider the nonlinear Volterra integro-differential equation*

$$x'(t) = 1 + \int_0^t x(s)x'(s)ds, \tag{3.2.8}$$

within the initial condition

$$x(0) = 0. \tag{3.2.9}$$

Constructiong a correction functional for Eq. (3.2.8) with using the Lagrange multiplier, we get

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\tau) \left[x_n'(\tau) - 1 - \int_0^\tau x_n(\xi)x_n'(\xi)d\xi \right] d\tau. \tag{3.2.10}$$

Taking the variation with respect to $x_n(t)$, we obtain

$$\begin{aligned}
\delta x_{n+1}(t) &= \delta x_n(t) + \delta \int_0^t \lambda(\tau) \left[x_n'(\tau) - 1 - x_n(\xi)x_n'(\xi)d\xi \right] d\tau \\
&= \delta x_n(t) + \int_0^t \lambda(\tau)\delta \left[x_n'(\tau) - 1 - \int_0^\tau x_n(\xi)x_n'(\xi)d\xi \right] d\tau \\
&= \delta x_n(t) + \int_0^t \lambda(\tau) \left[\delta x_n'(\tau) - \delta(1) - \delta \int_0^\tau x_n(\xi)x_n'(\xi)d\xi \right] d\tau.
\end{aligned}$$

Note that 1 and $x_n(\xi)x_n'(\xi)$ are restricted variables, i.e.

$$\begin{aligned}
\delta(1) &= 0, \\
\int_0^\tau x(\xi)x'(\xi)d\xi &= 0.
\end{aligned}$$

Now, we have

$$\delta x_{n+1}(t) = \delta x_n(t) + \int_0^t \lambda(\tau)\delta x_n'(\tau)d\tau.$$

Integrating the integral by parts, we get

$$\begin{aligned}
\delta x_{n+1}(t) &= \delta x_n(t) + \lambda(\tau)\delta x_n(\tau)|_0^t - \int_0^t \lambda'(\tau)\delta x_n(\tau)d\tau \\
&= (1 + \lambda(t))\,\delta x_n(t) + \int_0^t \lambda'(\tau)\delta x_n(\tau)d\tau.
\end{aligned}$$

The stationary conditions

$$1 + \lambda(t) = 0 \text{ and } \lambda'(\tau) = 0,$$

follow immediately. This in turn gives

$$\lambda(t) = -1.$$

Substituting this value of the Lagrange multiplier $\lambda(t) = -1$,, *into Eq. (3.2.10) gives the iteration formula*

$$x_{n+1}(t) = x_n(t) - \int_0^t \left[x_n'(\tau) - 1 - \int_0^\tau x_n(\xi) x_n'(\xi) d\xi \right] d\tau. \tag{3.2.11}$$

By the VIS, let

$$\mathcal{L}(x(t)) = x'(t) - y(t) = 0,$$

we can choose $x_0(0)$ *from the equation*

$$\mathcal{L}(x_0(t)) = x_0'(t) - 1 = 0. \tag{3.2.12}$$

From Eq. (3.2.12), we can select the initial approximation as $x_0(t) = t$.

Using this selection in Eq. (3.2.11), we obtain the following successive approximations

$$\begin{aligned}
x_0(t) &= t, \\
x_1(t) &= x_0(t) - \int_0^t \left[x_0'(\tau) - 1 - \int_0^\tau x_0(\xi) x_0'(\xi) d\xi \right] d\tau \\
&= t + \frac{t^3}{6}, \\
x_2(t) &= x_1(t) - \int_0^t \left[x_1'(\tau) - 1 - \int_0^\tau x_1(\xi) x_1'(\xi) d\xi \right] d\tau \\
&= t + \frac{t^3}{6} + \frac{t^5}{30} + \frac{t^7}{504}, \\
x_3(t) &= x_2(t) - \int_0^t \left[x_2'(\tau) - 1 - \int_0^\tau x_2(\xi) x_2'(\xi) d\xi \right] d\tau \\
&= t + \frac{t^3}{6} + \frac{t^5}{30} + \frac{17t^7}{2520} + \frac{19t^9}{22680} + \frac{67t^{11}}{831600} + \frac{67t^{13}}{196560} + \frac{t^{15}}{7620480}, \\
&\vdots
\end{aligned}$$

and so on.

As a result, the successive approximation solution can be found as follows

$$x_n(t) = t + \frac{t^3}{6} + \frac{t^5}{30} + \frac{17t^7}{2520} + \frac{19t^9}{22680} + \frac{67t^{11}}{831600} + \frac{67t^{13}}{196560} + \frac{t^{15}}{7620480} + ...$$

In consequence, the exact solution of Eq. (3.2.8) within the initial condition (3.2.9) is given by

$$\begin{aligned} x(t) &= \lim_{n\longrightarrow\infty} x_n(t) \\ &= \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}t\right). \end{aligned}$$

3.3 Daftardar-Gejji-Jafari strategy (DGJS)

Over the past few years, Daftardar-Gejji and Jafari **[14]** proposed a novel strategy for solving evolution equations and nonlinear functional equations. This strategy is known as the Daftardar-Gejji-Jafari strategy (DGJS). It has been successfully utilized to resolve numerous problems and it is simple and easy to apply to linear/nonlinear ordinary differential equations, partial differential equations, integal equations, and integro-differential equations. Furthermore, it is computationally efficient, requiring fewer resources than other strategies, such as the Adomian decomposition strategy. The DGJS converges to the exact solution through successive approximations. Among its advantages are its simplicity and ease of implementation, as well as its ability to yield superior results without imposing restrictive assumptions on the nonlinear terms, unlike some other strategies.

3.3.1 The DGJS description

In this portion, we present the basic concepts related to the DGJS, let us take the general nonlinear functional equation as

$$x(t) = \mathcal{N}(x(t)) + f(t), \tag{3.3.1}$$

where $\mathcal{N}$ is a nonlinear operator and f is a given function.

The solution $x(t)$ to Eq. (3.3.1) can be expressed in the form of the following series

$$x(t) = \sum_{i=0}^{\infty} x_i(t). \tag{3.3.2}$$

The nonlinear operator $\mathcal{N}$ can be decomposed as

$$\mathcal{N}\left(\sum_{i=0}^{\infty} x_i(t)\right) = \mathcal{N}\left(x_0(t)\right) + \sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} x_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} x_j(t)\right)\right\}. \tag{3.3.3}$$

Substituting Eqs. (3.3.2) and (3.3.3) in Eq. (3.3.1), we obtain

$$\sum_{i=0}^{\infty} x_i(t) = f(t) + \mathcal{N}\left(x_0(t)\right) + \sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} x_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} x_j(t)\right)\right\}. \tag{3.3.4}$$

From (3.3.4), the recurrence relation can be defined as

$$\begin{aligned} x_0(t) &= f(t), \\ x_1(t) &= \mathcal{N}\left(x_0(t)\right), \\ x_{n+1}(t) &= \mathcal{N}\left(\sum_{j=0}^{n} x_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{n-1} x_j(t)\right), n = 1, 2, ... \end{aligned} \tag{3.3.5}$$

So we see that

$$x_1(t) + x_2(t) + ... + x_{n+1}(t) = \mathcal{N}\left(x_0(t) + x_1(t) + ... + x_n(t)\right), n = 1, 2, ...,$$

and

$$\begin{aligned} x(t) &= \sum_{i=0}^{\infty} x_i(t) \\ &= f(t) + \mathcal{N}\left(\sum_{i=1}^{\infty} x_i(t)\right). \end{aligned}$$

The n-term approximation solution of equation (3.3.1) is given by

$$\begin{aligned} x(t) &= \sum_{i=0}^{n-1} x_i(t) \\ &= x_0(t) + x_1(t) + ... + x_{n-1}(t). \end{aligned}$$

Example 3.3.1 *Let us consider the nonlinear Fredholm integro-differential equation*

$$x'(t) = 6t^2 - \frac{t}{2} + \int_0^1 tsx^2(s)ds, \tag{3.3.6}$$

within the initial condition

$$x(0) = 0. \tag{3.3.7}$$

Integrating Eq. (3.3.6) from 0 *to* t *and using the initial condition (3.3.7), we obtain*

$$x(t) = 2t^3 - \frac{t^2}{4} + \int_0^t \left(\int_0^1 tsx^2(s)ds \right) dt. \tag{3.3.8}$$

Let

$$\begin{aligned} f(t) &= 2t^3 - \frac{t^2}{4}, \\ \mathcal{N}(x(t)) &= \int_0^t \left(\int_0^1 tsx^2(s)ds \right) dt, \end{aligned}$$

then Eq. (3.3.8) can be written in the form of a functional equation as

$$x(t) = f(t) + \mathcal{N}(x(t)),$$

Via the DGJS, the values of the first approximations are obtained as follows

$$\begin{aligned} x_0(t) &= 2t^3 - 0.25t^2, \\ x_1(t) &= \mathcal{N}(x_0(t)) = 0.183779781t^2, \\ x_2(t) &= \mathcal{N}(x_0(t) + x_1(t)) - \mathcal{N}(x_0(t)) = 0.047665597t^2, \\ x_3(t) &= \mathcal{N}(x_0(t) + x_1(t) + x_2(t)) - \mathcal{N}(x_0(t) + x_1(t)) = 0.013282005t^2 \\ &\vdots \end{aligned}$$

As

$$\begin{aligned} x(t) &= \sum_{i=0}^{3} x_i(t) \\ &= x_0(t) + x_1(t) + x_2(t) + x_3(t) \\ &= 2t^3 - 0.005272636t^2, \end{aligned}$$

it is clear that the iterations converge to the exact solution

$$x(t) = 2t^3,$$

as the number of iterations becomes large.

3.4 Differential transform strategy (DTS)

Professor J.K Zhou [39], was the first to propose the differential transform strategy (DTS) for solving linear and nonlinear boundary value problems, as well as initial value problems. This numerical strategy is based on Taylor series, transforming the boundary value problem involving ordinary differential equations into a recursive relation or a system of recursive relations for calculating the coefficients of the Taylor series. This strategy provides solutions in the form of convergent series with easily computable components.

In this portion, we wll introduce the basic definitions and fundamental theorems of DTS, which is used to resolve nonlinear integro-differential equations.

Definition 3.4.1 *[39] The differential transform of the function $x(t)$ is given as*

$$X(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}x(t)\right]_{t=t_0}, \tag{3.4.1}$$

where $x(t)$ is the original function and $X(k)$ the transformed function

Definition 3.4.2 *[39] The inverse differential transform of $X(k)$ is defined as*

$$x(t) = \sum_{k=0}^{\infty} X(k)(t-t_0)^k. \tag{3.4.2}$$

The combination of Eqs. (3.4.1) and (3.4.2) yields the following result

$$x(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k}{dx^k}x(t)\right]_{t=t_0}(t-t_0)^k. \tag{3.4.3}$$

Particularly, if $t_0 = 0$, then Eq. (3.4.3) becomes

$$x(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k}{dt^k}x(t)\right]_{t=0} t^k.$$

Based on the previous definitions, the essential operations of the DTS are described by the following theorem.

Theorem 3.4.1 *[39] Let $X(k), Y(k)$ and $Z(k)$ be the differential transforms of the functions $x(t), y(t)$ and $z(t)$ respectively, then*

1) If

$$z(t) = \lambda x(t) + \mu y(t), \lambda, \mu \in \mathbb{R},$$

then

$$Z(k) = \lambda X(k) + \mu Y(k).$$

2) If

$$z(t) = x(t)y(t),$$

then

$$Z(k) = \sum_{r=0}^{k} X(r)Y(k-r).$$

3) If $z(t) = x^1(t)x^2(t)...x^n(t)$, *then*

$$Z(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} X^1(k_1)X^2(k_2-k_1) \times ... \times X^{n-1}(k_{n-1}-k_{n-2})X^n(k-k_{n-1}).$$

4) If

$$z(t) = \frac{d^n}{dt^n}x(t),$$

then

$$Z(k) = \frac{(k+n)!}{k!}X(k+n),\ n = 1, 2, ...$$

5) If

$$z(t) = t^m,$$

then

$$Z(k) = \delta(k-m) = \begin{cases} 1, k = m \\ 0, k \neq m \end{cases}.$$

6) If

$$z(t) = \int_0^t x(s)ds,$$

then

$$Z(k) = \frac{X(k-1)}{k},\ k \geq 1,$$

with $Z(0) = 0$.

7) If

$$x(t) = \int_0^t x_1(s)x_2(s)...x_{n-1}(s)x_n(s)ds,$$

then

$$Z(k)=\frac{1}{k}\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}...\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}X^1(k_1)X^2(k_2-k_1)\times...\times X^{n-1}(k_{n-1}-k_{n-2})X^n(k-k_{n-1}),$$

with $Z(0)=0$.

8) If

$$z(t)=\int_0^t h(t-s)x(s)ds,$$

then

$$Z(k)=\sum_{r=0}^{k-1}\frac{r!(k-r-1)!}{k!}H(k)X(k-r-1),\ k\geq 1,$$

with $Z(0)=0$.

3.4.1 The DTS description

Now, to understand the concept of the DTS, let us take the general nonlinear differential equation

$$\mathcal{L}\left(x(t)\right)+\mathcal{N}\left(x(t)\right)=f(t), \tag{3.4.4}$$

within the initial conditions

$$x^{(k)}(0)=x_k, k=0,1,...,m-1, \tag{3.4.5}$$

where $\mathcal{L}=\dfrac{d^m}{dt^m}, m\in\mathbb{N}$ is a linear dfferential operator, $\mathcal{N}$ is nonlinear operators respectively, and $f(t)$ is a known analytical function.

Corresponding to the DTS and Theorem 3.4.1, we can construct the following iteration formula

$$X_{k+n}(k)=\frac{k!}{(k+n)!}\left[-R(k)+F(k)\right], \tag{3.4.6}$$

where $R(k)$ and $F(k)$ are the transformed form of the nonlinear term $\mathcal{N}\left(x(t)\right)$ and $f(t)$, rerspectively.

Through the initial condition (3.4.5), we can write

$$X(0)=x_k. \tag{3.4.7}$$

Exchanging Eq. (3.4.7) into Eq. (3.4.6) and using simple iterative computations, we attain the following $X(k)$ values for $k = 1, 2, ..., n$. The inverse transformation of the set of values $\{X(k)\}_{k=0}^{n}$ yields the following approximate solution

$$\widetilde{x_n}(t) = \sum_{k=0}^{n} X(k)t^k,$$

where n is order of approximation solution.

Thus, the exact solution of Eq. (3.4.4) within the initial conditions (3.4.5) is given as

$$x(t) = \lim_{n\to\infty} \widetilde{x_n}(t) = \sum_{k=0}^{\infty} X(k)t^k.$$

Example 3.4.1 *Let us consider the system of nonlinear Volterra integro-differential equation*

$$\begin{cases} x''(t) = 1 - \frac{1}{3}t^3 - \frac{1}{2}\left(y'(t)\right)^2 + \frac{1}{2}\int_0^t \left[x^2(s) + y^2(s)\right] ds, \\ y''(t) = -1 + t^2 - tx(t) + \frac{1}{4}\int_0^t \left[x^2(s) - y^2(s)\right] ds, \end{cases}, \tag{3.4.8}$$

within the initial conditions

$$\begin{cases} x(0) = 1 \\ y(0) = -1 \end{cases}, \quad \begin{cases} x'(0) = 2, \\ y'(0) = 0. \end{cases} \tag{3.4.9}$$

Taking the differential transform of systems (3.4.8) and (3.4.9), one gets

$$\begin{cases} X(0) = 1, \\ Y(0) = -1 \end{cases},$$

$$\begin{cases} X(1) = 2, \\ Y(1) = 0, \end{cases},$$

$$\begin{cases} X(k+2) = \dfrac{k!}{(k+2)!} \left(\begin{array}{c} \delta(k) - \frac{1}{3}\delta(k-3) \\ -\frac{1}{2}\sum_{r=0}^{k}(r+1)(k-r+1)Y(r+1)Y(k-r+1) \\ +\frac{1}{2}\left(\sum_{r=0}^{k}\frac{X(r)X(k-r)}{k} + \sum_{r=0}^{k}\frac{Y(r)Y(k-r)}{k}\right) \\ \frac{1}{k}\sum_{r=0}^{k}\frac{X(r)X(k-r)}{k} \end{array} \right) \\ Y(k+2) = \dfrac{k!}{(k+2)!} \left(\begin{array}{c} -\delta(k) + \delta(k-2) - X(k-2) \\ +\frac{1}{4}\left(\sum_{r=0}^{k}\frac{X(r)X(k-r)}{k} + \sum_{r=0}^{k}\frac{Y(r)Y(k-r)}{k}\right) \end{array} \right) \end{cases} \tag{3.4.10}$$

Based on the recurrence relation (3.4.10), we get

$$\begin{cases} X(0)=1 \\ Y(0)=-1 \end{cases},$$
$$\begin{cases} X(1)=2 \\ Y(1)=0 \end{cases},$$
$$\begin{cases} X(2)=\dfrac{1}{2!} \\ Y(2)=-\dfrac{1}{2!} \end{cases},$$
$$\begin{cases} X(3)=\dfrac{1}{3!} \\ Y(3)=-\dfrac{1}{3!} \end{cases},$$
$$\begin{cases} X(4)=\dfrac{1}{4!} \\ Y(4)=-\dfrac{1}{4!} \end{cases},$$
$$\vdots$$

and so on.

Then, the approximate analytical solution can be presented as

$$\begin{cases} \widetilde{x_n}(t)=\displaystyle\sum_{k=0}^{n} X(k)t^k = 1+2t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+....+\frac{t^n}{n!} \\ = t+\left(1+t+\dfrac{t^2}{2!}+\dfrac{t^3}{3!}+\dfrac{t^4}{4!}+....+\dfrac{t^n}{n!}\right)=t+\displaystyle\sum_{k=0}^{n}\frac{t^k}{k!}, \\ \widetilde{y_n}(t)=\displaystyle\sum_{k=0}^{n} Y(k)t^k = -1-\frac{t^2}{2!}-\frac{t^3}{3!}-\frac{t^4}{4!}-....-\frac{t^n}{n!} \\ = t-\left(1+t+\dfrac{t^2}{2!}+\dfrac{t^3}{3!}+\dfrac{t^4}{4!}+....+\dfrac{t^n}{n!}\right)=t-\displaystyle\sum_{k=0}^{n}\frac{t^k}{k!} \end{cases}.$$

Thus, the exact solution of Eq. (3.4.8) within the initial conditions (3.4.9) is given by

$$\begin{cases} x(t)=\lim\limits_{n\to\infty}\widetilde{x_n}(t)=\lim\limits_{n\to\infty}\left(t+\displaystyle\sum_{k=0}^{n}\frac{t^k}{k!}\right)=t+e^t \\ y(t)=\lim\limits_{n\to\infty}\widetilde{y_n}(t)=\lim\limits_{n\to\infty}\left(t-\displaystyle\sum_{k=0}^{n}\frac{t^k}{k!}\right)=t-e^t \end{cases}.$$

Chapter 4

Adaptation of decomposition strategies to resolve Caputo fractional integro-differential equations

In the present chapter, we focus on adapting the HAS, VIS, DGJS, and DTS to resolve fractional integro-differential equations using the Caputo fractional derivative. The accuracy and reliability of these strategies will be demonstrated by solving different kinds of proposed equations.

Let us consider the Caputo fractional Fredholm-Volterra integro-differential equation of the form

$$^{C}D^{\alpha}x(t) = f(t) + \int_0^1 K_1(t,s)\, G_1(x(s))ds + \int_0^t K_2(t,s)\, G_2(x(s))ds, \tag{4.1.1}$$

within the initial conditions

$$x^{(k)}(0) = C_k \in \mathbb{R}, k = 0, 1, 2, ..., m-1, \tag{4.1.2}$$

where $^{C}D^{\alpha} = \dfrac{d^{\alpha}}{dt^{\alpha}}$ indicates the Caputo fractional operator of the function $x(t)$ of order α with $m-1 < \alpha \leq m, m \in \mathbb{N}^*$, $x : I = [0,1] \to \mathbb{R}$ is the continuous function which has to be determined, $f : I \to \mathbb{R}$ and $K_i : I \times I \to \mathbb{R}$, $i = 1, 2$ are continuous functions, and $G_i : I \times I \to \mathbb{R}$, $i = 1, 2$ are nonlinear functions.

4.1 The HAS methodology

To investigate the HAS methodology, let us examine Eq. (4.1.1) within the initial conditions (4.1.2), then we can define

$$\mathcal{N}\left[\phi(t,q)\right] = {}^{C}D^{\alpha}\phi(t,q) - f(t) - \int_0^1 K_1(t,s)\, G_1(\phi(s,q))ds - \int_0^t K_2(t,s)\, G_2(\phi(s,q))ds. \tag{4.1.3}$$

Now, we construct the zero-order deformation equation

$$(1-q)\,{}^{C}D^{\alpha}\left[\phi(t,q) - x_0(t)\right] = qh\mathcal{N}\left[\phi(t,q)\right], \tag{4.1.4}$$

within the following initial conditions

$$x_0(t) = \phi^{(k)}(0,q) = C_k, k = 0,1,2,...,n-1, \tag{4.1.5}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $x_0(t)$ is an initial guess of the solution $x(t)$ and $\phi(t,q)$ is an unknown function on the independent variables t and q.

Also we assume that

$${}^{C}D^{\alpha}\left[C\right] = 0,$$

where C is arbitrary constant.

When the parameter q increases from 0 to 1, then the homotopy solution $\phi(t,q)$ varies from $x_0(t)$ to solution $x(t)$ of the original equation (4.1.1).

Using the parameter q, $\phi(t,q)$ can be expanded in Taylor series as follows

$$\phi(t,q) = x_0(t) + \sum_{n=1}^{+\infty} q^n x_n(t),$$

where

$$x_n(t) = \frac{1}{n!}\left.\frac{\partial^n \phi(t,q)}{\partial q^n}\right|_{q=0}.$$

Assuming that the auxiliary parameter h is properly selected so that the above series is convergent when $q = 1$, then the solution $x(t)$ can be given by

$$x(t) = \phi(t,1) = x_0(t) + \sum_{n=1}^{+\infty} x_n(t).$$

Differentiating Eqs. (4.1.4) and (4.1.5) $n-$times with respect to q, then setting $q = 0$, and finally dividing them by $n!$, we get the $n^{th}-$order deformation equation

$$ {}^{C}D^{\alpha}\left[x_n(t) - \chi_n x_{n-1}(t)\right] = h\mathcal{R}_n\left[\overrightarrow{x_n}(t)\right], \tag{4.1.6} $$

within the following initial conditions

$$ x_n(0) = 0, \tag{4.1.7} $$

where

$$ \begin{aligned} \mathcal{R}_{i,n}\left[\overrightarrow{x_n}(t)\right] &= \frac{1}{(n-1)!}\left.\frac{\partial^{n-1}\mathcal{N}\left[\phi(t,q)\right]}{\partial q^{n-1}}\right|_{q=0} \\ &= {}^{C}D^{\alpha}x_{n-1}(t) - \int_0^1 K_1(t,s)\,G_1(x_{n-1}(s))ds - \int_0^t K_2(t,s)\,G_2(x_{n-1}(s))ds \\ &\quad -(1-\chi_n)\,f(t). \end{aligned} $$

and

$$ \overrightarrow{x_n}(t) = \left\{x_0(t), x_1(t), x_2(t), ..., x_n(t)\right\}, $$

and

$$ \chi_n = \begin{cases} 0, n \leq 1 \\ 1, n > 1 \end{cases}. $$

Implementing the Riemann-Liouville fractional integral operator I^{α} to both sides of Eq.(4.1.6) and exploiting relation (1.1.17), we have

$$ x_n(t) = (\chi_n + h)\,x_{n-1}(t) - hI^{\alpha}\begin{bmatrix} \int_0^1 K_1(t,s)\,G_1(x_{n-1}(s))ds \\ +\int_0^t K_2(t,s)\,G_2(x_{n-1}(s))ds + (1-\chi_n)\,f(t) \end{bmatrix}, \tag{4.1.8} $$

Therefore, we acquire $x_1(t), x_2(t), x_3(t), ...$ using Eq.(4.1.8). The Nth-order approximation of $x(t)$ is given by

$$ x(t) = \sum_{n=0}^{N} x_n(t), \tag{4.1.9} $$

and for $N \longrightarrow \infty$, we get an accurate approximation of Eq. (4.1.1) within the initial conditions (4.1.2) as

$$ x(t) = \sum_{n=0}^{+\infty} x_n(t). $$

Example 4.1.1 *Let us examine the Caputo fractional Volterra integro-differential equation*

$$ {}^{C}D^{0.5}x(t) = \frac{32-3\sqrt{\pi}}{12\sqrt{\pi}}t^{1.5} + \int_0^t \frac{s}{t^{2.5}}x(s)ds, \tag{4.1.10} $$

within the initial condition

$$ x(0) = 0. \tag{4.1.11} $$

Based on Eqs. (4.1.3) and (4.1.10), can be written as

$$ \mathcal{N}\left[\phi(t,q)\right] = {}^{C}D^{0.5}\phi(t,q) - \frac{32-3\sqrt{\pi}}{12\sqrt{\pi}}t^{1.5} - \int_0^t \frac{s}{t^{2.5}}\phi(s,q)ds. $$

Now, using the nth-order deformation equation (4.1.6) and initial conditions (4.1.7), and iterative formula (4.1.8) we can write

$$ x_n(t) = (\chi_n + h)\, x_{n-1}(t) - hI^{0.5}\left[(1-\chi_n)\,\frac{32-3\sqrt{\pi}}{12\sqrt{\pi}}t^{1.5} + \int_0^t \frac{s}{t^{2.5}}x_{n-1}(s)ds\right]. $$

Then,

$$ \begin{aligned} x_0(t) &= 0, \\ x_1(t) &= h\left(\frac{3\sqrt{\pi}}{32}-1\right)t^2, \\ x_2(t) &= h\left(1-h\left(\frac{3\sqrt{\pi}}{32}-1\right)\right)\left(\frac{3\sqrt{\pi}}{32}-1\right)t^2, \\ x_3(t) &= h\left(1-h\left(\frac{3\sqrt{\pi}}{32}-1\right)\right)^2\left(\frac{3\sqrt{\pi}}{32}-1\right)t^2, \\ &\vdots \\ x_n(t) &= h\left(1-h\left(\frac{3\sqrt{\pi}}{32}-1\right)\right)^{n-1}\left(\frac{3\sqrt{\pi}}{32}-1\right)t^2. \end{aligned} $$

According to Eq. (4.1.9), the HAS series solution can be written as

$$ \begin{aligned} x(t) &= \sum_{n=0}^{N} x_n(t) \\ &= h\left(\frac{3\sqrt{\pi}}{32}-1\right)\left[1+\left(1-h\left(\frac{3\sqrt{\pi}}{32}-1\right)\right)+\ldots+\left(1-h\left(\frac{3\sqrt{\pi}}{32}-1\right)\right)^{N-1}\right]t^2. \end{aligned} $$

For $N \longrightarrow \infty$, the exact solution of Eq. (4.1.10) within the initial condition (4.1.11) when $\dfrac{64}{3\sqrt{\pi}-32} < h < 0$ is

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} x_n(t) \\ &= h\left(\frac{3\sqrt{\pi}}{32}-1\right)\left[1+\left(1-h\left(\frac{3\sqrt{\pi}}{32}-1\right)\right)+\left(1-h\left(\frac{3\sqrt{\pi}}{32}-1\right)\right)^2+...\right]t^2 \\ &= h\left(\frac{3\sqrt{\pi}}{32}-1\right)\left(\frac{1}{1-\left(1-h\left(\frac{3\sqrt{\pi}}{32}-1\right)\right)}\right)t^2 \\ &= t^2. \end{aligned}$$

Example 4.1.2 *Let us examine the Caputo fractional Fredholm-Volterra integro-differential equation*

$$^{C}D^{0.75}x(t) = \frac{6t^{2.25}}{\Gamma(3.25)} - \frac{t^2e^t}{5}x(t) + \int_0^1 (4-s^{-3})x(s)ds + \int_0^t e^t s x(s)ds, \tag{4.1.12}$$

within the initial condition

$$x(0) = 0. \tag{4.1.13}$$

Based on Eqs. (4.1.3) and (4.1.12), can be written as

$$\mathcal{N}\left[\phi(t,q)\right] = ^{C} D^{0.75}\phi(t,q) - \frac{6t^{2.25}}{\Gamma(3.25)} + \frac{t^2e^t}{5}\phi(t,q) - \int_0^1 (4-s^{-3})\phi(s,q)ds - \int_0^t e^t s\phi(s,q)ds.$$

Now, using the nth-order deformation equation (4.1.6) and initial conditions (4.1.7), and Iterative formulan (4.1.8) we can write

$$x_n(t) = (\chi_n + h)\, x_{n-1}(t) - hI^{0.75}\left[\begin{array}{c}(1-\chi_n)\dfrac{6t^{2.25}}{\Gamma(3.25)} - \dfrac{t^2e^t}{5}x_{n-1}(t) - \displaystyle\int_0^1 (4-s^{-3})x_{n-1}(s)ds \\ \qquad\qquad - \displaystyle\int_0^t e^t s x_{n-1}(s)ds\end{array}\right].$$

Then,

$$\begin{aligned} x_0(t) &= 0, \\ x_1(t) &= -ht^3, \\ x_2(t) &= -h\left(1+h\right)t^3, \\ x_3(t) &= -h\left(1+h\right)^2 t^3, \\ &\vdots \\ x_n(t) &= -h\left(1+h\right)^{n-1} t^3. \end{aligned}$$

According to Eq. (4.1.9), the HAS series solution can be written as

$$\begin{aligned} x(t) &= \sum_{n=0}^{N} x_n(t) \\ &= -h\left[1+1+\left(1+h\right)+\left(1+h\right)^2 ... +\left(1+h\right)^{N-1}\right] t^3. \end{aligned}$$

For $N \longrightarrow \infty$, the exact solution of Eq. (4.1.10) within the initial condition (4.1.11) when $-2 < h < 0$ is

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} x_n(t) \\ &= -h\left[1+1+\left(1+h\right)+\left(1+h\right)^2+...\right] t^3 \\ &= -h\left(\frac{1}{1-\left(1+h\right)}\right) t^3 \\ &= t^3. \end{aligned}$$

4.2 The VIS methodology

To instance the VIA methodology, let us consider Eq. (4.1.1) within the initial conditions (4.1.2), which may be rewritten as

$${}^{C}D^{\alpha}x(t) - f(t) - \int_0^1 K_1\left(t,s\right) G_1\left(x(s)\right)ds - \int_0^t K_2\left(t,s\right) G_2\left(x(s)\right)ds = 0. \tag{4.2.1}$$

Multiplying Eq. (4.2.1) by the general Lagrange multiplier $\lambda(t)$, we obtain

$$\lambda(t)\left[{}^{C}D^{\alpha}x(t) - f(t) - \int_0^1 K_1\left(t,s\right) G_1\left(x(s)\right)ds - \int_0^t K_2\left(t,s\right) G_2\left(x(s)\right)ds\right] = 0. \tag{4.2.2}$$

Now we take the Riemann-Liouville fractional integral operator I^{α} on both sides of Eq. (4.2.2), which gives us

$$I^{\alpha}\left(\lambda(t)\left[{}^{C}D^{\alpha}x(t)-f(t)-\int_0^1 K_1\left(t,s\right)G_1\left(x(s)\right)ds-\int_0^t K_2\left(t,s\right)G_2\left(x(s)\right)ds\right]\right)=0.$$

Based on the VIS, the descriptive formula for the correction functional for Eq. (4.1.1) is as

$$x_{n+1}(t)=x_n(t)+I^{\alpha}\left(\lambda(t)\left[\begin{array}{c}{}^{C}D^{\alpha}x_n(t)-f(t)-\int_0^1 K_1\left(t,s\right)G_1\left(x_n(s)\right)ds\\ -\int_0^t K_2\left(t,s\right)G_2\left(x_n(s)\right)ds\end{array}\right]\right). \quad (4.2.3)$$

In this case, it is difficult to determine the Lagrange multiplier through Eq. (4.2.3), because this equation contains a fractional derivative and an integral. To overcome this difficulty, we approximate terms I^{α} and ${}^{C}D^{\alpha}$ for $m-1<\alpha\leq m$ using $m-$order derivatives and integrals. Therefore, the correction functional becomes

$$x_{n+1}(t)=x_n(t)+\int_0^t\lambda(\tau)\left[\begin{array}{c}\frac{d^m}{d\tau^m}x_n(\tau)-f(\tau)-\int_0^1 K_1\left(\tau,s\right)G_1\left(x_n(s)\right)ds\\ -\int_0^t K_2\left(\tau,s\right)G_2\left(x_n(s)\right)ds\end{array}\right]d\tau.$$

Consequently, by taking the first variation with respect to the independent variable x_n and considering that $\delta x_n(0)=0$, we obtain

$$\delta x_{n+1}(t)=\delta x_n(t)+\delta\int_0^t\lambda(\tau)\left[\begin{array}{c}\frac{d^m}{d\tau^m}x_n(\tau)-f(\tau)-\int_0^1 K_1\left(\tau,s\right)G_1\left(x_n(s)\right)ds\\ -\int_0^t K_2\left(\tau,s\right)G_2\left(x_n(s)\right)ds\end{array}\right]d\tau, \quad (4.2.4)$$

where $f(\tau)$, $G_1\left(x_n(s)\right)$, and $G_2\left(x_n(s)\right)$ are considered as restricted variations, i.e, $\delta\, f(\tau)=0$, $\delta G_1\left(x_n(s)\right)=0$, and $\delta G_2\left(x_n(s)\right)=0$, therefore Eq.(4.2.4) will simplify to

$$\delta x_{n+1}(t)=\delta x_n(t)+\delta\int_0^t\lambda(\tau)\frac{d^m}{d\tau^m}x_n(\tau)d\tau. \quad (4.2.5)$$

Using the method of integration by parts on Eq. (4.2.5), we obtain the following Lagrange multipliers

$$\begin{aligned}\lambda(\tau) &= -1 \text{ for } m=1,\\ \lambda(\tau) &= \tau-t \text{ for } m=2,\end{aligned}$$

and in general, we have

$$\lambda(\tau) = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \text{ for } m \geq 1.$$

After substituting the value of $\lambda(\tau)$ into Eq. (4.2.3)„ we obtain the following iteration relation

$$x_{n+1}(t) = x_n(t) + I^\alpha \left(\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \left[\begin{array}{c} {}^C D^\alpha x_n(t) - f(t) - \int_0^1 K_1(t,s)\, G_1(x_n(s))ds \\ - \int_0^t K_2(t,s)\, G_2(x_n(s))ds \end{array} \right] \right).$$

Example 4.2.1 *Considering the nonlinear Caputo fractional Fredholm integro-differential equation*

$${}^C D^\alpha x(t) = 1 - \frac{t}{4} + \int_0^1 tsx^2(s)ds, \tag{4.2.6}$$

within the initial condition

$$x(0) = 0. \tag{4.2.7}$$

where ${}^C D^\alpha$ indicates the Caputo fractional derivative operator of order $0 < \alpha \leq 1$.

When the VIS methodology exhibited in portion 4.2 is implemented to Eqs. (4.2.6) and (4.2.7), the results obtained are

$$x_{n+1}(t) = x_n(t) - I^\alpha \left[{}^C D^\alpha x_n(t) - 1 + \frac{t}{4} - \int_0^1 tsx_n^2(s)ds \right]. \tag{4.2.8}$$

and the initial approximation

$$x_0(t) = x(0) + I^\alpha [1] = \frac{t^\alpha}{\Gamma(\alpha + 1)}.$$

According to relation (4.2.8) , we get

$$\begin{aligned} x_1(t) &= 0, \\ x_2(t) &= 0, \\ x_3(t) &= 0, \\ &\vdots \\ x_n(t) &= 0, n \geq 1. \end{aligned}$$

So that the analytic solution is

$$\begin{aligned} x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + ... \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)} + 0 + 0 + 0 + ... \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{4.2.9}$$

Setting $\alpha = 1$ in Eq. (4.2.9), we obtain the exact solution of Eq. (4.2.6) within the initial condition (4.2.7) as

$$x(t) = t.$$

Example 4.2.2 *Considering the nonlinear Caputo fractional Volterra integro-differential equation*

$${}^C D^{6/5} x(t) = \frac{5t^{4/5}}{2\Gamma\left(\frac{4}{5}\right)} - \frac{t^9}{252} + \int_0^t (t-s)\, x^3(s) ds, \tag{4.2.10}$$

within the initial conditions

$$x(0) = 0,\ x'(0) = 0. \tag{4.2.11}$$

When the VIS methodology exhibited in portion 4.2 is implemented to Eqs. (4.2.10) and (4.2.11), the results obtained are

$$x_{n+1}(t) = x_n(t) - I^{6/5}\left[{}^C D^{6/5} x_n(t) - \frac{5t^{4/5}}{2\Gamma\left(\frac{4}{5}\right)} + \frac{t^9}{252} - \int_0^t (t-s)\, x_n^3(s) ds\right]. \tag{4.2.12}$$

and the initial approximation

$$x_0(t) = x(0) + x'(0)t + I^{6/5}\left[\frac{5t^{4/5}}{2\Gamma\left(\frac{4}{5}\right)}\right] = t^2.$$

According to relation (4.2.12) , we get

$$\begin{aligned} x_1(t) &= 0, \\ x_2(t) &= 0, \\ x_3(t) &= 0, \\ &\vdots \\ x_n(t) &= 0, n \geq 1. \end{aligned}$$

So that the exact solution of Eq. (4.2.10) within the initial conditions (4.2.11) as

$$\begin{aligned} x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + ... \\ &= t^2 + 0 + 0 + 0 + ... \\ &= t^2. \end{aligned}$$

4.3 The DGJS methodology

To give the outlie of the DGJS methodology to resolve Eq. (4.1.1) within the initial conditions (4.1.2), we begin by applying the fractional Riemann-Liouville integral I^α on Eq. (4.1.1), which yields the following expression

$$I^\alpha \left[{}^C D^\alpha x(t)\right] = I^\alpha \left[f(t)\right] + I^\alpha \left[\int_0^1 K_1(t,s)\, G_1(x(s))ds + \int_0^t K_2(t,s)\, G_2(x(s))ds\right]. \quad (4.3.1)$$

Now, empolying relation (1.1.17) and the given initial conditions (4.1.2), we can write Eq. (4.3.1) as

$$x(t) = \sum_{k=0}^{m-1} \frac{x^{(k)}(0)}{k!} t^k + I^\alpha \left[f(t)\right] + I^\alpha \left[\int_0^1 K_1(t,s)\, G_1(x(s))ds + \int_0^t K_2(t,s)\, G_2(x(s))ds\right]. \quad (4.3.2)$$

Next assume that

$$\begin{aligned} g(t) &= \sum_{k=0}^{m-1} \frac{C_k}{k!} t^k + I^\alpha \left[f(t)\right], \\ \mathcal{N}(x(t)) &= I^\alpha \left[\int_0^1 K_1(t,s)\, G_1(x(s))ds + \int_0^t K_2(t,s)\, G_2(x(s))ds\right]. \end{aligned}$$

Thus, Eq. (4.3.2) can be written in the following form

$$x(t) = g(t) + \mathcal{N}(x(t)), \quad (4.3.3)$$

where g is a known function and $\mathcal{N}$ is given nonlinear operator of $x(t)$.

The DGJS represent the solution $x(t)$ as an infinite series as

$$x(t) = \sum_{i=0}^{\infty} x_i(t). \quad (4.3.4)$$

Inserting Eq (4.3.4) into Eq.(4.3.3) gives

$$\sum_{i=0}^{\infty} x_i(t) = g(t) + \mathcal{N}\left(\sum_{i=0}^{\infty} x_i(t)\right).$$

The nonlinear operator $\mathcal{N}$ is decomposed as

$$\mathcal{N}\left(\sum_{i=0}^{\infty} x_i(t)\right) = \mathcal{N}\left(x_0(t)\right) + \sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} x_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} x_j(t)\right)\right\}.$$

Thus, Eq. (4.3.3) is given as

$$\sum_{i=0}^{\infty} x_i(t) = g(t) + \mathcal{N}\left(x_0(t)\right) + \sum_{i=1}^{\infty}\left\{\mathcal{N}\left(\sum_{j=0}^{i} x_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} x_j(t)\right)\right\}. \tag{4.3.5}$$

The DGJS is based on the assumption that the function g, resulting from the source term and the given initial conditions, can be divided into two parts, g_1 and g_2. Based on this assumption, we define $g = g_1 + g_2$ and by applying a slight modification to the components $x_0(t)$ and $x_1(t)$, the recurrence sequence is defined as follows

$$\begin{aligned} x_0(t) &= g_1(t), \\ x_1(t) &= g_2(t) + \mathcal{N}\left(x_0(t)\right), \\ x_{n+1}(t) &= \mathcal{N}\left(\sum_{j=0}^{n} x_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{n-1} x_j(t)\right), n = 1, 2, ... \end{aligned} \tag{4.3.6}$$

Finally, we find the solution of Eq. (4.1.1) within the initial conditions (4.1.2) by calculating the terms $x_0(t), x_1(t), x_2(t), ...$ of the recurrence sequence (4.3.6) and using Eq.(4.3.4).

Example 4.3.1 *Examine the nonlinear Caputo fractional Volterra integro-differential equation that is as follows*

$${}^{C}D^{\alpha}x(t) = 2t - \frac{t^6}{6} + 5\int_0^t (t-s)\, x^2(s)ds, \tag{4.3.7}$$

within the initial condition

$$x(0) = 1. \tag{4.3.8}$$

where ${}^{C}D^{\alpha}$ indicates the Caputo fractional derivative operator of order $0 < \alpha \leq 1$.

We follow the steps outlined in the DGJS methodology, as indicated in portion 4.3, to obtain the result

$$\begin{aligned} g(t) &= I^{\alpha}\left[2t-\frac{t^6}{6}\right], \\ \mathcal{N}(x(t)) &= 5I^{\alpha}\left[\int_0^t (t-s)\,x^2(s)ds\right]. \end{aligned} \tag{4.3.9}$$

By utilizing the recurrence sequence (4.3.6) and (4.3.9), we get

$$\begin{aligned} x_0(t) &= I^{\alpha}\left[2t\right], \\ x_1(t) &= I^{\alpha}\left[\frac{t^6}{6}\right]+\mathcal{N}\left(x_0(t)\right), \\ x_{n+1}(t) &= \mathcal{N}\left(\sum_{j=0}^{n} x_j(t)\right)-\mathcal{N}\left(\sum_{j=0}^{n-1} x_j(t)\right), n=1,2,... \end{aligned} \tag{4.3.10}$$

By exploiting the recurrence sequence (4.3.10) we can arrive at

$$\begin{aligned} x_0(t) &= \frac{2}{\Gamma(\alpha+2)}t^{\alpha+1}, \\ x_1(t) &= \frac{20\Gamma(2\alpha+3)}{\Gamma^2(\alpha+2)\Gamma(3\alpha+5)}t^{3\alpha+4}-\frac{5!}{\Gamma(\alpha+7)}t^{\alpha+6}, \\ x_2(t) &= \frac{400\Gamma(2\alpha+3)\Gamma(4\alpha+6)}{\Gamma^3(\alpha+2)\Gamma(3\alpha+5)\Gamma(5\alpha+8)}t^{5\alpha+7}-\frac{20.5!\Gamma(2\alpha+8)}{\Gamma(\alpha+2)\Gamma(\alpha+7)\Gamma(3\alpha+10)}t^{3\alpha+9} \\ &\quad+\frac{2000\Gamma^2(2\alpha+3)\Gamma(6\alpha+9)}{\Gamma^4(\alpha+2)\Gamma^2(3\alpha+5)\Gamma(7\alpha+11)}t^{7\alpha+10}-\frac{200.5!\Gamma(2\alpha+3)\Gamma(4\alpha+11)}{\Gamma^2(\alpha+2)\Gamma(3\alpha+5)\Gamma(5\alpha+13)}t^{5\alpha+12} \\ &\quad+\frac{(5!)^2\Gamma^2(2\alpha+13)}{\Gamma^2(\alpha+7)\Gamma^2(3\alpha+15)}t^{3\alpha+14} \\ &\quad\vdots \end{aligned}$$

and so on.

In the case of $\alpha=1$*, we get*

$$\begin{aligned} x_0(t) &= t^2, \\ x_1(t) &= 0, \\ x_2(t) &= 0, \\ &\vdots \\ x_n(t) &= 0, n\geq 1. \end{aligned}$$

Hence, the exact solution of Eq. (4.3.7) within the initial condition (4.3.8) is

$$\begin{aligned} x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + ... \\ &= t^2 + 0 + 0 + 0 + ... \\ &= t^2. \end{aligned}$$

Example 4.3.2 *Examine the nonlinear Caputo fractional Volterra integro-differential equation that is as follows*

$$^{C}D^{\alpha}x(t) = 1 + \int_0^t e^{-s}x^2(s)ds, \tag{4.3.11}$$

within the initial condition

$$x(0) = 1,\ x'(0) = 1,\ x''(0) = 1,\ x'''(0) = 1, \tag{4.3.12}$$

where $^{C}D^{\alpha}$ *indicates the Caputo fractional derivative operator of order* $3 < \alpha \leq 4$.

We follow the steps outlined in the DGJS methodology, as indicated in portion 4.3, to obtain the result

$$\begin{aligned} g(t) &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + I^{\alpha}\left[1\right], \\ \mathcal{N}(x(t)) &= I^{\alpha}\left[\int_0^t e^{-s}x^2(s)ds\right]. \end{aligned} \tag{4.3.13}$$

Similarly to the procedure in the preceding example, we take the truncated Taylor expansions for the exponential term in (4.3.13), i.e.

$$e^{-s} \approx 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}$$

By utilizing the recurrence sequence (4.3.6) and (4.3.13), we get

$$\begin{aligned} x_0(t) &= 1, \\ x_1(t) &= t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \mathcal{N}(x_0(t)), \\ x_{n+1}(t) &= \mathcal{N}\left(\sum_{j=0}^{n} x_j(t)\right) - \mathcal{N}\left(\sum_{j=0}^{n-1} x_j(t)\right), n = 1, 2, ... \end{aligned} \tag{4.3.14}$$

By exploiting the recurrence sequence (4.3.14) we can arrive at

$$\begin{aligned} x_0(t) &= 1, \\ x_1(t) &= t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\ &\quad + \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} \\ &\quad \vdots \end{aligned}$$

and so on.

Hence, the approximate solution is given by

$$\begin{aligned} x(t) &= x_0(t) + x_1(t) + x_2(t) + x_3(t) + ... \\ &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\ &\quad + \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} + ... \end{aligned}$$

In the case of $\alpha = 4$, we get

$$\begin{aligned} x(t) &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!} + ... \\ &= e^t, \end{aligned}$$

that is the exact solution of Eq. (4.3.11) within the initial condition (4.3.12).

4.4 The FDTS methodology

Considering the Caputo fractional Fredholm-Volterra integro-differential equation of the form

$$^C D^{n\alpha} x(t) = f(t) + \int_0^1 K_1(t,s)\, G_1(x(s))ds + \int_0^t K_2(t,s)\, G_2(x(s))ds, \tag{4.4.1}$$

within the initial conditions

$$D^{i\alpha} x(0) = C_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, \tag{4.4.2}$$

where $^C D^{n\alpha} = \dfrac{d^{n\alpha}}{dt^{n\alpha}}$ indicates the Caputo fractional operator of the function $x(t)$ of order $n\alpha$ with $\dfrac{n-1}{n} < \alpha \leq 1, n \in \mathbb{N}^*$, $x : I = [0,1] \to \mathbb{R}$ is the continuous function which has

to be determined, $f : I \to \mathbb{R}$ and $K_i : I \times I \to \mathbb{R}$, $i = 1, 2$ are continuous functions, and $G_i : I \times I \to \mathbb{R}$, $i = 1, 2$ are nonlinear functions.

To adapt the FDTS methodology to resolve Eq. (4.4.1) within the initial conditions (4.4.2), we need some definitions and properties of the fractional differential transform (FDT) that we will use to achieve our goal.

Definition 4.4.1 ***[16]*** *(a) The FDT of the function $x(t)$ is defined as*

$$X(k) = \frac{1}{\Gamma(k\alpha + 1)} \left[\frac{d^{k\alpha}}{dt^{k\alpha}} x(t) \right]_{t=0},$$

where α is a parameter describing the order of fractional derivative in the Caputo sense, $x(t)$ is the original function and $X(k)$ the transformed function.

(b) The inverse fractional differential transform of $X(k)$ is defined as

$$x(t) = \sum_{k=0}^{\infty} X(k) t^{k\alpha}.$$

(c) From (a) and (b), we have

$$x(t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \left[\frac{d^{k\alpha}}{dt^{k\alpha}} x(t) \right]_{t=0} t^{k\alpha}.$$

The basic property of FDTS is given by the following theorem.

Theorem 4.4.1 ***[16]*** *Let $X(k)$ and $V(k)$ be the FDTs of the functions $x(t)$ and $v(t)$ respectively. If*

$$v(t) = \frac{d^{n\alpha}}{dt^{n\alpha}} x(t),$$

then

$$V(k) = \frac{\Gamma\left((k+n)\,\alpha + 1\right)}{\Gamma\left(k\alpha + 1\right)} X(k+n), n = 1, 2, ...$$

Now we study the solution of Eq. (4.4.1) within the initial conditions (4.4.2) using the FDTS.

The FDTS gives the solution of Eqs. (4.4.1) and (4.4.2) in the form of an infinite series which converges rapidly to the exact solution as follows

$$x(t) = \sum_{k=0}^{\infty} X(k) t^{k\alpha},$$

where $X(k)$ is the FDT function of $x(t)$.

To achieve this result, we consider the kernels $K_i(t,s)$, $i = 1,2$ of Eq. (4.4.1) as a difference kernels based on the difference $t-s$. Then Eq. (4.4.1) becomes

$$ {}^{C}D^{n\alpha}x(t) = f(t) + \int_0^1 K_1\left(t-s\right) G_1\left(x(s)\right)ds + \int_0^t K_2\left(t-s\right) G_2\left(x(s)\right)ds, \tag{4.4.3} $$

By applying the FDTS to Eq. (4.4.3) and using Theorems 3.4.1 and 4.4.1 together, we obtain the following recurrence relation formula

$$ \frac{\Gamma\left(\left(k+n\right)\alpha+1\right)}{\Gamma\left(k\alpha+1\right)} X(k+1) = F(k) + A(k) + B(k), \tag{4.4.4} $$

Here, $F(k)$, $A(k)$, and $B(k)$ are the FDTs of the functions $f(t)$, $\int_0^1 K_1\left(t-s\right) G_1\left(x(s)\right)ds$, and $\int_0^t K_2\left(t-s\right) G_2\left(x(s)\right)ds$, where

$$ \begin{aligned} A(k) &= \sum_{r=0}^{k-1} \frac{r!(k-r-1)!}{k!} K_{11}(k) A(k-r-1), \\ B(k) &= \sum_{r=0}^{k-1} \frac{r!(k-r-1)!}{k!} K_{22}(k) B(k-r-1). \end{aligned} $$

The recurrence relation (4.4.4) can be written as

$$ X(k+1) = \frac{\Gamma\left(k\alpha+1\right)}{\Gamma\left(\left(k+n\right)\alpha+1\right)} \left[F(k) + A(k) + B(k)\right], \tag{4.4.5} $$

Employing the FDTS at the initial condition (4.1.2), gives

$$ X(0) = C_i. \tag{4.4.6} $$

Inserting Eq. (4.4.6) into Eq. (4.4.5), we obtain the values of $X(k), k = 0, 1, 2, ...$ successively as

$$\begin{aligned} X(0) &= C_i, \\ X(1) &= \frac{1}{\Gamma(n\alpha+1)}\left[F(0)+A(0)+B(0)\right], \\ X(2) &= \frac{\Gamma(\alpha+1)}{\Gamma((1+n)\alpha+1)}\left[A(1)+B(1)\right], \\ X(3) &= \frac{\Gamma(2\alpha+1)}{\Gamma((2+n)\alpha+1)}\left[A(2)+B(2)\right], \\ X(4) &= \frac{\Gamma(3\alpha+1)}{\Gamma((3+n)\alpha+1)}\left[A(3)+B(3)\right], \\ &\vdots \end{aligned}$$

and so on.

Then, the inverse FDT of the set of values $\{X(k)\}_{k=0}^{n}$ admits the $n-$term approximation solution as

$$x(t) = \sum_{k=0}^{n} X(k) t^{k\alpha}.$$

Thus, the final solution of Eqs. (4.4.1) and 4.4.2) can be obtained as

$$x(t) = \lim_{n\to\infty} \sum_{k=0}^{n} X(k) t^{k\alpha} = \sum_{k=0}^{\infty} X(k) t^{k\alpha}.$$

Example 4.4.1 *Considering the nonlinear Caputo fractional Volterra integro-differential equation of the following form*

$$^{C}D^{\alpha}x(t) = -1 + \int_0^t x^2(s)ds, \tag{4.4.7}$$

within the initial condition

$$x(0) = 0, \tag{4.4.8}$$

where $^{C}D^{\alpha}$ *indicates the Caputo fractional derivative operator of order* $0 < \alpha \leq 1$.

According to the description of the FDTS presented in portion 4.4, we have the following recurrence relation formula

$$X(k+1) = \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}\left[-\delta(k)+A(k)\right], \tag{4.4.9}$$

where

$$A(k) = \frac{1}{k}\sum_{r=0}^{k-1} X(r)X(k-r-1), k \geq 1 \text{ with } A(0) = 0,$$

is the FDT of the nonlinear term $\int_0^t x^2(s)ds$.

From Theorem 3.4.1, the first nonlinear terms are the following

$$\begin{aligned} A(0) &= 0, \\ A(1) &= X^2(0), \\ A(2) &= X(0)X(1), \\ A(3) &= \frac{1}{3}\left(2X(0)X(2) + X^2(1)\right), \\ A(4) &= \frac{1}{4}\left(2X(0)X(3) + 2X(1)X(2)\right). \end{aligned}$$

Based on the recurrence relation (4.4.9), we have

$$\begin{aligned} X(0) &= 0, \\ X(1) &= -\frac{1}{\Gamma(\alpha+1)}, \\ X(2) &= 0, \\ X(3) &= 0, \\ X(4) &= \frac{1}{3}\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)\Gamma^2(\alpha+1)}t^{4\alpha}, \\ X(5) &= 0, \\ X(6) &= 0, \\ &\vdots \end{aligned}$$

and so on.

Thus, the approximate solution of Eqs. (4.4.7) and (4.4.8) is given by

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} X(k)t^{k\alpha} = X(0) + X(1)t^{\alpha} + X(2)t^{2\alpha} + X(3)t^{3\alpha} + ... \\ &= -\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{1}{3}\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)\Gamma^2(\alpha+1)}t^{4\alpha} + ... \end{aligned} \qquad (4.4.10)$$

Inserting $\alpha = 1$ *in Eq. (4.4.10), we obtain*

$$x(t) = -t + \frac{1}{12}t^4 - \frac{1}{252}t^7 + ...$$

Example 4.4.2 *Considering the nonlinear Caputo fractional Volterra integro-differential equation of the following form*

$$^{C}D^{\alpha}x(t) = 1 + \int_0^t x(s)x'(s)ds, \tag{4.4.11}$$

within the initial condition

$$x(0) = 0, \tag{4.4.12}$$

where $^{C}D^{\alpha}$ *indicates the Caputo fractional derivative operator of order* $0 < \alpha \leq 1$.

According to the description of the FDTS presented in portion 4.4, we have the following recurrence relation formula

$$X(k+1) = \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}\left[\delta(k) + A(k)\right], \tag{4.4.13}$$

where

$$A(k) = \frac{1}{k}\sum_{r=0}^{k-1} X(r)X(k-r-1), k \geq 1 \text{ with } A(0) = 0,$$

is the FDT of the nonlinear term $\int_0^t x(s)x'(s)ds$.

From Theorem 3.4.1, the first nonlinear terms are the following

$$\begin{aligned}
A(0) &= 0,\\
A(1) &= X(0)X'(0),\\
A(2) &= \frac{1}{2}\left[X(0)X'(1) + X(1)X'(0)\right],\\
A(3) &= \frac{1}{3}\left(X(0)X'(2) + (X'(1))^2 + X(2)X'(0)\right),\\
A(4) &= \frac{1}{4}\left(X(0)X'(3) + X(1)X'(2) + X(2)X'(1) + X(3)X'(0)\right).
\end{aligned}$$

Based on the recurrence relation (4.4.13), we have

$$\begin{aligned}
X(0) &= 0,\\
X(1) &= \frac{1}{\Gamma(\alpha+1)},\\
X(2) &= 0,\\
X(3) &= \frac{\Gamma(2\alpha+1)}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)},\\
X(4) &= 0,\\
X(5) &= \frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{2\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)},\\
&\vdots
\end{aligned}$$

Thus, the approximate solution is given by

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} X(k)t^{k\alpha} = X(0) + X(1)t^{\alpha} + X(2)t^{2\alpha} + X(3)t^{3\alpha} + ... \\ &= \frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{\Gamma(2\alpha+1)}{2\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha} \\ &\quad + \frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{2\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}t^{5\alpha} + ... \end{aligned} \tag{4.4.1}$$

Inserting $\alpha = 1$ *in Eq. (4.4.14), we obtain*

$$\begin{aligned} u(t) &= t + \frac{t^4}{6} + \frac{t^5}{30} + ... \\ &= \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}t\right). \end{aligned}$$

which is the exact solution of Eqs. (4.4.11) within the initial condition (4.4.12).

Chapter 5

Investigation of solutions to nonlinear fractional Volterra integro-differential equations under the Caputo-Fabrizio fractional derivative

In the present chapter, we investigate the solutions of nonlinear fractional Volterra integro-differential equations under the Caputo-Fabrizio fractional derivative using a novel coupling strategy called the Khalouta decomposition strategy (KhDS). This strategy is a coupling of two different strategies: the Khalouta transform strategy (KhTS) and the Adomian decomposition strategy (ADS). We will start by presenting the fundamental definitons of the Caputo-Fabrizio fractional derivative, then provide the definition of KhTS and its fundamental properties. We then determine the necessary conditions for the existence and uniqueness of solutions to the nonlinear fractional Volterra integro-differential equations. Finally, we introduce the methodology of the proposed strategy, as well as three numerical experiments on the nonlinear fractional Volterra integro-differential equations.

5.1 Caputo-Fabrizio fractional derivative

The Caputo-Fabrizio fractional derivative, originally introduced by Michele Caputo and Mauro Fabrizio [12] in 2015, is a type of fractional derivative with a non-singular exponential kernel, defined as an integral involving the exponential function of the difference between the variables. Unlike the original Caputo derivative, it does not have a singularity at the beginning of the interval, making it useful for modeling physical phenomena like viscoelasticity and in some engineering applications. It was introduced as a more practical alternative to traditional fractional derivatives.

In Eq. (1.1.16) if the changes occur as follows

$$(t-s)^{-\alpha} \longrightarrow \exp\left(-\frac{\alpha\,(t-s)}{1-\alpha}\right) \text{ and } \frac{1}{\Gamma(1-\alpha)} \longrightarrow \frac{(2-\alpha)\mathcal{CF}(\alpha)}{2(1-\alpha)},$$

we obtain the following definition.

Definition 5.1.1 *[12] Let $x \in H^1(\mathbb{R}^+)$, then the Caputo-Fabrizio fractional derivative of $x(t)$ of order $0 < \alpha < 1$ is given by*

$${}^{CF}D^{\alpha}x(t) = \frac{(2-\alpha)\mathcal{CF}(\alpha)}{2(1-\alpha)} \int_0^t x'(s) \exp\left(-\frac{\alpha\,(t-s)}{1-\alpha}\right) ds.$$

where $\mathcal{CF}(\alpha)$ is a normalization function that satisfies $\mathcal{CF}(0) = \mathcal{CF}(1) = 1$. This fractional operator has no singularities at $t = s$.

For $\mathcal{CF}(\alpha) = \dfrac{2}{2-\alpha}$, we have

$${}^{CF}D^{\alpha}x(t) = \frac{1}{1-\alpha} \int_0^t x'(s) \exp\left(-\frac{\alpha\,(t-s)}{1-\alpha}\right) ds. \tag{5.1.1}$$

Proposition 5.1.1 *[12] Let $x \in H^1(\mathbb{R}^+)$. If $n \geq 1$ and $0 < \alpha < 1$, then the Caputo-Fabrizio fractional derivative of order $\alpha + n$ is given by*

$${}^{CF}D^{\alpha+n}x(t) = {}^{CF}D^{\alpha}(D^n x(t)). \tag{5.1.2}$$

Definition 5.1.2 *[12] Let $x \in H^1(\mathbb{R}^+)$, then the Caputo-Fabrizio fractional integral of $x(t)$ of order $0 < \alpha < 1$ is given by*

$${}^{CF}I^{\alpha}x(t) = \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}x(t) + \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)} \int_0^t x(s)ds. \tag{5.1.3}$$

For the Caputo-Fabrizio fractional integral and Caputo-Fabrizio fractional derivative, we have the following relation

$$ {}^{CF}I^{\alpha\,CF}D^{\alpha}x(t) = x(t) - x(0). \tag{5.1.4}$$

5.2 KhTS definitions and results

In reference [22], Ali Khalouta proposed a new integral transform in 2023. This new generalized integral transform, which will be abbreviated as KhS in the rest of this chapter, includes a variety of the family of known integral transforms and is particularly useful for solving differential equations, partial differential equations, integral equations, and integro-differential equations.

Definition 5.2.1 *[22] Let $x(t)$ is a real-valued function of one variable t.*

1) The KhT of $x(t)$ is given by

$$\mathbb{KH}\left[x(t)\right] = \mathcal{K}(s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^{\infty} \exp\left(-\frac{st}{\gamma\eta}\right) x(t)dt,$$

where $s,\gamma,\eta > 0$ are the KhT variables.

2) The inverse KhT of $x(t)$ is given by

$$\mathbb{KH}^{-1}\left[\mathcal{K}(s,\eta,\gamma)\right] = x(t),\ \textit{for } t \geq 0,$$

which is equivalent to

$$x(t) = \mathbb{KH}^{-1}\left[\mathcal{K}(s,\eta,\gamma)\right] = \frac{1}{2\pi i}\int_{\varphi-i\infty}^{\varphi+i\infty} \frac{1}{s}\exp\left(\frac{st}{\gamma\eta}\right)\mathcal{K}(s,\eta,\gamma)ds,$$

where φ is a real constant and the integral is taken along $s = \varphi$ in the complex plane $s = u + iv$.

Theorem 5.2.1 *[22] The fundamental properties of KhT are*

(1) Property of the linearity

$$\mathbb{KH}\left[ax(t) + by(t)\right] = a\mathbb{KH}\left[x(t)\right] + b\mathbb{KH}\left[y(t)\right], a, b \in \mathbb{R}$$

(2) Property of the derivative

$$\mathbb{KH}\left[x^{(n)}(t)\right] = \left(\frac{s}{\gamma\eta}\right)^{n} \mathbb{KH}\left[x(t)\right] - \sum_{k=0}^{n-1} \left(\frac{s}{\gamma\eta}\right)^{n-k} x^{(k)}(0), n \geq 1.$$

where $x^{(n)}(t)$ is the nth derivative of $x(t)$.

(3) Property of the convolution

$$\mathbb{KH}\left[(x * y)(t)\right] = \frac{\gamma\eta}{s}\mathbb{KH}\left[x(t)\right]\mathbb{KH}\left[y(t)\right],$$

where $\mathbb{KH}\left[(x * y)(t)\right]$ is the Khalouta convolution of the functions $x(t)$ and $y(t)$.

(4) KhTs for some common functions

$$\begin{aligned}
\mathbb{KH}\left[1\right] &= 1, \\
\mathbb{KH}\left[t\right] &= \frac{\gamma\eta}{s}, \\
\mathbb{KH}\left[\frac{t^n}{n!}\right] &= \left(\frac{\gamma\eta}{s}\right)^{n}, n \geq 0, \\
\mathbb{KH}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] &= \left(\frac{\gamma\eta}{s}\right)^{\alpha}, \alpha > -1,
\end{aligned}$$

Proof. The proof of this theorem is found in detail in the reference [22]. ■

Theorem 5.2.2 *[24] The KhT of the Caputo-Fabrizio fractional derivative of the function $x(t)$ is given as*

$$\mathbb{KH}\left[{}^{CF}D^{\alpha+n}x(t)\right] = \frac{s}{(1-\alpha)s + \alpha\gamma\eta}\left(\left(\frac{s}{\gamma\eta}\right)^{n}\mathbb{KH}\left[x(t)\right] - \sum_{k=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{n-k}x^{(k)}(0)\right).$$

Proof. Utilizing Eqs. (5.1.1) and (5.1.2), we get

$$\begin{aligned}
\mathbb{KH}\left[{}^{CF}D^{\alpha+n}x(t)\right] &= \mathbb{KH}\left[{}^{CF}D^{\alpha}(D^{n}x(t))\right] \\
&= \frac{1}{1-\alpha}\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\left(\int_0^\varpi x^{(n+1)}(s)\exp\left(-\frac{\alpha(t-s)}{1-\alpha}\right)ds\right)dt \\
&= \frac{1}{1-\alpha}\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{st}{\gamma\eta}\right)\left(x^{(n+1)}(t) * \exp\left(-\frac{\alpha t}{1-\alpha}\right)\right)dt \\
&= \frac{1}{1-\alpha}\mathbb{KH}\left[x^{(n+1)}(t) * \exp\left(-\frac{\alpha t}{1-\alpha}\right)\right].
\end{aligned}$$

Exploiting Theorem 5.2.1, we obtain

$$\begin{aligned}\mathbb{KH}\left[{}^{CF}D^{\alpha+n}x(t)\right] &= \frac{1}{1-\alpha}\frac{\gamma\eta}{s}\mathbb{KH}\left[x^{(n+1)}(t)\right]\mathbb{KH}\left[\exp\left(-\frac{\alpha t}{1-\alpha}\right)\right]\\ &= \frac{\gamma\eta}{(1-\alpha)s+\alpha\gamma\eta}\left[\left(\frac{s}{\gamma\eta}\right)^{n+1}\mathbb{KH}\left[x(t)\right]-\sum_{k=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{n-k}x^{(k)}(0)\right]\\ &= \frac{s}{(1-\alpha)s+\alpha\gamma\eta}\left[\left(\frac{s}{\gamma\eta}\right)^{n}\mathbb{KH}\left[x(t)\right]-\sum_{k=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{n-k}x^{(k)}(0)\right].\end{aligned}$$

Hence the desired result. ■

5.3 Existence and uniqueness results

In this protion, we present the necessary conditions that guarantee the existence and uniqueness of the solution to the nonlinear fractional Volterra integro-differential equation in the form

$${}^{CF}D^{\alpha}x(t) = f(t) + \int_0^t K(t,s)F(x(s))ds, \tag{5.3.1}$$

within the initial condition

$$x(0) = x_0 \in \mathbb{R}. \tag{5.3.2}$$

where ${}^{CF}D^{\alpha}$ is the Caputo-Fabrizio fractional derivative operator of order α with $0 < \alpha \leq 1$, $x(t)$ is an unknown real function of the variable t, f, K are known real functions and F is nonlinear function.

Let $B = \{x(t)/x(t) \in C(I)\}$ be the Banach space equipped with the norm $\|x(t))\|_B = \max_{t\in I}|x(t)|$. To prove our results, we need the following hypotheses

(**H1**) There exists a constant $L_F > 0$ such that, for all $x, y \in C(I)$

$$|F(x(t)) - F(y(t))| \leq L_F\,|x(t) - y(t)|.$$

(**H2**) There exists a function $K^* \in C(D, \mathbb{R}^+)$, the set of any positive function continuous on $D = \{(t,s) \in \mathbb{R}\times\mathbb{R} : 0 \leq s \leq t \leq 1\}$ such that

$$K^* = \sup_{t\in I}\int_0^t |K(t,s)|\,ds < \infty.$$

(**H3**) The function $f : I \to \mathbb{R}$ is continuous.

Lemma 5.3.1 *If a function $x(t) \in C(I)$ satisfies Eqs. (5.3.1)-(5.3.2), then the problems (5.3.1)-(5.3.2) are equivalent to the problem of finding a continuous solution of the integral equation*

$$\begin{aligned} x(t) &= x(0) + \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)}\left(f(t) + \int_0^t K(t,s)F(x(s))ds\right) \\ &\quad + \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}\int_0^t \left(f(s) + \int_0^s K(s,\tau)F(x(\tau))d\tau\right) ds. \end{aligned}$$

Proof. To prove this lemma, it is sufficient to apply the Caputo-Fabrizio fractional integral to Eq. (5.3.1) and use Eqs. (5.1.3) and (5.1.4). ■

Theorem 5.3.1 *Suppose that the hypotheses (**H1**)-(**H3**) are satisfied and*

$$\left(\Phi(\alpha) + \Psi(\alpha)\right) K^* L_F < 1. \tag{5.3.3}$$

Then, there is a unique solution $x(t) \in C(I)$ to Eqs. (5.3.1)-(5.3.2).

Proof. Firs, we define an operator $\mathbb{P} : C(I) \to C(I)$ by

$$\begin{aligned} \mathbb{P}x(t) &= x(0) + \Phi(\alpha)\left(f(t) + \int_0^t K(t,s)F(x(s))ds\right) \\ &\quad + \Psi(\alpha)\int_0^t \left(f(s) + \int_0^s K(s,\tau)F(x(\tau))d\tau\right) ds, \end{aligned}$$

where

$$\Phi(\alpha) = \frac{2(1-\alpha)}{(2-\alpha)\mathcal{CF}(\alpha)} \text{ and } \Psi(\alpha) = \frac{2\alpha}{(2-\alpha)\mathcal{CF}(\alpha)}.$$

Let $x(t), y(t) \in C(I)$, then we have

$$\begin{aligned} |\mathbb{P}x(t) - \mathbb{P}y(t)| &\leq \Phi(\alpha)\left(\int_0^t |K(t,s)|\,|F(x(s)) - F(y(s))|\,ds\right) \\ &\quad + \Psi(\alpha)\int_0^t \left(\int_0^s |K(s,\tau)|\,|F(x(\tau)) - F(y(\tau))|\,d\tau\right) ds \\ &\leq \Phi(\alpha)K^*L_F\,|x(t) - y(t)| + \Psi(\alpha)K^*L_F\,|x(t) - y(t)| \int_0^t 1ds \\ &\leq \left(\Phi(\alpha) + \Psi(\alpha)\right) K^* L_F\,|x(t) - y(t)|\,. \end{aligned}$$

Hence, we have

$$\|\mathbb{P}x(t) - \mathbb{P}y(t)\|_B \leq \left(\Phi(\alpha) + \Psi(\alpha)\right) K^* L_F\,\|x(t) - y(t)\|_B\,.$$

According to (5.3.3), $\mathbb{P}$ is a contraction.

Due to Banach's fixed point theorem 1.2.1, $\mathbb{P}$ has only one fixed point x in $C(I)$, so $x(t)$ is a unique solution to Eqs (5.3.1)-(5.3.2).

Hence the desired result. ■

5.4 Khalouta decomposition strategy (KhDS)

In this portion, we explore the fundamental idea of KhDS.

Theorem 5.4.1 *[18] Consider the nonlinear fractional Volterra integro-differential equation (5.3.1) within the initial condition (5.3.2).*

By implementing KhDS, the solution to Eqs (5.3.1)-(5.3.2) can be expanded as an infinite series as follows

$$x(t)=\sum_{n=0}^{\infty}x_n(t). \tag{5.4.1}$$

Proof. Firstly, we consider the kernel $K(t,s)$ of Eq. (5.3.1) as a difference kernel based on the difference $t-s$. Then Eq. (5.3.1) becomes

$${}^{CF}D^{\alpha}x(t)=f(t)+\int_0^t K(t-s)F(x(s))ds. \tag{5.4.2}$$

Operating the KhT to Eq. (5.4.2) and using Theorem 5.2.2 and the convolution property (3), we attain

$$\begin{aligned}\mathbb{KH}\left[x(t)\right] &= x(0)+\left(\frac{(1-\alpha)\,s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(t)+\int_0^t K(t-s)F(x(s))ds\right]\\ &= x(0)+\left(\frac{(1-\alpha)\,s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(t)\right]\\ &\quad+\left(\frac{(1-\alpha)\,s\gamma\eta+\alpha\left(\gamma\eta\right)^2}{s^2}\right)\mathbb{KH}\left[K(t)\right]\mathbb{KH}\left[F(x(t))\right].\end{aligned} \tag{5.4.3}$$

Taking the inverse KhT to Eq. (5.4.3) we acquire

$$\begin{aligned}x(t) &= x(0)+\mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(t)\right]\right]\\ &\quad+\mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s\gamma\eta+\alpha\left(\gamma\eta\right)^2}{s^2}\right)\mathbb{KH}\left[K(t)\right]\mathbb{KH}\left[F(x(t))\right]\right].\end{aligned} \tag{5.4.4}$$

The ADS [1] defines the solution by an infinite series of linear tems as

$$x(t) = \sum_{n=0}^{\infty} x_n(t), \tag{5.4.5}$$

and the nonlinear terms as

$$F(x(t)) = \sum_{n=0}^{\infty} \mathcal{A}_n(x(t)), \tag{5.4.6}$$

where $\mathcal{A}_n(x(t))$ are the Adomian polynomials to be determined by the following formula

$$\mathcal{A}_n(x(t)) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i x_i\right)\right]_{\lambda=0}, \ n = 0, 1, 2, ...$$

Replacing Eqs. (5.4.5) and (5.4.6) into Eq. (5.4.4) leads to

$$\begin{aligned}\sum_{n=0}^{\infty} x_n(t) &= x(0) + \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s + \alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(t)\right]\right] \\ &\quad + \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s\gamma\eta + \alpha\,(\gamma\eta)^2}{s^2}\right)\mathbb{KH}\left[K(t)\right]\mathbb{KH}\left[\sum_{n=0}^{\infty}\mathcal{A}_n(x(t))\right]\right].\end{aligned} \tag{5.4.7}$$

Equating the terms on both sides of Eq. (5.4.7), wa attain the following recurrence relation

$$\begin{aligned}x_0(t) &= x(0) + \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s + \alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(t)\right]\right], \\ x_{n+1}(t) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s\gamma\eta + \alpha\,(\gamma\eta)^2}{s^2}\right)\mathbb{KH}\left[K(t)\right]\mathbb{KH}\left[\mathcal{A}_n(x(t))\right]\right], \ n \geq 0.\end{aligned} \tag{5.4.8}$$

Using the recurrence relation (5.4.8) gives the components of $x_n(t), n \geq 0$.

We define the $m-$terms approximate of the KhDS solution $x(t)$ by

$$\begin{aligned}\phi_m\left[x(t)\right] &= \sum_{n=0}^{m-1} x_n(t) \\ &= x_0(t) + x_1(t) + x_2(t) + ... + x_{m-1}(t),\end{aligned}$$

with

$$\lim_{m\to\infty}\phi_m\left[x(t)\right] = x(t).$$

Hence the desired result. ■

5.5 Convergence analysis of the KhDS

In this portion, we give the sufficient condition for the convergence analysis of our strategy.

Theorem 5.5.1 *[18] Let $x_n(t)$ and $x(t)$ be in Banach space B. Then the KhDS series solution $x(t)$ defined by Eq. (5.4.1) converges to the exact solution of Eqs. (5.3.1) and (5.3.2) if $\exists \varrho \in (0,1)$ such as $\|x_{n+1}(t)\|_B \leq \varrho \|x_n(t)\|_B$ and $x_0(t) \in B$ is bounded.*

Proof. Considering the sequence of partial sum $\{\mathcal{S}_n(t)\}_{n=0}^{\infty}$ of the form

$$\begin{aligned}
\mathcal{S}_0(t) &= x_0(t), \\
\mathcal{S}_1(t) &= x_0(t) + x_1(t), \\
\mathcal{S}_2(t) &= x_0(t) + x_1(t) + x_2(t), \\
&\vdots \\
\mathcal{S}_n(t) &= x_0(t) + x_1(t) + x_2(t) + ... + x_n(t).
\end{aligned}$$

To obtain the desired result, we will prove that $\{\{\mathcal{S}_n(t)\}_{n=0}^{\infty}$ is a Cauchy sequence in B. From the last hypothesis of the theorem we have $\varrho \in (0,1)$, then

$$\begin{aligned}
\|\mathcal{S}_{n+1}(t) - \mathcal{S}_n(t)\|_B &\leq \|x_{n+1}(t)\|_B \leq \varrho \|x_n(t)\|_B \\
&\leq \varrho^2 \|x_{n-1}(t)\|_B \leq ... \leq \varrho^{n+1} \|x_0(t)\|_B .
\end{aligned}$$

For any $n, m \in \mathbb{N}$ with $n \geq m$, we have

$$\begin{aligned}
\|\mathcal{S}_{n+1}(t) - \mathcal{S}_n(t)\|_B &= \|\mathcal{S}_n(t) - \mathcal{S}_{n-1}(t) + \mathcal{S}_{n-1}(t) - \mathcal{S}_{n-2}(t) + ... + \mathcal{S}_{m+1}(t) - \mathcal{S}_m(t)\|_B \\
&\leq \|\mathcal{S}_n(t) - \mathcal{S}_{n-1}(t)\|_B + \|\mathcal{S}_{n-1}(t) - \mathcal{S}_{n-2}(t)\|_B + ... + \|\mathcal{S}_{m+1}(t) - \mathcal{S}_m(t)\|_B \\
&\leq \varrho^n \|x_0(t)\|_B + \varrho^{n-1} \|x_0(t)\|_B + ... + \varrho^{m+1} \|x_0(t)\|_B \\
&= \varrho^{m+1} \left(1 + \varrho + ... + \varrho^{n-m-1}\right) \|x_0(t)\|_B \\
&\leq \varrho^{m+1} \left(\frac{1 - \varrho^{n-m}}{1 - \varrho}\right) \|x_0(t)\|_B .
\end{aligned}$$

Since $0 < \varrho < 1$, then $1 - \varrho^{n-m} < 1$. So from inequality (4.4.2), we have

$$\|\mathcal{S}_n(t) - \mathcal{S}_m(t)\|_B \leq \frac{\varrho^{m+1}}{1 - \varrho} \|x_0(t)\|_B .$$

Because $x_0(t)$ is bounded, i.e.$\|x_0(t)\|_B < \infty$, then we have

$$\lim_{n,m\to\infty} \|\mathcal{S}_n(t) - \mathcal{S}_m(t)\|_B = 0.$$

Accordingly $\{\mathcal{S}_n(t)\}_{n=0}^{\infty}$ is a Cauchy sequence in Banach space B. It concludes that the solution of Eqs. (5.3.1) and (5.3.2) as series is convergent.

Hence the desired result. ■

Theorem 5.5.2 *[18] The maximum absolute truncation error of the series solution (5.4.1) for Eqs. (5.3.1) and (5.3.2) is estimated to be*

$$\left\| x(t) - \sum_{n=0}^{r} x_n(t) \right\|_B \leq \frac{\varrho^{r+1}}{1-\varrho} \|x_0(t)\|_B .$$

Proof. Let the series $\sum_{n=0}^{r} x_n(t)$ be finite. Then

$$\begin{aligned}
\left\| x(t) - \sum_{n=0}^{r} x_n(t) \right\|_B &\leq \left\| \sum_{n=r+1}^{\infty} x_n(t) \right\|_B \\
&\leq \sum_{n=r+1}^{\infty} \|x_n(t)\|_B \\
&\leq \sum_{n=r+1}^{\infty} \varrho^r \|x_0(t)\|_B \\
&\leq \varrho^{r+1} \left(1 + \varrho + \varrho^2 + \varrho^3 + ...\right) \|x_0(t)\|_B \\
&\leq \frac{\varrho^{r+1}}{1-\varrho} \|x_0(t)\|_B .
\end{aligned}$$

Hence the desired result. ■

5.6 Numerical experiments

In this portion, we will demonstrate the usefulness of KhDM in solving three numerical experiments for nonlinear fractional Volterra integro-differential equations involving the Caputo-Fabrizio fractional derivative.

Example 5.6.1 *Suppose that the nonlinear fractional Volterra integro-differential*

$$^{CF}D^{\alpha}x(t) = -1 + \int_0^t x^2(s)ds, \tag{5.6.1}$$

within the initial condition

$$x(0) = 0. \tag{5.6.2}$$

where ${}^{CF}D^{\alpha}x(t)$ *is the Caputo-Fabrizio fractional derivative of order* α *with* $0 < \alpha \leq 1$.

The exact solution of Eqs. (5.6.1)-(5.6.2) does not exist, fortunately it is considered as an approximate solution.

To resolve this problem with the proposed strategy, we apply the same steps in KhDS described in portion 5.4 to Eqs. (5.6.1)-(5.6.2), we attain the following recurrence relation

$$\begin{aligned} x_0(t) &= -\mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s + \alpha\gamma\eta}{s}\right)\mathbb{KH}\left[1\right]\right], \\ x_{n+1}(t) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s + \alpha\gamma\eta}{s}\right)\mathbb{KH}\left[\int_0^t \mathcal{A}_n(x(s))ds\right]\right],\ n \geq 0, \end{aligned} \tag{5.6.3}$$

where

$$x^2(s) = \sum_{n=0}^{\infty} \mathcal{A}_n(x(s)),$$

and the first few of $\mathcal{A}_n$ *are given by*

$$\begin{aligned} \mathcal{A}_0 &= x_0^2, \\ \mathcal{A}_1 &= 2x_0x_1, \\ \mathcal{A}_2 &= 2x_0x_2 + x_1^2 \\ &\vdots \end{aligned}$$

and so on.

From the recurrence relation (5.6.3), we have

$$\begin{aligned} x_0(t) &= -\left((1-\alpha) + \alpha t\right), \\ x_1(t) &= (1-\alpha)^3\, t + 3\alpha\,(1-\alpha)^2\,\frac{t^2}{2!} + 4\alpha^2\,(1-\alpha)\,\frac{t^3}{3!} + 2\alpha^3\frac{t^4}{4!}, \\ &\vdots \end{aligned}$$

and so on.

Consequently, the KhDS approximate solution of Eqs. (5.6.1)-(5.6.2) is expressed as

$$x(t) = -(1-\alpha) + \left((1-\alpha)^3 - \alpha\right)t + 3\alpha\,(1-\alpha)^2\,\frac{t^2}{2!} + 4\alpha^2\,(1-\alpha)\,\frac{t^3}{3!} + 2\alpha^3\frac{t^4}{4!} + \ldots \tag{5.6.4}$$

Inserting $\alpha = 1$ in Eq. (5.6.4), we get

$$x(t) = -t + \frac{t^4}{12} - \frac{t^7}{252} + ...$$

which is the same result as in [38].

Example 5.6.2 *Suppose that the nonlinear fractional Volterra integro-differential*

$$^{CF}D^{\alpha}x(t) = 1 + \int_0^t x'(s)x(s)ds, \tag{5.6.5}$$

within the initial condition

$$x(0) = 0, \tag{5.6.6}$$

where $^{CF}D^{\alpha}x(t)$ is the Caputo-Fabrizio fractional derivative of order α with $0 < \alpha \leq 1$.

To resolve this problem with the proposed strategy, we apply the same steps in KhDS described in portion 5.4 to Eqs. (5.6.5)-(5.6.6), we attain the following recurrence relation

$$\begin{aligned} x_0(t) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s + \alpha\gamma\eta}{s}\right)\mathbb{KH}\left[1\right]\right], \\ x_{n+1}(t) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s + \alpha\gamma\eta}{s}\right)\mathbb{KH}\left[\int_0^t \mathcal{A}_n(x(s))ds\right]\right],\ n \geq 0, \end{aligned} \tag{5.6.7}$$

where

$$x'(s)x(s) = \sum_{n=0}^{\infty} \mathcal{A}_n(x(s)),$$

and the first few of $\mathcal{A}_n$ are given by

$$\begin{aligned} \mathcal{A}_0 &= x_0'x_0, \\ \mathcal{A}_1 &= x_0'x_1 + x_1'x_0, \\ \mathcal{A}_2 &= x_0'x_2 + x_1'x_1 + x_0'x_2, \\ &\vdots \end{aligned}$$

and so on.

From the recurrence relation (5.6.7), we have

$$\begin{aligned} x_0(t) &= (1-\alpha) + \alpha t, \\ x_1(t) &= \alpha\,(1-\alpha)^2\,t + 2\alpha^2\,(1-\alpha)\,\frac{t^2}{2} + \alpha^3\frac{t^3}{3!}, \\ &\vdots \end{aligned}$$

and so on.

Consequently, the KhDS approximate solution of Eqs. (5.6.5)-(5.6.6) is expressed as

$$x(t) = (1-\alpha) + \left(\alpha + \alpha\,(1-\alpha)^2\right) t + 2\alpha^2\,(1-\alpha)\,\frac{t^2}{2} + \alpha^3\frac{t^3}{3!} + \ldots \tag{5.6.8}$$

Inserting $\alpha = 1$ in Eq. (5.6.8), we get the exact solution in the standard case

$$\begin{aligned} x(t) &= t + \frac{t^3}{6} + \frac{t^5}{30} + \ldots \\ &= \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}t\right), \end{aligned}$$

which is the same result as in [38].

Example 5.6.3 *Suppose that the nonlinear fractional Volterra integro-differential*

$${}^{CF}D^{\alpha}x(t) = -\frac{1}{2} + \int_0^t (x'(s))^2\,ds, \tag{5.6.9}$$

within the initial condition

$$x(0) = 0, \tag{5.6.10}$$

where ${}^{CF}D^{\alpha}x(t)$ is the Caputo-Fabrizio fractional derivative of order α with $0 < \alpha \leq 1$.

To resolve this problem with the proposed strategy, we apply the same steps in KhDS described in portion 5.4 to Eqs. (5.6.5)-(5.6.6), we attain the following recurrence relation

$$\begin{aligned} x_0(t) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s + \alpha\gamma\eta}{s}\right)\mathbb{KH}\left[-\frac{1}{2}\right]\right], \\ x_{n+1}(t) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s + \alpha\gamma\eta}{s}\right)\mathbb{KH}\left[\int_0^t \mathcal{A}_n(x(s))ds\right]\right],\ n \geq 0, \end{aligned} \tag{5.6.11}$$

where

$$(x'(s))^2 = \sum_{n=0}^{\infty}\mathcal{A}_n(x(s)),$$

and the first few of $\mathcal{A}_n$ are given by

$$\begin{aligned} \mathcal{A}_0 &= (x_0')^2, \\ \mathcal{A}_1 &= 2x_0'x_1', \\ \mathcal{A}_2 &= 2x_0'x_2' + x_1'x_1', \\ &\vdots \end{aligned}$$

and so on.

From the recurrence relation (5.6.11), we have

$$\begin{aligned} x_0(t) &= -\frac{1}{2}\left((1-\alpha)+\alpha t\right), \\ x_1(t) &= \frac{1}{4}\alpha^2(1-\alpha)\,t+\frac{1}{8}\alpha^3 t^2, \\ x_2(t) &= -\left(\frac{1}{4}\alpha^3(1-\alpha)^2\,t+\frac{1}{4}\alpha^4(1-\alpha)\,t^2+\frac{1}{24}\alpha^5 t^3\right), \\ &\vdots \end{aligned}$$

Consequently, the KhDS approximate solution of Eqs. (5.6.9)-(5.6.10) is expressed as

$$\begin{aligned} x(t) &= -\frac{1}{2}(1-\alpha)-\frac{1}{4}\left(\alpha^3(1-\alpha)^2-\alpha^2(1-\alpha)+2\alpha\right)t \\ &\quad +\frac{1}{8}\left(\alpha^3-2\alpha^4(1-\alpha)^2\right)t^2-\frac{1}{24}\alpha^5 t^3+\ldots \end{aligned} \tag{5.6.12}$$

Inserting $\alpha = 1$ in Eq. (5.6.12), we get the exact solution in the standard case

$$\begin{aligned} x(t) &= -\frac{1}{2}t+\frac{1}{8}t^2-\frac{1}{24}t^3+\ldots \\ &= -\ln\left(\frac{1}{2}t+1\right), \end{aligned}$$

which is the same result as in [38].

5.7 Numerical results and simulations

In this portion, we discuss the numerical results obtained using MATLAB programming to evaluate the performance of the KhDS. Numerical simulations enable us to better understand the practical implications of the theoretical results.

Figure 1 depicts the graphs of the approximate solutions using KhDS at various values of fractional order α corresponding to Example 5.6.1. Figure 2 depicts the graphs of the exact and approximate solutions using KhDS at various values of fractional order α corresponding to Example 5.6.1. From this figure, it is obvious that when $\alpha = 1$, the exact and approximate solution coincide. Table 1 contains the numerical values of KhDS approximate solution for various values of α corresponding to Example 5.6.1. Tables 2 and 3 report the numerical values of the exact and KhDS approximate solutions for various values of α, corresponding

to Examples 5.6.2 and 5.6.3, respectevely. It is observed that the results obtained for $\alpha = 1$ coincides with the exact solution, while for other values of α, the approximate solution is in good agreement with the exact solution.

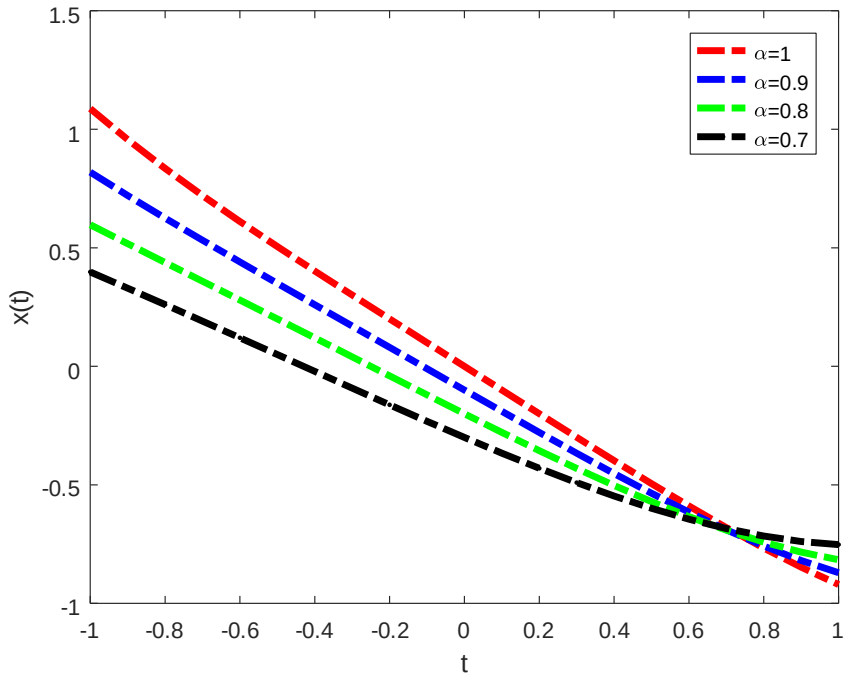

Figure 1: The graphs of the KhDS approximate solutions for Example 5.6.1 at various values of α.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.1	-0.36625	-0.27863	-0.18970	-0.099992
0.2	-0.42999	-0.35573	-0.27873	-0.199870
0.3	-0.49052	-0.43063	$-0.366\,53$	-0.29933
0.4	-0.54708	-0.50257	-0.45243	-0.39787
0.5	-0.59884	-0.57067	-0.53558	-0.49482

Table 1: Numerical values of the KhDS approximate solutions for Example 5.6.1 at various values of α.

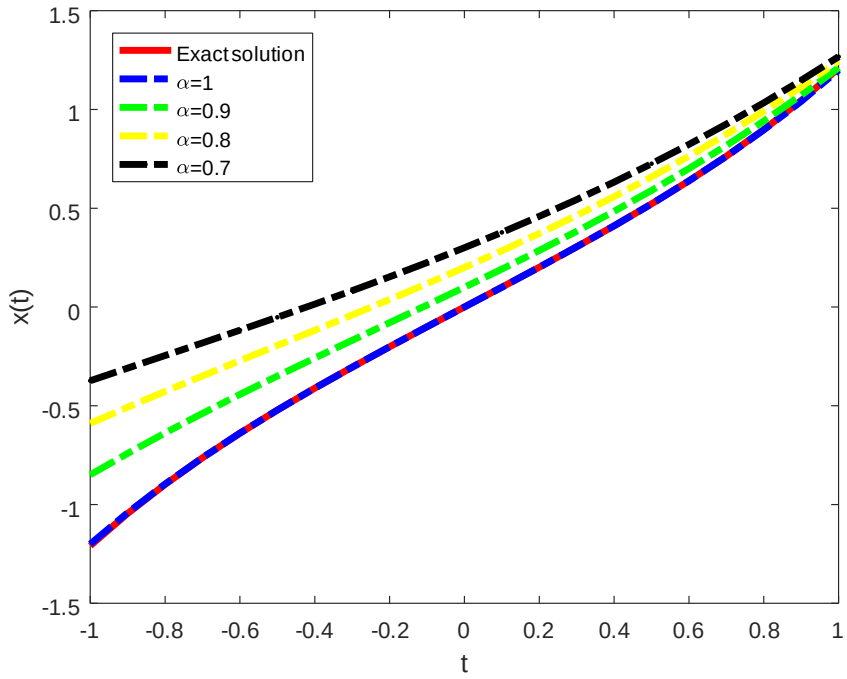

Figure 2: The graphs of the exact and KhDS approximate solutions for Example 5.6.2 at various values of α.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	*Exact solution*	$\lvert x_{exact} - x_{KhDS} \rvert$
0.1	0.37783	0.28457	0.19183	0.10017	0.10017	6.7597×10^{-10}
0.2	0.45894	0.37220	0.28601	0.20134	0.20134	8.7055×10^{-8}
0.3	0.54367	0.46342	0.38327	0.30458	0.30458	1.5028×10^{-6}
0.4	0.63238	0.55874	0.48434	0.41101	0.41102	1.1423×10^{-5}
0.5	0.72540	0.65867	0.58994	0.52188	0.52193	5.5515×10^{-5}

Table 2: Numerical values of the exact and KhDS approximate solutions for Example 5.6.2 at various values of α.

t	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	*Exact solution*	$\lvert x_{exact} - x_{KhDS} \rvert$
0.1	-0.13673	-0.092287	-0.0049875	-0.0049875	1.5563×10^{-10}
0.2	-0.17234	$-0.132\,93$	-0.0099503	-0.0099503	2.4802×10^{-9}
0.3	-0.20691	-0.17208	-0.014889	-0.014889	1.2506×10^{-8}
0.4	-0.24054	-0.20989	-0.019803	-0.019803	3.9370×10^{-8}
0.5	-0.27329	-0.24649	-0.024693	-0.024693	9.5743×10^{-8}

Table 3: Numerical values of the exact and KhDS approximate solutions for Example 5.6.3 at various values of α.

Conclusions and future research

In this thesis, we find the approximate analytical solutions for nonlinear fractional integro-differential equations with fractional derivatives of singular and non-singular kernels, using various semi-analytic strategies. We apply the Caputo and Caputo-Fabrizio fractional derivatives and their properties. Furthermore, we present a new coupling strategy, called Khalouta decomposition strategy (KhDS), which is applied to the study of solutions of nonlinear fractional Volterra integro-differential equations. Two-dimensional graphs and tables are provided to understand the effect of fractional order. The advantage of KhDS lies in its efficiency, accuracy, and ease of application, with reduced computational effort. The obtained results demonstrate the computational efficiency of the current strategy. Finally, the accuracy and efficiency of the proposed strategy indicate its applicability to several linear and nonlinear fractional integro-differential equations.

These results open up avenues for further research and applications in related fields of science and mathematics. Future research directions could include:

- Investigating the stability and relative error analysis of KhDS.

- Expanding the applicability of KhDS to more complex problems, like fractional Fredholm-Volterra integro-differential equations.

- Expanding the Khalouta integral transform to other fractional derivatives, like conformable, Liouville-Caputo, and Atangana-Baleanu-Caputo fractional derivative orders.

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ملخص:

نظرًا لأن العديد من المسائل الفيزيائية تُنمذج باستخدام معادلات تكاملية تفاضلية كسرية غير خطية، فقد حظيت حلول هذا النوع من المعادلات بدراسة مكثفة من قِبل العديد من الباحثين. تكمن أهمية هذه الرسالة في عرضها لاستراتيجية جديدة وفعّالة لإيجاد حلول لمعادلات تكاملية تفاضلية كسرية غير خطية، وخاصةً معادلات فولتيرا التكاملية التفاضلية الكسرية الغير خطية. وُصفت المشتقة الكسرية وفقًا لمفهوم كابوتو-فابريزيو. واستُخدمت تجارب عددية لإثبات دقة هذه الاستراتيجية وصلاحيتها.

كلمات مفتاحية: المعادلات التكاملية التفاضلية الكسرية، المشتقة الكسرية كابوتو-فابريزيو، تحويل خلوطة

Résumé:

Comme de nombreux problèmes physiques sont modélisés par des équations intégro-différentielles fractionnaires non-linéaires, les solutions de ce type d'équations ont été largement étudiées par de nombreux auteurs. L'importance de cette thèse réside dans la présentation d'une nouvelle stratégie efficace pour trouver des solutions aux équations intégro-différentielles fractionnaires non-linéaires, en particulier aux équations intégro-différentielles fractionnaires non-linéaires de Volterra. La dérivée fractionnaire est décrite au sens de Caputo-Fabrizio. Des expériences numériques sont utilisées pour démontrer la précision et la validité de cette stratégie.

Mots clés: Équations intégro-différentielles fractionnaires, Dérivée fractionnaire de Caputo-Fabrizio, Transformation de Khalouta.

Abstract:

Since many physical problems are modeled by nonlinear fractional integro-differential equations, the solutions of this type of equations have been extensively studied by many authors. The importance of this thesis lies in the presentation of a new and efficient strategy for finding solutions to nonlinear fractional integro-differential equations, in particular to the nonlinear fractional Volterra integro-differential equations. The fractional derivative is described in the Caputo-Fabrizio sense. Numerical experiments are used to demonstrate the accuracy and validity of this strategy.

Key words: Fractional integro-differential equations, Caputo-Fabrizio fractional derivative, Khalouta transform.