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ON THE SOLUTIONS OF NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATION ARISING IN MATHEMATICAL BIOLOGY

Presented by

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بسم الله الرحمن الرحيم

شكر وعرفان

الحمد لله وحده، والصلاة والسلام على من لا نبيّ بعده، وبعد:

الشكر لله -عزوجل- الذي أنار لي الدرب، وفتح لي أبواب العلم وأمدني بالصبر والإرادة لإتمام هذه الرسالة, فله الحمد والشكر حمدًا طيبًا مُباركًا يليق بجلاله، عملا بقول المُصطفى صلى الله عليه وسلم

"لا يشكر الله من لا يشكر الناس".

فإن الوفاء يقتضي أن يُرد الفضل لأهله، لذلك أتقدم بجزيل الشكر والعرفان إلى من كان له الفضل بعد الله في إخراج هذا البحث العلمي, الأستاذ **الدكتور علي خلوطة** لتفضله بالإشراف على هذه الرسالة, والذي وجدت فيه أستاذًا فاضلا معطًاء, بذل الجهد، وقدَّم التوجيه السليم والرأي السديد، الذي ساعدني في تخطي الكثير من الصعاب فجزاه الله عنى خير الجزاء وأمده الله بدوام الصحة والعافية.

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Introduction

In recent years, mathematical models of fractional calculus have been widely used in various disciplines, including mathematics, physics, chemistry, technology, computer science, medicine, biology, geology, economics and other areas of natural sciences and engineering. Over the past decades and in a large part of the literature, we can find the concept of "memory" as the main advantage for dealing with fractional differential equations. This property has a significant impact on the behavior of solutions to the models studied **[35],[37],[40]**,[**41**]**,[50],[51],[54]**. The property of nonlocality of fractional derivatives which gives priority to the use of fractional calculus, means that computing the time-fractional derivative of a function $f(t)$ at a time $t = t_1$ requires all of the previous history. This effect justifies the use of fractional calculus to better explain real-life models. In general, these models have made greater progress than those without the concept of memory. Recently, several authors have conducted important research in the field of fractional calculus and its applications in wide range of issues in the fields of science and engineering, such as thermodynamics [**43**], electrodynamics [**56**], network dynamics [**58**], finance [**55**], mathematical biology [**46**], and optimal control theory [**4**].

There are many natural phenomena that appear in various areas of science and engineering represented by differential equations. Today, there is special interest in fractional partial differential equations, especially nonlinear equations, due to their influence in real-world problems like diffusion processes, fluid flow, electromagnetic waves, fractional stochastic systems, control processing, etc. The majority of scientific problems in physics, engineering and biological systems are nonlinear and it is very difficult to obtain their exact solutions. For this reason, numerical and approximate techniques must be applied.

In 2023, Ali Khalouta [**36**] developed a new general integral transform called Khalouta transform (KHT) which is a generalization of many well-known integral transforms such as: Laplace transform, Sumudu transform, ZZ transform, ZMA transform, Elzaki transform, Aboodh transform, Natural transform and Shehu transform. KHT introduces a distinctive mathematical method that provides a different viewpoint and technique for dealing with mathematical expressions and problems in the time domain for $t \geq 0$. This is so because KHT provides a simple mathematical method. KHT is currently recognized as a basic approach for solving linear differential equations, but it lacks the ability to handle nonlinear equations.

The reduced differential transform method (RDTM) was first proposed by the Turkish mathematician Keskin [**34**] in 2009, to construct a suitable solution of some highly nonlinear partial differential equations of mathematical physics. RDTM is an iterative procedure for obtaining the Taylor series solution of differential equations. This method reduces the size of the computational work and can be easily applied to many nonlinear physical problems.

The goal of this thesis is to propose a novel hybrid technique called Khalouta reduced differential transform method (KHRDTM) to solve a nonlinear time-fractional biological population model and evaluate the approximate solution.

The nonlinear time-fractional biological population model is given as follows:

$$^{C}D_t^{\alpha} u = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + h u^{\mu}\left(1 - q u^{\delta}\right),$$

subject to the initial condition

$$u(x, y, 0) = u_0(x, y),$$

where $^{C}D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ represents the time-fractional derivative operator in the Caputo's sense of the function u of order α with $0 < \alpha \leq 1$ and $u = \left\{u(x, y, t), (x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{+}\right\}$ represents the population density and h, μ, q, δ are real numbers.

Our methodology combines the Khalouta transform (KHT) and the reduced differential transform method (RDTM), avoiding linearization or discretization of variables and thus providing both approximate and accurate solutions.

The remainder of this thesis is arranged as follows:

Chapter 1 covers important definitions, properties, lemmas, and theorems, as well as some useful results on the theory of fractional calculus, which is a very important area to study fractional partial differential equations.

In Chapter 2, we provide some results regarding the existence and uniqueness of the solution of the fractional differential equation under Caputo's fractional derivative using Banach's fixed point theorem.

In Chapter 3, we introduce some approximate analytical techniques for solving integer-order nonlinear differential equations. These techniques are: Adomian decomposition technique (ADT), homotopy perturbation technique (HPT), variational iteration technique (VIT), and Daftardar-Gejji-Jafari technique (DGJT).

Chapter 4 discusses the methods referred to in resolving nonlinear Caputo time-fractional partial differential equations. This discussion is strengthened with many diverse numerical examples.

In Chapter 5, we presente the methodology of KHRDTM for solving general nonlinear time-fractional partial differential equation. Furthermore, the proposed method is applied to determine the exact solution of the nonlinear time-fractional biological population model, accompanied by three illustrative applications. The exact solution is described in terms of the Mittag-Leffler functions.

Finally, the last part presents the conclusions drawn from this study and future work.

Chapter 1

Fundamentals of fractional calculus theory

In this chapter, we present definitions and theorems concerning fractional calculus such as specific functions of fractional integration, fractional derivation, and other notions to better understand our work. We will start by providing some examples of applications of fractional calculus theory in specific scientific fields.

1.1 Fractional systems and their applications

Fractional systems appear increasingly in different research domains. However, the growing interest in these systems is their applications in various types of problems in pure mathematics, applied mathematics and mathematical physics by modeling a wide range of natural phenomena. It can be seen that for the most of the domains described below, fractional operators are used to take into account memory effects. Let us mention the works **[27]**,**[54]** that combine different applications of fractional calculus.

1.1.1 Automatic systems

In automatic systems, some researchers have utilised control laws introducing fractional derivatives. Podlubny **[51]** showed that the best method for ensuring efficient control of fractional systems is the use of fractional controllers. It proposes a generalization of traditional PID controllers. The CRONE group, founded by Oustaloup in the 70s, applies these methods to numerous industrial systems : spectroscope, car suspension [**48**], robot-picker, electro-hydraulic plow, car battery, etc...

1.1.2 Physics

One of the most notable applications of fractional calculus in physics has been in the context of classical mechanics. Fred Riewe [**53**] showed that the Lagrangian containing time-fractional derivatives leads to an equation of motion with non-conservative forces such as friction. This result is remarkable because frictional forces and non-conservative forces are essential in the usual macroscopic variational treatment, and therefore in the most advanced methods of classical mechanics. Riewe generalized the usual calculation of Lagrangian variations which depends on fractional derivatives [**52**] in order to deal with the usual non-conservative forces. On the other hand, several approaches have been developed to generalize the principle of least action and the Euler-Lagrange equation in the case of fractional derivatives [**3**],[**5**].

1.1.3 Mechanics of continuous media

The deformation of continuous media (solid or liquid) is often described using of two tensors, the deformations denoted by ε_{ij} and those of stresses are denoted by σ_{ij}.Some materials, such as polymers (eraser, rubber,...) show an intermediate between viscous and elastic properties, which are described as viscoelastic. Such systems can be modeled using the following relationship between the two tensors

$$\sigma_{ij} = E\varepsilon_{ij}\left(t\right) + \eta D^{\alpha}\varepsilon_{ij}\left(t\right), 0 < \alpha < 1.$$

This law is justified by Bagley and Torvik in [**7**],[**8**] (for $\alpha = \frac{1}{2}$). In [**49**], the introduction of fractional derivatives in the case of polymers is motivated by the following analysis: due to the length of the fibers, the applied deformations take time to be communicated step by step (the length of the wound fibers being much greater than the geometric distance). They are gradually amortized and induce memory effects (the state at time t will depend on previous states). If the constraint decreases when $t^{-(1+\alpha)}$, this can lead to a fractional order derivative. This operator thus makes it possible to give a simple macroscopic description (requiring few parameters) of complex microscopic phenomena. A presentation of viscoelasticity via fractional derivation is given in [**18**].

1.1.4 Acoustics

For certain wind musical instruments, visco-thermal losses can be effectively modeled using fractional derivatives in time [**26**].

1.2 Functional spaces

In this part, we provide some fundamental definitions, notations, and properties of functional analysis theory, which is an indispensable tool in fractional calculus theory.

1.2.1 Spaces of integrable functions

Definition 1.2.1 *[12] Let* $\Omega = [0, T]$ $(0 < T < +\infty)$ *a finite interval of* $\mathbb{R}$ *and* $1 \leq p \leq \infty$.

1) For $1 \leq p < \infty$, *the space* $L^p(\Omega)$ *is the space of real functions* f *on* Ω *such as* f *is measurable and*

$$\int_0^T |f(t)|^p \, dt < \infty.$$

2) For $p = \infty$, *the space* $L^\infty(\Omega)$ *is the space of measurable functions* f *bounded almost everywhere (a.e) on* Ω.

Theorem 1.2.1 *[12] Let* $\Omega = [0, T]$ $(0 < T < +\infty)$ *a finite interval of* $\mathbb{R}$.

1) For $1 \leq p < \infty$, *the space* $L^p(\Omega)$ *is a Banach space with the norm*

$$\|f\|_p = \left(\int_0^T |f(t)|^p \, dt \right)^{1/p} < \infty.$$

2) The space $L^\infty(\Omega)$ *is a Banach space with the norm*

$$\|f\|_\infty = \inf \{ M \geq 0 : |f(t)| \leq M \ \ a.e \ on \ \Omega \}.$$

1.2.2 Spaces of continuous and absolutely continuous functions

Definition 1.2.2 *[41] Let $\Omega = [0,T]$ $(0 < T < +\infty)$ a finite interval of $\mathbb{R}$ and $n \in \mathbb{N}$.*

We designate by $C^n(\Omega)$ the space of functions f which have their derivatives of order less than or equal to n continuous on Ω , with the norm

$$\|f\|_{C^n(\Omega)} = \sum_{k=0}^{n} \left\|f^{(k)}\right\|_{C(\Omega)} = \sum_{k=0}^{n} \max_{t\in\Omega} \left|f^{(k)}(t)\right|, \ n \in \mathbb{N}.$$

In particular, if $n = 0$ we have $C^0(\Omega) = C(\Omega)$ is the space of continuous functions f on Ω , with the norm

$$\|f\|_{C(\Omega)} = \max_{t\in\Omega} |f(t)|\,.$$

Definition 1.2.3 *[41] Let $\Omega = [0,T]$ $(0 < T < +\infty)$ a finite interval of $\mathbb{R}$.*

We designate by $AC(\Omega)$ the space of primitive functions of integrable functions, i.e.

$$AC(\Omega) = \left\{ f/\exists\varphi \in L^1(\Omega) : f(t) = c + \int_0^t \varphi(s)ds \right\},$$

and we call $AC(\Omega)$ the space of absolutely continuous functions on Ω.

Definition 1.2.4 *[41] For $n \in \mathbb{N}^*$ we designate by $C^n_\mu(\Omega)$ the space of functions f which have continuous derivatives on Ω up to order $(n-1)$ such as $f^{(n-1)} \in AC(\Omega)$, i.e.*

$$AC^n(\Omega) = \{f : \Omega \longrightarrow \mathbb{C},\ f^{(k)} \in C(\Omega), k \in \{0,1,2...n-1\}\,,\ f^{(n-1)} \in AC(\Omega)\}.$$

In particular, for $n = 1$ we have $AC^1(\Omega) = AC(\Omega)$.

A characterization of the functions of this space is given by the following lemma.

Lemma 1.2.1 *[41] A function $f \in AC^n(\Omega), n \in \mathbb{N}^*$ if and only if it is represented as follows*

$$f(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f^{(n)}(\tau)d\tau + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

1.2.3 Spaces of continuous functions with weights

Definition 1.2.5 *[41] Let $\Omega = [0, T]$ $(0 < T < +\infty)$ a finite interval of $\mathbb{R}$ and $\mu \in \mathbb{C}$ $(0 \leq \mathrm{Re}(\mu) < 1)$.*

We designate by $C_\mu(\Omega)$ the space of functions f definite on Ω such as the function $t^\mu f(t) \in C(\Omega)$, i.e.

$$C_\mu(\Omega) = \{f : \Omega \longrightarrow \mathbb{C}, (.)^\mu f(.) \in C(\Omega)\},$$

with the norm

$$\|f\|_{C_\mu(\Omega)} = \|t^\mu f(t)\|_{C(\Omega)} = \max_{t\in\Omega} |t^\mu f(t)|.$$

The space $C_\mu(\Omega)$ is called the space of continuous functions with weights.

In particular, for $\mu = 0$ we have $C_0(\Omega) = C(\Omega)$.

Definition 1.2.6 *[41] For $n \in \mathbb{N}^*$, we designate by $C_\mu^n(\Omega)$ the space of functions f which have continuous derivatives on Ω up to order $(n-1)$ such as $f^{(n)} \in C_\mu(\Omega)$, i.e.*

$$C_\mu^n(\Omega) = \{f : \|f\|_{C_\mu^n(\Omega)} = \sum_{k=0}^{n-1} \left\|f^{(k)}\right\|_{C(\Omega)} + \left\|f^{(n)}\right\|_{C_\mu(\Omega)}\}.$$

In particular, for $n = 0$ we have $C_\mu^0(\Omega) = C_\mu(\Omega)$.

1.2.4 Fixed point theorem

Definition 1.2.7 *Let X be a Banach space and $T : X \longrightarrow X$ continuous application. We say that T is contracting if T is Lipschitzian with ratio $K < 1$, such as*

$$\forall x, y \in X : \|T(x) - T(y)\| \leq K \|x - y\|.$$

Theorem 1.2.2 ***(Banach theorem)*** ***[30]*** *Let X be a Banach space and $T : X \longrightarrow X$ a contracting operator, then T admits a unique fixed point, i.e. $\exists! u^* \in X$ such as*

$$Tu^* = u^*.$$

In addition, If T^k , $k \in \mathbb{N}$ is an operator sequence defined by

$$T^1 = T \text{ et } T^k = TT^{k-1}, k \in \mathbb{N} \setminus \{1\},$$

then for every $u_0 \in X$ a sequence $\left\{T^k u_0\right\}_{k=0}^{\infty}$ converges to the fixed point u^*.

$$\lim_{k\to\infty} \left\|T^k u_0 - u^*\right\| = 0.$$

1.3 Special functions for fractional derivation

In this part, we introduce some special functions such as the Gamma function, the Bêta function, and the Mittag-Leffler function. These functions play a very important role in the fractional calculus theory and its applications.

1.3.1 The Gamma function

One of the fundamental functions of fractional calculus is **the Gamma function** $\Gamma(z)$ which naturally extends the factorial to positive real numbers (and even to complex numbers with positive real parts).

Definition 1.3.1 *[50] For $z \in \mathbb{C}$ such that $\operatorname{Re}(z) > 0$. The Gamma function $\Gamma(z)$ is given by the following integral*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt, \tag{1.3.1}$$

with $\Gamma(1) = 1$ and $\Gamma(0^+) = +\infty$.

The Gamma function $\Gamma(z)$ is a monotonic and strictly decreasing function for $0 < z < 1$.

An important property of the Gamma function $\Gamma(z)$ is the following recurrence relation

$$\Gamma(z+1) = z\Gamma(z), Re(z) > 0, \tag{1.3.2}$$

which can be demonstrated by integration by parts

$$\Gamma(z+1) = \int_0^{+\infty} e^{-t} t^{z} dt = \left[-e^{-t} t^{z}\right]_0^{+\infty} + z \int_0^{+\infty} e^{-t} t^{z-1} dt = z\Gamma(z).$$

The Gamma function generalizes the factorial because $\Gamma(n+1) = n!,\ \forall n \in \mathbb{N}$.

Indeed $\Gamma(1) = 1$ and using relation (1.3.2) we get

$$\begin{aligned}
\Gamma(2) &= 1\Gamma(1) = 1!,\\
\Gamma(3) &= 2\Gamma(2) = 2.1! = 2!,\\
\Gamma(4) &= 3\Gamma(3) = 3.2! = 3!,\\
&\ \ \vdots\\
\Gamma(n+1) &= n\Gamma(n) = n(n-1)! = n!.
\end{aligned}$$

1.3.2 The Bêta function

The Bêta function is one of the fundamental functions of fractional calculus. This function plays an important role when combined with the Gamma function.

Definition 1.3.2 *[50] The Bêta function is a type of Euler integral defined for complex numbers z and w by*

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt, \mathrm{Re}(z)>0, \mathrm{Re}(w)>0. \tag{1.3.3}$$

The Bêta function is related to the Gamma function by the following relation

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, Re(z)>0, Re(w)>0, \tag{1.3.4}$$

it follows from (1.3.4) that

$$B(z,w)=B(w,z), Re(z)>0, Re(w)>0.$$

1.3.3 The Mittag-Leffler function

The Mittag-Leffler function plays a very important role in the theory of integer differential equations. It is also widely used in the search for solutions of fractional differential equations, this function was introduced by G.M. Mittag-Leffler G.M. Mittag-Leffler **[44]**,**[45]**.

Definition 1.3.3 *[50] For $z\in\mathbb{C}$. The Mittag-Leffler function $E_\alpha(z)$ of a parameter α is defined as*

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \alpha>0. \tag{1.3.5}$$

In particular

$$E_1(z)=e^z \text{ and } E_2(z)=\cosh(\sqrt{z}).$$

This function can be generalized for two parameters α and β to give

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha>0,\ \beta>0. \tag{1.3.6}$$

1.4 Fractional integrals and derivatives

In the literature, there are several definitions of fractional integral and derivative. Among these definitions, the most widely used is the Riemann-Liouville and Caputo sense. Therefore, in this part, we give some basic definitions and important properties of fractional calculus theory such as Riemann-Liouville fractional integral and Caputo fractional derivative.

1.4.1 Fractional integral in the sense of Riemann-Liouville

The idea of the fractional integral of order $\alpha \in \mathbb{C}$ $(Re(\alpha) > 0)$, according to the Riemann-Liouville approach, generalizes the Cauchy formula for an integral repeated n-times.

Let f be a continuous function on the interval $[0, T], T > 0$. A primitive of f is given by

$$I^1 f(t) = \int_0^t f(\tau)d\tau.$$

For a second primitive and according to Fubini's theorem we will have

$$\begin{aligned} I^2 f(t) &= \int_0^t I^1 f(u)du = \int_0^t \left(\int_0^u f(\tau)d\tau \right) du = \int_0^t \left(\int_\tau^t du \right) f(\tau)d\tau \\ &= \int_0^t (t-\tau)\, f(\tau)d\tau. \end{aligned}$$

By repeating n-times, we obtain at the n^{ieme} primitive of the function f in the form

$$I^n f(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)d\tau, t > 0, n \in \mathbb{N}^*. \tag{1.4.1}$$

This formula is called Cauchy formula, and since the generalization of the factorial by the Gamma function $\Gamma(n) = (n-1)!$, Riemann realized that the second member of equation (1.4.1) could have meaning even when n takes a non-integer value, he defined the fractional integral in the following way.

Definition 1.4.1 *[41],[50] Let $f \in L^1([0,T]), T > 0$. The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ of the function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ t > 0, \tag{1.4.2}$$

where I^α represents the Riemann–Liouville fractional integral operator and $\Gamma(.)$ is the Gamma function given by (1.3.1).

Theorem 1.4.1 *[41],[50] If $f \in L^1([0,T]), T > 0$, then $I^\alpha f$ exists for almost all $t \in [0,T]$ and moreover $I^\alpha f \in L^1([0,T])$.*

Proof. Using equation (1.4.2) and Fubini's theorem, we find

$$\begin{aligned}
\int_0^t |I^\alpha f(t)| \, dt &\leq \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^t (t-\tau)^{\alpha-1} |f(\tau)| \, d\tau dt \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^T |f(\tau)| \left(\int_\tau^T (t-\tau)^{\alpha-1} \, dt \right) d\tau \\
&\leq \frac{1}{\alpha\Gamma(\alpha)} \int_0^T |f(\tau)| \, (T-\tau)^\alpha d\tau \\
&\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \int_0^T |f(\tau)| \, d\tau.
\end{aligned}$$

Since $f \in L^1([0,T])$, the last quantity is finite, which establishes the result. ■

Example 1.4.1 ***The integral of** $f(t) = t^\beta$ **in the sense of Riemann-Liouville.***

Let $\alpha > 0, \beta > -1$, then we have

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^\beta d\tau. \tag{1.4.3}$$

By changing the variable

$$\tau = ts,$$

where $s = 0$ when $\tau = 0$ and $s = 1$ when $\tau = t$ and $d\tau = tds$, then equation (1.4.3) becomes

$$\begin{aligned} I^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (t - ts)^{\alpha-1} (ts)^{\beta} tds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 [t(1-s)]^{\alpha-1} t^{\beta+1} s^{\beta} ds \\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{\beta} ds \\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{(\beta+1)-1} ds. \end{aligned}$$

Using the definition of the Bêta function (1.3.3) and relation (1.3.4), we obtain

$$\begin{aligned} I^{\alpha} f(t) &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)} B(\alpha, \beta+1) \\ &= \frac{t^{\alpha+\beta}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \end{aligned}$$

So the fractional integral in the sense of Rieman–Liouville of the function $f(t) = t^{\beta}$ is given as

$$I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{1.4.4}$$

In particular, relation (1.4.4) shows that the fractional integral in the sense of Rieman–Liouville of order α of a constant $c \in \mathbb{R}$ is given as

$$I^{\alpha} c = \frac{c}{\Gamma(\alpha+1)} t^{\alpha}.$$

Proposition 1.4.1 ***[41],[50]*** *Let $\alpha, \beta \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0)$. For any function $f \in L^1([0,T]), T > 0$ we have*

$$I^{\alpha}\left(I^{\beta} f(t)\right) = I^{\alpha+\beta} f(t) = I^{\beta}\left(I^{\alpha} f(t)\right),$$

for almost all $t \in [0,T]$. Moreover, if $f \in C([0,T])$, then this identity is $\forall t \in [0,T]$.

Proof. Suppose that $f \in L^1([0,T])$, then we have

$$\begin{aligned} I^{\alpha}\left(I^{\beta} f(t)\right) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} I^{\beta} f(\tau) d\tau \\ &= \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \int_0^t (\tau-\zeta)^{\beta-1} f(\zeta) d\zeta d\tau. \end{aligned}$$

According to Theorem 1.4.1, the integrals appearing in the previous equality exist for almost all $t \in [0,T]$ and Fubini's theorem allows us to establish

$$I^{\alpha}\left(I^{\beta} f(t)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \left(\int_{\zeta}^t (t-\tau)^{\alpha-1} (\tau-\zeta)^{\beta-1} d\tau \right) f(\zeta) d\zeta.$$

By changing the variable

$$\tau = \zeta + (t-\zeta)s,$$

where $s = 0$ when $\tau = \zeta$ and $s = 1$ when $\tau = t$ and $d\tau = (t-\zeta)ds$, we get

$$I^{\alpha}\left(I^{\beta} f(t)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-\zeta)^{\alpha+\beta-1} \left(\int_0^1 (1-s)^{\alpha-1} s^{\beta-1} ds \right) f(\zeta) d\zeta.$$

Finally, according to the definition of the beta function (1.3.3) and the relation (1.3.4), we get

$$I^{\alpha}\left(I^{\beta} f(t)\right) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-\zeta)^{\alpha+\beta-1} f(\zeta) d\zeta = \left(I^{\alpha+\beta} f\right)(t).$$

Now we assume that $f \in C([0,T])$, then by the theorems on integrals depending on the parameters $I^{\alpha} f \in C([0,T])$, and hence we have

$$I^{\alpha+\beta} f, I^{\alpha} I^{\beta} f \ \in C([0,T]).$$

Thus, according to the above, the two continuous functions $I^{\alpha+\beta} f, I^{\alpha} I^{\beta} f$ coincide almost everywhere on $[0,T]$, therefore they must coincide everywhere on $[0,T]$.

The proof of this Proposition is complete. ■

The following theorem provides a result concerning the inversion of the limit and the fractional integral.

Theorem 1.4.2 *[41],[50] Let $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ and $(f_k)_{k=1}^{+\infty}$ be a sequence of continuous and simply convergent functions on [0; T]. Then we can invert the fractional integral in the sense of Riemann-Liouville and the limit sign as follows*

$$\left[I^{\alpha}(\lim_{k \longrightarrow +\infty} f_k)\right](t) = \lim_{k \longrightarrow +\infty} I^{\alpha} f_k(t).$$

Proof. Let $f_k \longrightarrow f$ be simply convergent and

$$\begin{aligned}
|I^{\alpha} f_k(t) - I^{\alpha} f(t)| &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f_k(\tau) - f(\tau)| \, d\tau \\
&\leq \frac{\|f_k - f\|_{\infty}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \, d\tau \\
&\leq \frac{\|f_k - f\|_{\infty}}{\Gamma(\alpha)} \frac{1}{\alpha} t^{\alpha} \\
&\leq \frac{t^{\alpha}}{\alpha \Gamma(\alpha)} \|f_k - f\|_{\infty} \\
&\leq \frac{1}{\Gamma(\alpha+1)} \|f_k - f\|_{\infty} T^{\alpha} \underset{k \longrightarrow +\infty}{\longrightarrow} 0.
\end{aligned}$$

This is the desired result. ■

1.4.2 Fractional derivative in the sense of Riemann–Liouville

Definition 1.4.2 *[41],[50] Let $f \in L^1([0,T]), T > 0$ an integrable function on $[0,T]$. The fractional derivative in the sense of Riemann-Liouville of order $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$ of the function f is defined as*

$$\begin{aligned}
D^{\alpha} f(t) &= D^n I^{n-\alpha} f(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, t > 0,
\end{aligned} \qquad (1.4.5)$$

where D^{α} represents the Riemann–Liouville fractional derivative operator and $n-1 < [\mathrm{Re}(\alpha)] < n, n \geq 1$ with $[\mathrm{Re}(\alpha)]$ is the integer part of $\mathrm{Re}(\alpha)$.

In particular, if $\alpha = 0$, then

$$D^0 f(t) = \frac{1}{\Gamma(1)} \left(\frac{d}{dt}\right) \int_0^t f(\tau) d\tau = f(t).$$

If $\alpha = n \in \mathbb{N}$, then

$$D^n f(t) = \frac{1}{\Gamma(1)}\left(\frac{d^{n+1}}{dt^{n+1}}\right)\int_0^t f(\tau)d\tau = \frac{d^n}{dt^n}f(t) = f^{(n)}(t).$$

Therefore, the fractional derivative in the sense of Riemann–Liouville coincides with the classical derivative for $\alpha \in \mathbb{N}$.

Moreover, if $0 < \alpha < 1$, then $n = 1$ and therefore we have

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha} f(\tau)d\tau,\ t > 0.$$

Example 1.4.2 ***The derivative of*** $f(t) = t^\beta$ ***in the sense of Riemann-Liouville.***

Let $\alpha > 0$ *such that* $n-1 < \alpha < n$ *and* $\beta > -1$, *according to the definition of the fractional derivative in the sense of Riemann-Liouville in equation (1.4.5) and relation (1.4.4), we get*

$$D^\alpha t^\beta = D^n I^{n-\alpha} t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)} D^n t^{\beta+n-\alpha}. \tag{1.4.6}$$

Taking into account

$$\begin{aligned} D^n t^{\beta+n-\alpha} &= (\beta+n-\alpha)(\beta+n-\alpha-1).......(\beta-\alpha+1)t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \end{aligned} \tag{1.4.7}$$

Substituting the result (1.4.7) into the formula (1.4.6), we get

$$\begin{aligned} D^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}\frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \end{aligned}$$

So the fractional derivative in the sense of Riemann-Liouville of the function $f(t) = t^\beta$ *is given as*

$$D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \tag{1.4.8}$$

In particular, if $\beta = 0$ *and* $\alpha > 0$, *then the fractional derivative in the sense of Riemann-Liouville of order* α *of a constant* $c \in \mathbb{R}$ *is non-zero, and its value given as*

$$D^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}.$$

It is easy to determine the following result.

Lemma 1.4.1 *Let $n-1<\alpha<n$ with $n=[\alpha]+1$ and f be a function verifying*

$$D^{\alpha}f(t)=0,$$

then

$$f(t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-n)}t^{j+\alpha-n},\ \ \forall c_0,c_1,...,c_{n-1}\in\mathbb{R}.$$

In particular, if $0<\alpha<1$, then

$$f(t)=ct^{\alpha-1},\forall c\in\mathbb{R}.$$

Proof. Let $D^{\alpha}f(t)=0$, according to (1.4.5) we have

$$D^{\alpha}f(t)=D^{n}I^{n-\alpha}f(t)=0.$$

And as a result we have

$$I^{n-\alpha}f(t)=\sum_{j=0}^{n-1}c_jt^j. \tag{1.4.9}$$

Now, applying the Riemann–Liouville fractional integral operator I^{α} to equation (1.4.9) we get

$$I^{n}f(t)=\sum_{j=0}^{n-1}c_jI^{\alpha}t^j.$$

Using the relation (1.4.4) (See Example 1.4.1), we get

$$I^{n}f(t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}t^{j+\alpha}. \tag{1.4.10}$$

Applying the operator D^n to equation (1.4.10), we obtain

$$f(t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)}D^{n}t^{j+\alpha}.$$

The use of the classical derivation of the formula

$$D^{n}t^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}t^{\alpha-n},$$

gives

$$f(t)=\sum_{j=0}^{n-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+\alpha+1)} \frac{\Gamma(j+\alpha+1)}{\Gamma(j+\alpha-n+1)} t^{j+\alpha-n}.$$

Finally we get

$$f(t)=\sum_{j=0}^{n-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-n)} t^{j+\alpha-n}.$$

The proof of this Lemma is complete. ■

The following proposition establishes a sufficient condition for the existence of the fractional derivative in the sense of Riemann-Liouville.

Proposition 1.4.2 *[41] Let $\alpha \geq 0$ and $n=[\alpha]+1$. If $f \in AC^n([0,T]), T>0$, then the fractional derivative in the sense of Riemann-Liouville $D^\alpha f$ exists almost everywhere on $[0,T]$. Moreover it is represented as*

$$D^\alpha f(t)=\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}+\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. \qquad (1.4.11)$$

1.4.3 Properties of the fractional derivative in the Riemann-Liouville sense

The Riemann–Liouville fractional derivative operator has properties that can be summarized in the following propositions.

Proposition 1.4.3 *[41],[50] For $n-1<\alpha \leq n$ and $m-1<\beta \leq m$ we have*

1) The Riemann–Liouville fractional derivative operator is linear

$$D^\alpha(\lambda f+g)(t)=\lambda(D^\alpha f)(t)+(D^\alpha g)(t), \lambda \in \mathbb{R}.$$

2) Generally

$$D^\alpha(D^\beta f(t)) \neq D^\beta(D^\alpha f(t)).$$

Proof. 1) Let $f, g \in L^1([0,T])$ and $\lambda \in \mathbb{R}$, so we have

$$\begin{aligned} D^{\alpha}\left(\lambda f+g\right)(t) &= D^{n}I^{n-\alpha}\left(\lambda f(t)+g(t)\right) \\ &= \lambda D^{n}I^{n-\alpha}\left(\left(f+g\right)(t)\right). \end{aligned}$$

As the n^{th} derivative and the integral are linear, then

$$\begin{aligned} D^{\alpha}\left(\lambda f+g\right)(t) &= \lambda D^{n}I^{n-\alpha}f(t)+D^{n}I^{n-\alpha}g(t) \\ &= \lambda\left(D^{\alpha}f\right)(t)+\left(D^{\alpha}g\right)(t). \end{aligned}$$

2) We have

$$D^{\alpha}(D^{\beta}f(t))=D^{\alpha+\beta}f(t)-\sum_{k=0}^{m-1}D^{\beta-k}f(0)\frac{t^{-\alpha-k}}{\Gamma(1-\alpha-k)},$$

and

$$D^{\beta}(D^{\alpha}f(t))=D^{\alpha+\beta}f(t)-\sum_{k=0}^{n-1}D^{\alpha-k}f(0)\frac{t^{-\beta-k}}{\Gamma(1-\beta-k)}.$$

Consequently, the two fractional derivation operators commute only if $\alpha=\beta$ and $D^{\alpha-k}f(0)=0$ for all $k=1,2,...,n$ and $D^{\beta-k}f(0)=0$ for all $k=1,2,...,m$.

Which completes the proof. ■

Proposition 1.4.4 *[41],[50] Let $\alpha,\beta>0$ such that $n-1<\alpha\leq n$ and $m-1<\beta\leq m$ where $n,m\in\mathbb{N}^{*}$.*

1) For $f\in L^{1}([0,T]),T>0,$ the following equality

$$D^{\alpha}\left(I^{\alpha}f(t)\right)=f(t),$$

is true for almost all $t\in[0,T]$.

2) If $\alpha>\beta>0,$ then for $f\in L^{1}([0,T]),T>0,$ the following relation

$$D^{\beta}(I^{\alpha}f(t))=I^{\alpha-\beta}f(t),$$

is true almost everywhere on $t\in[0,T]$.

In particular, if $\beta=k\in\mathbb{N}$ and $\alpha>k,$ then

$$D^{k}(I^{\alpha}f(t))=I^{\alpha-k}f(t).$$

3) If $\beta\geq\alpha>0$ and the fractional derivative $D^{\beta-\alpha}f$ exists, then we have

$$D^{\beta}(I^{\alpha}f(t))=D^{\beta-\alpha}f(t).$$

4) For $\alpha > 0$, $k \in \mathbb{N}^*$. If the fractional derivatives $D^\alpha f$ and $D^{k+\alpha} f$ exist, then

$$D^k(D^\alpha f(t)) = D^{k+\alpha} f(t).$$

Proof. 1) Using equation (1.4.5) and Proposition 1.4.1, we have for $n = [\alpha] + 1$

$$D^\alpha(I^\alpha f(t)) = D^n I^{n-\alpha} I^\alpha f(t) = f(t), \text{ a.e on } [0, T].$$

2) According to equation (1.4.5) and Proposition 1.4.1 we obtain for $\alpha > \beta > 0$, then $n \geq m$ and we have

$$\begin{aligned} D^\beta(I^\alpha f(t)) &= D^n I^{n-\beta}(I^\alpha f)(t) \\ &= D^n \left(I^{n+\alpha-\beta} f\right)(t) \\ &= D^n I^n \left(I^{\alpha-\beta} f\right)(t) \\ &= \left(I^{\alpha-\beta} f\right)(t). \end{aligned}$$

3) For $\beta \geq \alpha > 0$, we have

$$\begin{aligned} D^\beta(I^\alpha f(t)) &= D^m I^{m-\beta}(I^\alpha f)(t) \\ &= D^m I^{m-(\beta-\alpha)} f(t) \\ &= \left(D^{\beta-\alpha} f\right)(t), \end{aligned}$$

exists for $i - 1 \leq \beta - \alpha < i$ and $i \leq m$.

4) For $\alpha > 0$ and $k \in \mathbb{N}^*$, we have

$$\begin{aligned} D^k(D^\alpha f(t)) &= D^k D^n I^{n-\alpha} f(t) \\ &= D^{k+n} I^{n-\alpha+k-k} f(t) \\ &= D^{k+n} I^{k+n-(k+\alpha)} f(t) \\ &= (D^{k+\alpha} f)(t). \end{aligned}$$

The proof of this Proposition is complete. ■

1.4.4 Fractional derivative in the Caputo sense

Although the fractional derivative in the Riemann-Liouville sense has played an important role in the development of fractional calculus, several authors including Caputo (1967-1969) have reported that this definition needs to be revised **[13]**, because the problems applied in viscoelasticity, solid mechanics and rheology, require initial conditions physically interpretable by classical derivatives such as $f(0)$, $f'(0), f''(0)$ etc..., which is not the case in modeling by the Riemann-Liouville approach which requires knowledge of the initial conditions of the fractional derivatives.

Despite the fact that initial value problems with such initial conditions can be solved mathematically, the solution of this problem was proposed by M. Caputo in his definition which he adapted with F. Mainardi in the structure of the viscoelasticity theory. **[14]**.

In this part, we give the definition of the fractional derivative in the Caputo sense as well as some basic properties.

Let $[0,T]$ be a finite interval of $\mathbb{R}$ and let I^α and D^α be the Riemann–Liouville fractional integral and derivative operators given by (1.4.2) and (1.4.5) respectively.

Definition 1.4.3 *[41] The Caputo fractional derivative ${}^CD^\alpha f(t)$ order $\alpha \in \mathbb{C}$ $(\mathrm{Re}(\alpha) \geq 0)$ on the interval $[0,T]$ is defined via the Riemann-Liouville fractional derivative by*

$$ {}^CD^\alpha f(t) = D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right), \tag{1.4.12}$$

where

$$ n = [\mathrm{Re}(\alpha)] + 1 \textit{ for } \alpha \notin \mathbb{N} \textit{ and } n = \alpha \textit{ for } \alpha \in \mathbb{N}. \tag{1.4.13}$$

If $\alpha = 0$, then we have

$$ {}^CD^0 f(t) = f(t). $$

In particular, if $0 < Re(\alpha) < 1$, the equation (1.4.12) takes the form

$$ {}^CD^\alpha f(t) = D^\alpha \left[f(t) - f(0) \right]. $$

The Caputo fractional derivative (1.4.12) is defined for functions $f(t)$ for which Riemann-Louville derivative (1.4.2) exists in particular, it is defined for functions $f \in AC^n([0,T])$.

Now, we have the following theorem.

Theorem 1.4.3 *[41] Let* $\operatorname{Re}(\alpha) \geq 0$ *and* n *be given by the relation (1.4.13). If* $f \in AC^n([0,T])$*, then the Caputo fractional derivative* ${}^C D^\alpha f(t)$ *exists almost everywhere on* $[0,T]$*.*

1) If $\alpha \notin \mathbb{N}$*, then* ${}^C D^\alpha f(t)$ *is given by*

$$\begin{aligned} {}^C D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \\ &= I^{n-\alpha} D^n f(t), t > 0. \end{aligned} \tag{1.4.14}$$

In particular, if $0 < \operatorname{Re}(\alpha) < 1$ *and* $f \in AC([0,T])$*, then*

$$\begin{aligned} {}^C D^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau \\ &= I^{1-\alpha} f'(t), t > 0. \end{aligned} \tag{1.4.15}$$

2) If $\alpha \in \mathbb{N}$*, then*

$${}^C D^\alpha f(t) = f^{(n)}(t).$$

Proof. From Definition 1.4.3, we have

$$\begin{aligned} {}^C D^\alpha f(t) &= D^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right) \\ &= D^n I^{n-\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right). \end{aligned}$$

Using

$$\begin{aligned} D^\alpha f(t) &= D^n I^{n-\alpha} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \ t > 0. \end{aligned}$$

Let's assume that

$$F(t) = I^{n-\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k \right).$$

From equation (1.4.2), we have

$$F(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \left(f(\tau) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \tau^k \right) d\tau.$$

Integrating by parts, we will have

$$\begin{aligned}
F(t) &= \frac{1}{\Gamma(n-\alpha)}\int\limits_0^t (t-\tau)^{n-\alpha-1}\left(f(\tau)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\tau^k\right)d\tau \\
&= \frac{1}{\Gamma(n-\alpha)}\left\{-\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\left(f(\tau)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\tau^k\right)\Bigg|_{\tau=0}^{\tau=t}\right. \\
&\left. + \int\limits_0^t \frac{(t-\tau)^{n-\alpha}}{n-\alpha}D\left(f(\tau)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\tau^k\right)d\tau\right\} \\
&= \frac{1}{\Gamma(n-\alpha+1)}\int\limits_0^t (t-\tau)^{n-\alpha+1-1}D\left(f(\tau)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\tau^k\right)d\tau \\
&= I^{n-\alpha+1}D\left(f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k\right).
\end{aligned}$$

Repeating this process n-times, we find:

$$\begin{aligned}
F(t) &= I^{n-\alpha+n}D^n\left(f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k\right) \\
&= I^nI^{n-\alpha}D^n\left(f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k\right).
\end{aligned}$$

where $\sum\limits_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k$ is a polynomial of degree $n-1$.

Consequently

$$F(t) = I^nI^{n-\alpha}D^nf(t).$$

Thus

$$\begin{aligned}
{}^CD^\alpha f(t) &= D^nF(t) \\
&= D^nI^nI^{n-\alpha}D^nf(t) \\
&= I^{n-\alpha}D^nf(t) \\
&= \frac{1}{\Gamma(n-\alpha)}\int\limits_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau.
\end{aligned}$$

The proof of this Theorem is complete. ■

Theorem 1.4.4 *[41] Let* $\mathrm{Re}(\alpha) \geq 0$ *and* n *be given by the relation (1.4.13) and* $f \in C^n([0,T])$. *Then the Caputo fractional derivative* ${}^C D^\alpha f(t)$ *is continuous on* $[0,T], T > 0$.

1) If $\alpha \notin \mathbb{N}$, *then* ${}^C D^\alpha f(t)$ *is given by equation (1.4.14). In particular, if* $0 < \alpha < 1$, *then* ${}^C D^\alpha f(t)$ *takes the form (1.4.15).*

2) If $\alpha \in \mathbb{N}$, *then*

$$ {}^C D^\alpha f(t) = f^{(n)}(t). $$

Example 1.4.3 ***The derivative of*** $f(t) = t^\beta$ ***in the Caputo sense.***

Let n *be an integer and* $0 \leq n-1 < \alpha < n$ *with* $\beta > n-1$, *then according to equation (1.4.14) we have*

$$ f^{(n)}(\tau) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)} \tau^{\beta-n}. \tag{1.4.16} $$

where

$$ {}^C D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} \int_0^t (t-\tau)^{n-\alpha-1} \tau^{\beta-n} d\tau. \tag{1.4.17} $$

By changing the variable

$$ \tau = ts, $$

where $s = 0$ *when* $\tau = 0$ *and* $s = 1$ *when* $\tau = t$ *and* $d\tau = tds$.

Equation (1.4.17) becomes as follows

$$ \begin{aligned} {}^C D^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} \int_0^t (t-\tau)^{n-\alpha-1} \tau^{\beta-n} d\tau \\ &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} t^{\beta-\alpha} \int_0^1 (1-s)^{n-\alpha-1} s^{\beta-n} ds. \end{aligned} $$

Using the definition of the Bêta function (1.3.3) and the relation (1.3.4) we obtain

$$ \begin{aligned} {}^C D^\alpha t^\beta &= \frac{\Gamma(\beta+1)B(n-\alpha,\beta-n+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \end{aligned} $$

So the Caputo fractional derivative of the function $f(t) = t^\beta$ is given as

$$^{C}D^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}. \tag{1.4.18}$$

In particular, the use of formula (1.4.12) or (1.4.14) to calculate the fractional derivative in the sense of Caputo of order $\alpha > 0$ of a constant $c \in \mathbb{R}$ expresses that this derivative is zero, i.e.

$$^{C}D^{\alpha}c = 0.$$

1.4.5 Properties of the fractional derivative in the Caputo sense

The fractional derivative in the Caputo sense has properties that can be summarized in the following propositions.

Proposition 1.4.5 *[41],[50] Let $\alpha \in \mathbb{C}$ such that $n-1 < \mathrm{Re}(\alpha) < n$ with $n \in \mathbb{N}^*$ and let the two functions $f(t)$ and $g(t)$ such that $^{C}D^{\alpha}f(t)$ and $^{C}D^{\alpha}g(t)$ exist. The Caputo fractional derivative is a linear operator*

$$^{C}D^{\alpha}\left(\lambda f + \mu g\right)(t) = \lambda\left(^{C}D^{\alpha}f\right)(t) + \mu\left(^{C}D^{\alpha}g\right)(t),\ \ \lambda, \mu \in \mathbb{R}.$$

Proof. According to (1.4.14), we have

$$\begin{aligned} ^{C}D^{\alpha}\left(\lambda f + \mu g\right)(t) &= I^{n-\alpha}D^{n}\left(\left(\lambda f + \mu g\right)(t)\right) \\ &= I^{n-\alpha}D^{n}\left(\lambda f(t)\right) + I^{n-\alpha}D^{n}\left(\mu g(t)\right). \end{aligned}$$

As the n^{th} derivative and the integral are linear, then

$$\begin{aligned} ^{C}D^{\alpha}\left(\lambda f + \mu g\right)(t) &= \lambda I^{n-\alpha}D^{n}f(t) + \mu I^{n-\alpha}D^{n}g(t) \\ &= \lambda\left(^{C}D^{\alpha}f\right)(t) + \mu\left(^{C}D^{\alpha}g\right)(t). \end{aligned}$$

Hence the result. ■

Proposition 1.4.6 *[41],[50] Assume that $n-1 < \mathrm{Re}(\alpha) < n, m, n \in \mathbb{N}^*$ and let the function $f(t)$ such that $^{C}D^{\alpha}f(t)$ exists, then*

$$^{C}D^{\alpha}D^{m}f(t) = ^{C}D^{\alpha+m}f(t) \neq D^{m}\,{}^{C}D^{\alpha}f(t).$$

1.4.6 Relationship between the Riemann-Liouville and Caputo approaches

The following theorem establishes the relationship between the fractional derivative in the Caputo sense and the Riemann-Liouville sense.

Theorem 1.4.5 *[41] Let* $\mathrm{Re}(\alpha) > 0$ *with* $n-1 < \mathrm{Re}(\alpha) < n$ $(n \in \mathbb{N}^*)$ *and let* f *be a funtion such that fractional derivatives* ${}^C D^\alpha f(t)$ *and* $D^\alpha f(t)$ *exist, then*

$$^C D^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}.$$

Proof. Consider the limited Taylor series development of the function f at $t = 0$ as follows

$$\begin{aligned} f(t) &= f(0) + \frac{f'(0)}{1!}t + \frac{f''(0)}{1!}t^2 + ... + \frac{f^{(n-1)}(0)}{(n-1)!}t^{n-1} + R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k+1)}t^k + R_{n-1}, \end{aligned}$$

with

$$R_{n-1} = \int_0^t \frac{f^{(n)}(\tau)}{(n-1)!}(t-\tau)^{n-1}d\tau.$$

Using the properties of n^{th}-order integration, we get

$$R_{n-1} = \frac{1}{\Gamma(n)} \int_a^t f^{(n)}(\tau)(t-\tau)^{n-1}d\tau = I^n f^{(n)}(t).$$

Using the linearity of the Riemann-Liouville operator and relation (1.4.8) we obtain

$$\begin{aligned} D^\alpha f(t) &= D^\alpha \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k+1)}t^k + R_{n-1} \right) \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k+1)} D_a^\alpha t^k + D^\alpha R_{n-1} \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)\Gamma(k+1)}{\Gamma(k-\alpha+1)\Gamma(k+1)} t^{k-\alpha} + D^\alpha I^n f^{(n)}(t) \end{aligned}$$

$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha} + I^{n-\alpha} f^{(n)}(t)$$
$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha} + \left({}^{c}D^{\alpha} f\right)(t).$$

Therefore

$${}^{C}D^{\alpha} f(t) = D^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}.$$

Hence the result. ■

Remark 1.4.1 *If $f^{(k)}(0) = 0$ for $k = 0, 1, 2, ..., n-1$, then we have*

$${}^{C}D^{\alpha} f(t) = D^{\alpha} f(t).$$

Proposition 1.4.7 ***(Composition with the Riemann–Liouville fractional integral operator)***

The Caputo fractional derivative operator is a left-inverse of the Riemann-Liouville fractional integral operator but not a right-hand inverse. Because, if f is a continuous function on $[0,T]$, then we have

$${}^{C}D^{\alpha} \left(I^{\alpha} f(t)\right) = f(t) \ \text{and} \ I^{\alpha} \left({}^{C}D^{\alpha} f(t)\right) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^{k}. \tag{1.4.19}$$

Remark 1.4.2 *The main advantage of the Caputo approach is that the initial conditions of fractional differential equations involving Caputo fractional derivatives take the same form as for integer differential equations.*

Chapter 2

Existence and uniqueness theorem for fractional differential equation with Caputo fractional derivative

In this chapter, we study the existence and uniqueness of the solution to the fractional differential equation involving Caputo fractional derivative by employing the Banach fixed point theorem.

We consider the following fractional differential equation (FDE).

$$^{C}D^{\alpha}u(t) = h(t, u(t)),\ t \in \Omega = [0, T]\,, \tag{2.1.1}$$

subject to the initial conditions

$$u^{(k)}(0) = u_k \in \mathbb{R}, k = 0, 1, ..., n-1, \tag{2.1.2}$$

where $^{C}D^{\alpha}$ is the Caputo fractional derivative operator of order $n-1 < \alpha \leq n$, with $n = [\alpha]+1$ and $[\alpha]$ is the integer part of α, $h(., u) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function with respect to $t \in \Omega$ for all $u \in \mathbb{R}$.

2.1 Equivalence between the FDE and the Volterra integral equation

In this part, we present an equivalence result between the FDE and theVolterra integral equation in the space $C^{n-1}(\Omega)$. According to this result, the existence and uniqueness of the solution of the proposed equation are demonstrated.

Theorem 2.1.1 *[41] Let $\alpha > 0$ with $\alpha \notin \mathbb{N}$, $n = [\alpha] + 1$ and $h(.,u) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with respect to $t \in \Omega$ for all $u \in \mathbb{R}$. Then $u \in C^{n-1}(\Omega)$ is a solution of equations (2.1.1) and (2.1.2) if and only if u is a solution of the following Volterra integral equation*

$$u(t) = \sum_{k=0}^{n-1} \frac{u_k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau, u(\tau)) d\tau, 0 \leq t \leq T. \tag{2.1.3}$$

Proof. Let $\alpha > 0$ with $\alpha \notin \mathbb{N}$ and $n = [\alpha] + 1$.

1) We assume that $u \in C^{n-1}(\Omega)$ is a solution of equations (2.1.1) and (2.1.2). Since $h(.,u) \in C(\Omega)$ for all $u \in \mathbb{R}$ and from equation (2.1.1), we have ${}^C D^\alpha u(t) \in C(\Omega)$.

Using relation (1.4.19), we get

$$I^\alpha \left({}^C D^\alpha u(t)\right) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k.$$

Therefore, we have

$$\begin{aligned} u(t) &= \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k + I^\alpha \left({}^C D^\alpha u(t)\right) \\ &= \sum_{k=0}^{n-1} \frac{u_k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau, u(\tau)) d\tau. \end{aligned}$$

2) Suppose $u \in C^{n-1}(\Omega)$ is a solution of the Volterra integral equation (2.1.3). By deriving equation (2.1.3) k-times ($k = 1, ..., n-1$) and Proposition 1.4.2, we find for all $k = 0, 1, ..., n-1$

$$u^{(k)}(t) = \sum_{j=k}^{n-1} \frac{u_j}{(j-k)!} t^{j-k} + \frac{1}{\Gamma(\alpha-k)} \int_0^t (t-\tau)^{\alpha-k-1} h(\tau, u(\tau)) d\tau.$$

By changing the variable $\tau = ts$, we get

$$u^{(k)}(t) = \sum_{j=k}^{n-1} \frac{u_j}{(j-k)!} t^{j-k} + \frac{t^{\alpha-k}}{\Gamma(\alpha-k)} \int_0^1 (1-s)^{\alpha-k-1} h(ts, u(ts)) ds.$$

By implementing the limit $t \longrightarrow 0$ and using the continuity of the function h, we obtain equation (2.1.2). On the other hand, by applying the Riemann-Liouville fractional derivative operator D^α on the Volterra integral equation (2.1.3) and with (2.1.2), we get

$$D^\alpha \left(u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k \right) = D^\alpha I^\alpha h(t, u(t)).$$

From Definition 1.4.3 and Proposition 1.4.2, we obtain equation (2.1.1).

The proof of this Theorem is complete. ■

Corollary 2.1.1 *Let $0 < \alpha < 1$ and $h(., u) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with respect to $t \in \Omega$ for all $u \in \mathbb{R}$. Then $u \in C(\Omega)$ is a solution of to the following Caputo fractional differential equation*

$$^C D^\alpha u(t) = h(t, u(t)),$$

subject to the initial condition

$$u(0) = u_0,$$

if and only if u is a solution of the following Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau, u(\tau)) d\tau.$$

2.2 Existence and uniqueness theorem

In this part, we demonstrate the existence and uniqueness of the solution of equations (2.1.1) and (2.1.2) in the space of functions $C^{n-1,\alpha}(\Omega))$ given by

$$C^{n-1,\alpha}(\Omega) = \left\{ u \in C^{n-1}(\Omega), ^C D^\alpha u \in C(\Omega), n = [\alpha] + 1 \right\}.$$

To prove the existence and uniqueness of the solution to equations (2.1.1) and (2.1.2), we need the following lemma.

Lemma 2.2.1 *If $\alpha > 0$ with $\alpha \notin \mathbb{N}$ and $n = [\alpha] + 1$, then the fractional integral operator $I^\alpha : C(\Omega) \longrightarrow C^{n-1}(\Omega)$ in the Riemann-Liouville sense is bounded, i.e.*

$$\|I^\alpha f\|_{C^{n-1}(\Omega)} \leq M \|f\|_{C(\Omega)} \text{ with } M = \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)}. \tag{2.2.1}$$

Proof. Let $f \in C(\Omega)$. Proposition 1.4.4, we get:

$$D^k I^\alpha f(t) = I^{\alpha-k} f(t), \text{ for all } k = 0, 1, ..., n-1.$$

For all $t \in \Omega$, we have

$$\begin{aligned} \|I^\alpha f\|_{C^{n-1}(\Omega)} &= \sum_{k=0}^{n-1} \left\|D^k I^\alpha f\right\|_{C(\Omega)} = \sum_{k=0}^{n-1} \left\|I^{\alpha-k} f\right\|_{C(\Omega)} \\ &\leq \sum_{k=0}^{n-1} \frac{\|f\|_{C(\Omega)}}{\Gamma(\alpha-k)} \int_0^t (t-\tau)^{\alpha-k-1} d\tau \\ &\leq \sum_{k=0}^{n-1} \frac{\|f\|_{C(\Omega)}}{(\alpha-k)\Gamma(\alpha-k)} t^{\alpha-k} \\ &\leq \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)} \|f\|_{C(\Omega)}. \end{aligned}$$

We put

$$M = \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)},$$

we get

$$\|I^\alpha f\|_{C^{n-1}(\Omega)} \leq M \|f\|_{C(\Omega)}.$$

The proof of this Lemma is complete. ∎

Theorem 2.2.1 *[41] Let $\alpha > 0$ with $\alpha \notin \mathbb{N}$ and $n = [\alpha] + 1$ and G an open set of $\mathbb{R}$. We suppose that $h : \Omega \times G \longrightarrow \mathbb{R}$ is a function such as*

1) For all fixed $u \in G$, we have $h(., u) \in C(\Omega)$.

2) The function $h : \Omega \times G \longrightarrow \mathbb{R}$ satisfies the Lipschitz condition with respect to u, i.e. there is $L > 0$ such as

$$|h(t, u_1) - h(t, u_2)| \leq L |u_1 - u_2|, \text{ for all } t \in \Omega, u_1, u_2 \in G. \tag{2.2.2}$$

If

$$L\sum_{k=0}^{n-1}\frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)}<1, \tag{2.2.3}$$

then the fractional differential equation (2.1.1) has unique solution $u\in C^{n-1,\alpha}(\Omega)$, which satisfies the initial conditions given in equation (2.1.2).

Proof. To prove this theorem, we begin by demonstrating the existence of a unique solution $u\in C^{n-1}(\Omega)$ of equations (2.1.1) and (2.1.2). By Theorem 2.1.1, it suffices to prove the existence of a unique solution $u\in C^{n-1}(\Omega)$ of the Volterra integral equation (2.1.3). To do this, we use Theorem 1.2.2 of the Banach fixed point in the space $C^{n-1}(\Omega)$ with the following norm

$$\|u_1-u_2\|_{C^{n-1}(\Omega)}=\sum_{k=0}^{n-1}\left\|u_1^{(k)}-u_2^{(k)}\right\|_{C(\Omega)}. \tag{2.2.4}$$

We rewrite the Volterra integral equation (2.1.3) in the form of the following operator

$$u(t)=(Tu)(t),$$

where

$$(Tu)(t)=u_0(t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(\tau,u(\tau))d\tau, \tag{2.2.5}$$

with

$$u_0(t)=\sum_{j=0}^{n-1}\frac{u_j}{j!}t^j. \tag{2.2.6}$$

To apply the Banach fixed point theorem, it is necessary to prove

1- If $u\in C^{n-1}(\Omega)$ then $Tu\in C^{n-1}(\Omega)$.

2- For each

$$u_1,u_2\in C^{n-1}(\Omega),\|Tu_1-Tu_2\|_{C^{n-1}(\Omega)}\leq\phi\|u_1-u_2\|_{C(\Omega)},\ \text{with } 0<\phi<1. \tag{2.2.7}$$

Let $u\in C^{n-1}(\Omega)$. By deriving equation (2.2.5) k-times ($k=1,...,n-1$) and using Proposition 1.4.4, we find for all $k=0,1,...,n-1$

$$(Tu)^{(k)}(t)=u_0^{(k)}(t)+\frac{1}{\Gamma(\alpha-k)}\int_0^t(t-\tau)^{\alpha-k-1}h(\tau,u(\tau))d\tau, \tag{2.2.8}$$

with

$$u_0^{(k)}(t) = \sum_{j=k}^{n-1} \frac{u_j}{(j-k)!} t^{j-k}.$$

For all $k = 0, 1, ..., n-1$, the first term on the right of equation (2.2.8) is a continuous function on $[0, T]$, and by Lemma 2.2.1, the second term is continuous on $[0, T]$.

Therefore, we have

$$\left\| \frac{1}{\Gamma(\alpha - k)} \int_0^t (t-\tau)^{\alpha-k-1} h(\tau, u(\tau)) d\tau \right\|_{C^{n-1}(\Omega)} \leq \frac{T^{\alpha-k}}{\Gamma(\alpha - k + 1)} \left\| h(\tau, u(\tau)) \right\|_{C(\Omega)}, \quad (2.2.9)$$

for all $k = 0, 1, ..., n-1$. Consequently $Tu \in C^{n-1}(\Omega)$.

Using equations (2.2.4), (2.2.8), (2.2.9) and the Lipschitz condition (2.2.2), we obtain

$$\begin{aligned}
\|Tu_1 - Tu_2\|_{C^{n-1}(\Omega)} &= \sum_{k=0}^{n-1} \left\| (Tu_1)^{(k)} - (Tu_2)^{(k)} \right\|_{C(\Omega)} \\
&\leq \sum_{k=0}^{n-1} \left\| \frac{1}{\Gamma(\alpha-k)} \int_0^t (t-\tau)^{\alpha-k-1} \left[h(\tau, u_1(\tau)) - h(\tau, u_2(\tau)) \right] d\tau \right\|_{C(\Omega)} \\
&\leq \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)} \left\| h(\tau, u_1(\tau)) - h(\tau, u_2(\tau)) \right\|_{C(\Omega)} \\
&\leq L \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)} \left\| u_1(\tau) - u_2(\tau) \right\|_{C(\Omega)}.
\end{aligned}$$

Using relation (2.2.3) we obtain (2.2.7) and we have

$$\phi = L \sum_{k=0}^{n-1} \frac{T^{\alpha-k}}{\Gamma(\alpha-k+1)}.$$

Therefore, by the Banach fixed point theorem 1.2.2, there exists a unique solution $u^* \in C^{n-1}(\Omega)$ of the Volterra integral equation (2.1.3) on the interval $[0, T]$.

According to Banach fixed point theorem 1.2.2, the solution $u^*(t)$ is a limit of the convergent sequence $(T^n u_0^*)(t)$

$$\lim_{n\to\infty} \|T^n u_0^* - u^*\|_{C^{n-1}(\Omega)} = 0. \quad (2.2.10)$$

We take

$$u_0^* = u_0,$$

with $u_0(t)$ given by (2.2.6).

From equation (2.2.5), the sequence $\left(T^n u_0^*\right)(t)$ is defined by the recurrence formula

$$\left(T^n u_0^*\right)(t) = u_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h\left(\tau, \left(T^{n-1} u_0^*\right)(\tau)\right) d\tau, n = 1, 2, ... \tag{2.2.11}$$

Noting $u_n(t) = T^n u_0^*(t)$, then equation (2.2.11) takes the following form

$$u_n(t) = u_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau, u_{n-1}(\tau)) d\tau, n \in \mathbb{N},$$

and equation (2.2.10) becomes

$$\lim_{n \longrightarrow \infty} \|u_n - u\|_{C^{n-1}(\Omega)} = 0. \tag{2.2.12}$$

Using equation (2.1.1) and the Lipschitz condition (2.2.2), we get

$$\begin{aligned} \left\|{}^C D^\alpha u_n - {}^C D^\alpha u\right\|_{C(\Omega)} &= \|h(t, u_n(t)) - h(t, u(t))\|_{C(\Omega)} \\ &\leq L \|u_n - u\|_{C(\Omega)} \\ &\leq L \|u_n - u\|_{C(\Omega)} . \end{aligned}$$

According to relation (2.2.12), we obtain

$$\lim_{n \longrightarrow \infty} \left\|{}^C D^\alpha u_n - {}^C D^\alpha u\right\|_{C(\Omega)} = 0.$$

Then ${}^C D^\alpha u \in C(\Omega)$ and we have $u \in C^{n-1,\alpha}(\Omega)$.

The proof of this Theorem is complete. ■

Example 2.2.1 *Let us consider the following fractional differential equation*

$${}^C D^\alpha u(t) = t^2 - 1, \tag{2.2.13}$$

subject to the initial condition

$$u(0) = 1. \tag{2.2.14}$$

Here $t \in \Omega = [0, 1]$ and ${}^C D^\alpha$ is the Caputo fractional derivative operator of order $\alpha =$ 1/2.

We find a continuous function $u : [0,1] \longrightarrow R$ satisfying (2.2.13) and (2.2.13).

By solving the proposed equation, we obtain

$$\begin{aligned} u(t) &= 1 + \frac{1}{\Gamma(1/2)} \int\limits_0^t (t-\tau)^{-1/2} \tau^2 d\tau - \frac{1}{\Gamma(1/2)} \int\limits_0^t (t-\tau)^{-1/2} d\tau \\ &= 1 + \frac{16}{15\sqrt{\pi}} t^{5/2} - \frac{2}{\sqrt{\pi}} t^{1/2}. \end{aligned}$$

Example 2.2.2 *Let us consider the following fractional differential equation*

$$^{C}D^{\alpha}u(t) = t^2 - 2t + 1, \tag{2.2.15}$$

subject to the initial condition

$$u(1) = 0. \tag{2.2.16}$$

Here $t \in \Omega = [1,2]$ and $^{C}D^{\alpha}$ is the Caputo fractional derivative operator of order $\alpha =$ 1/3.

We find a continuous function $u : [1,2] \longrightarrow R$ satisfying (2.2.15) and (2.2.16).

By solving the proposed equation, we obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(1/3)} \int\limits_1^t (t-\tau)^{-2/3} (\tau - 1)^2 d\tau \\ &= \frac{27}{14\Gamma(1/3)} (t-1)^{7/3}. \end{aligned}$$

Chapter 3

Approximate analytical techniques and their applications

In this chapter we deal with some approximate analytical techniques to find the approximate solution of classical differential equations, some numerical examples are given to compare the exact and approximate solutions.

3.1 Adomian decomposition technique (ADT)

The ADT was established by George Adomian [**1**] in the 1980s. The ADT has received considerable attention in recent years in applied mathematics, and in the field of series solutions in particular. Moreover, it is known to be a powerful and efficient technique, easily solves many types of linear or nonlinear ordinary or partial differential equations, and integral equations; see [**2**],[**16**]. This technique generates a solution in the form of a series whose terms are determined by a recursive relation using these Adomian polynomials.

3.1.1 Basic idea of ADT

In order to explain the basic idea of the ADT, we consider the following functional equation of the form

$$F\,(u) = h(t), t \in \Omega \subset \mathbb{R}^*, \tag{3.1.1}$$

where $u = \{u(t), t \in \Omega \subset \mathbb{R}^*\}$ is an unknown function, F is an ordinary or partial differential operator consisting of linear and nonlinear terms and h is a given function. The linear part is usually decomposed into $L + R$, where L is a readily invertible differential operator and R denotes the remainder of the linear operator. Under these conditions, equation (3.1.1) can be written as follows

$$L(u) + R(u) + N(u) = h(t), \tag{3.1.2}$$

where N is a nonlinear differential operator.

Equation (3.1.2) can be written as

$$L(u) = h(t) - R(u) - N(u). \tag{3.1.3}$$

Multiplying both sides of equation (3.1.3) by L^{-1}, we get

$$L^{-1}(L(u)) = L^{-1}(h(t)) - L^{-1}(R(u)) - L^{-1}(N(u)), \tag{3.1.4}$$

where

$$L^{-1} = \int\int \ldots \int (.)\,(dt)^n,$$

is the inverse operator of operator L.

On the other hand we have

$$L^{-1}(L(u)) = u - \varpi,$$

where ϖ is the constant of integration.

Thus, equation (3.1.4) is written in the form

$$u = \varpi + L^{-1}(h(t)) - L^{-1}(R(u)) - L^{-1}(N(u)). \tag{3.1.5}$$

For the ADM, the solution u is expressed in terms of a series form

$$u = \sum_{n=0}^{\infty} u_n, \tag{3.1.6}$$

and the nonlinear term Nu is represented by the series

$$Nu = \sum_{n=0}^{\infty} A_n. \tag{3.1.7}$$

A_n are called Adomian polynomials which depend on $u_0, u_1, ..., u_n$ and can be formulated by

$$A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ..., \tag{3.1.8}$$

where λ is a real parameter.

Replacing (3.1.6) and (3.1.7) in (3.1.5), we get the following equation

$$\sum_{n=0}^{\infty} u_n = \varpi + L^{-1}(h(t)) - L^{-1}R\left(\sum_{n=0}^{\infty} u_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{3.1.9}$$

Equating the terms in (3.1.9), we find

$$\begin{cases} u_0 = \varpi + L^{-1}(h(t)), \\ u_1 = L^{-1}(R(u_0)) - L^{-1}(A_0), \\ u_2 = L^{-1}(R(u_1)) - L^{-1}(A_1), \\ \vdots \\ u_{n+1} = L^{-1}(R(u_n)) - L^{-1}(A_n). \end{cases} \tag{3.1.10}$$

Remark 3.1.1 *Identification (3.1.10) is not unique but it is the only one that allows to explicitly define the values of* $\{u_n\}_{n\geq 0}$. *Relation (3.1.10) allows to calculate all the terms of the series without ambiguity because the* A_n *depend only on* $u_0, u_1, ..., u_n$.

In general, it is almost impossible to compute the sum of the series $\sum_{n=0}^{\infty} u_n$ (except in very special cases). Therefore, the solution is usually approximated in the form of a truncated series as follows

$$\varphi_n = \sum_{i=0}^{n-1} u_i, n \geq 1.$$

The question is: How can we determine $(A_n)_{n\geq 0}$ and under what conditions does the technique converge?

3.1.2 Adomian polynomials

Definition 3.1.1 *[2] Adomian polynomials are defined by the relation*

$$\begin{cases} A_0(u_0) = N(u_0), \\ A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}. \end{cases} \tag{3.1.11}$$

The relation (3.1.11) makes it possible to compute the Adomian polynomials $(A_n)_{n\geq 0}$ as follows

$$\begin{aligned}
A_0(u_0) &= N(u_0),\\
A_1(u_0,u_1) &= u_1\frac{\partial}{\partial u}N(u_0),\\
A_2(u_0,u_1,u_2) &= u_2\frac{\partial}{\partial u}N(u_0)+\frac{1}{2!}u_1^2\frac{\partial^2}{\partial u^2}N(u_0),\\
A_3(u_0,u_1,u_2,,u_3) &= u_3\frac{\partial}{\partial u}N(u_0)+u_1u_2\frac{\partial^2}{\partial u^2}N(u_0)+\frac{1}{3!}u_1^3\frac{\partial^3}{\partial u^3}N(u_0),\\
&\vdots
\end{aligned}$$

This formula is written as follows

$$A_n=\sum_{\nu=0}^{n}c(\nu,n)N^{(\nu)}(u_0), n\geq 1,$$

where $c(\nu,n)$ denoted the sum of all the products divided by $m!$ of the ν terms u_i whose sum of the indices i is equal to n, m being the number of repetitions of the same terms in the product.

3.1.3 Convergence analysis of ADT

Important theorems are presented which indicate that there exist sufficient conditions for the convergence of ADT and all these conditions concern the nonlinear operator N.

Indeed, from relation (3.1.10) we have

Theorem 3.1.1

$$\textit{If} \sum_{n\geq 0}A_n<+\infty \textit{ then } \sum_{n\geq 0}u_n<+\infty, \tag{3.1.12}$$

and vice versa.

The first proofs of convergence were given by Yves Cherruault, and are based on the fixed point theorem.

Let us give the main points of the demonstration (For more details see **[15]**).

Note that the decomposition technique applied to (3.1.1) is reduced to find a sequence

$$S_n=u_1+u_2+...+u_n,$$

with $S_0 = 0$ and verifying the following recurring relation

$$\begin{cases} u_0 = h, \\ S_0 = 0, & , n = 0, 1, 2, ... \\ S_{n+1} = N(u_0 + S_n), \end{cases} \tag{3.1.13}$$

Now we give the convergence result.

Theorem 3.1.2 *If N is a contraction operator, i.e. verifies $0 < \|N\| < \theta < 1$, then the sequence $(S_n)_{n\geq 0}$ satisfying the recurrence relation (3.1.13), converges to S solution of $S = N(u_0 + S)$.*

Proof. According to relation (3.1.13), we have

$$\begin{aligned} \|S_n - S\| &= \|N(u_0 + S_n) - N(u_0 + S)\| \\ &\leq \|N\| \|S_n - S\| \leq \theta \|S_n - S\| \\ &\leq \theta^n \|S_1 - S\| . \end{aligned}$$

Since $0 < \theta < 1$, so the sequence $(S_n)_{n\geq 0}$ converges to S. ∎

Furthermore, we have

$$\sum_{n\geq 0} A_n = \sum_{n\geq 1} u_n,$$

and as $\sum_{n\geq 1} u_n$ is convergent according to Theorem 3.1.1, we then have the following result.

Corollary 3.1.1 *If N is a contraction operator then the series of u_n and A_n are convergent. In addition $\sum_{n\geq 0} u_n$ is solution of the equation*

$$F(u) = h.$$

Example 3.1.1 *Let us consider the nonlinear differential equation*

$$u' + u^2 = 0, t \geq 0, \tag{3.1.14}$$

subject to the initial condition

$$u(0) = 1. \tag{3.1.15}$$

We have

$$L(u) = u' \text{ and } N(u) = u^2,$$

with $L = \dfrac{d}{dt}(.)$.

The symbol L^{-1} *is a simple integral of* 0 *at* t.

Now, applying the ADT to equations (3.1.14)-(3.1.15) gives

$$u = \sum_{n=0}^{\infty} u_n = u(0) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{3.1.16}$$

where A_n *are the Adomian polynomials of the nonlinear term* $N(u) = u^2$.

The first few components of the Adomian polynomials are

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ &\vdots \end{aligned}$$

and the recurrence relation in (3.1.16) yields

$$\begin{aligned} u_0 &= 1, \\ u_1 &= -L^{-1}(A_0) = -t, \\ u_2 &= -L^{-1}(A_1) = t^2, \\ u_3 &= -L^{-1}(A_2) = -t^3, \\ u_4 &= -L^{-1}(A_3) = t^4, \\ &\vdots \end{aligned}$$

As a result, the series solution of equations (3.1.14)-(3.1.15) can be found as

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n = 1 - t + t^2 - t^3 + t^4 - .... \\ &= \sum_{n=0}^{\infty} (-t)^n = \frac{1}{1+t}. \end{aligned}$$

Example 3.1.2 *Let us consider the nonlinear differential equation*

$$u' - e^u = 0, t \geq 0, \tag{3.1.17}$$

subject to the initial condition

$$u(0) = 0. \tag{3.1.18}$$

We have

$$L(u) = u' \text{ and } N(u) = e^u,$$

with $L = \dfrac{d}{dt}(.)$.

The symbol L^{-1} *is a simple integral of* 0 *at* t.

Now, applying the ADT to equations (3.1.17)-(3.1.18) gives

$$u = \sum_{n=0}^{\infty} u_n = u(0) + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{3.1.19}$$

where A_n *are the Adomian polynomials of the nonlinear term* $N(u) = e^u$.

The first few components of the Adomian polynomials are

$$\begin{aligned} A_0 &= e^{u_0}, \\ A_1 &= u_1, \\ A_2 &= u_2 + \frac{1}{2!}u_1^2, \\ A_3 &= u_3 + u_1 u_2 + \frac{1}{3!}u_1^3, \\ &\vdots \end{aligned}$$

and the recurrence relation in (3.1.19) yields

$$\begin{aligned} u_0 &= 0, \\ u_1 &= L^{-1}(A_0) = t, \\ u_2 &= L^{-1}(A_1) = \frac{t^2}{2}, \\ u_3 &= L^{-1}(A_2) = \frac{t^3}{3}, \\ u_4 &= L^{-1}(A_3) = \frac{t^4}{4}, \\ &\vdots \end{aligned}$$

As a result, the series solution of equations (3.1.17)-(3.1.18) can be found as

$$u = \sum_{n=0}^{\infty} u_n = t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + ...$$

3.2 Homotopy perturbation technique (HPT)

The HPT was defined and developed by the Chinese mathematician Ji-Haun-He in 1999 **[20]**,**[21]**,**[22]**. The HPT has been widely used for solving nonlinear boundary and initial value problems and is a powerful mathematical tool to study a wide variety of problems arising in in various fields. It is successfully obtained by combining homotopy theory in topology and perturbation theory. The main advantage of the HPT is that it provides an approximate analytical solution to a wide range of linear and nonlinear problems, without requiring linearization, discretization, assumptions, and calculus of Adomian polynomials. **[33].**

3.2.1 Basic idea of HPT

In order to illustrate the basic idea of the HPT, let us consider the nonlinear functional equation

$$A(u) - g(t) = 0, t \in \Omega \subset \mathbb{R}^*, \tag{3.2.1}$$

subject to boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, t \in \Gamma, \tag{3.2.2}$$

where A is a general differential operator, B is an operator that defines the boundary condition, $u = \{u(t), t \in \Omega \subset \mathbb{R}^*\}$ is an unknown function is an unknown function, $g(t)$ is a non-homogeneous term, and Γ is the boundary of the domain $\Omega \subset \mathbb{R}^*$.

In general, operator A can be divided into two factors operators L and N, where L represents a linear operator and N represents a nonlinear operator. Therefore, equation (3.2.1) can be rewritten as

$$L(u) + N(u) - g(t) = 0.$$

We construct a homotopy $v(t,p) : \Omega \times [0,1] \longrightarrow \mathbb{R}$, which verifies

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - g(t)\right] = 0, p \in [0,1], t \in \Omega, \tag{3.2.3}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - g(t)\right] = 0, \tag{3.2.4}$$

where $p \in [0,1]$ is the homotopy parameter and u_0 is an initial approximation of equation (3.2.1) which satisfied the boundary conditions (3.2.2).

According to equations (3.2.3) and (3.2.4) we have

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0, \\ H(v,1) &= A(v) - g(t) = 0. \end{aligned}$$

Changing p from zero to unity transforms $u_0(t)$ into $u(t)$.In topology with this property, the function $v(t,p)$ is called homotopy. According to the HPT, we can use the parameter p as a small parameter, and assume that the solutions of equations (3.2.3) and (3.2.4) can be written as a power series of p as follows

$$v = v_0 + pv_1 + p^2 v_2 + ... = \sum_{i=0}^{\infty} p^i v_i. \tag{3.2.5}$$

For $p = 1$, the approximate solution of equation (3.2.1) becomes

$$u = \lim_{p \longrightarrow 1} v = v_0 + v_1 + v_2 + ... = \sum_{i=0}^{\infty} v_i. \tag{3.2.6}$$

3.2.2 Convergence analysis of HPT

Here we study the convergence analysis of the HPT **[6]**,**[11]**.

First, we have the relation (3.2.4) which can be written as follows

$$L(v) - L(u_0) = p\left[g(t) - L(u_0) - N(v)\right]. \tag{3.2.7}$$

Replacing (3.2.5) in (3.2.7) yields

$$L\left(\sum_{i=0}^{\infty} p^i v_i\right) - L(u_0) = p\left[g(t) - L(u_0) - N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]. \tag{3.2.8}$$

Thus

$$\sum_{i=0}^{\infty} L\left(v_i\right) - L(u_0) = p\left[g(t) - L(u_0) - N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right]. \tag{3.2.9}$$

According to Maclaurin's formula of $N\left(\sum_{i=0}^{\infty} p^i v_i\right)$ with respect to p, we have

$$N\left(\sum_{i=0}^{\infty} p^i v_i\right) = \sum_{i=n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right)_{p=0} p^i. \tag{3.2.10}$$

After **[29],** we have

$$\left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{\infty} p^i v_i\right)\right)_{p=0} = \left(\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i v_i\right)\right)_{p=0}.$$

Then

$$N\left(\sum_{i=0}^{\infty} p^i v_i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i v_i\right)\right)_{p=0} p^i.$$

We put

$$H_n(v_0, v_1, ..., v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n} p^i v_i\right)\right]_{p=0}, n = 0, 1, 2, ..., \tag{3.2.11}$$

where H_n are called He's polynomials **[29].**

Then

$$N\left(\sum_{i=0}^{\infty} p^i v_i\right) = \sum_{i=0}^{\infty} H_i p^i. \tag{3.2.12}$$

Substituting (3.2.12) into (3.2.9) yields

$$\sum_{i=0}^{\infty} L\left(v_i\right) - L(u_0) = p\left[g(t) - L(u_0) - \sum_{i=0}^{\infty} H_i p^i\right]. \tag{3.2.13}$$

Identifying the terms with those of the same power p, we obtain the following result

$$\begin{aligned}
p^0 &: L\left(v_0\right) - L(u_0) = 0,\\
p^1 &: L\left(v_1\right) = g(t) - L(u_0) - H_0,\\
p^2 &: L\left(v_2\right) = -H_1,\\
p^3 &: L\left(v_3\right) = -H_2,\\
&\vdots\\
p^{n+1} &: L\left(v_{n+1}\right) = -H_n.
\end{aligned} \tag{3.2.14}$$

Appying the inverse operator L^{-1} to (3.1.4), we conclude

$$\begin{aligned}
p^0 &: v_0 = u_0, \\
p^1 &: v_1 = L^{-1}(g(t)) - u_0 - L^{-1}(H_0), \\
p^2 &: v_2 = -L^{-1}(H_1), \\
p^3 &: v_3 = -L^{-1}(H_2), \\
&\vdots \\
p^{n+1} &: v_{n+1} = -L^{-1}(H_n).
\end{aligned} \tag{3.2.15}$$

Theorem 3.2.1 *The solution of equation (3.2.1) obtained by the HPT is equivalent to the determination of the series S_n defined by:*

$$\begin{cases} S_0 = 0, \\ S_n = v_1 + v_2 + ... + v_n. \end{cases} \tag{3.2.16}$$

using the iterative scheme

$$S_{n+1} = -L^{-1}N(S_n + v_0) - u_0 + L^{-1}(g(t)), \tag{3.2.17}$$

where

$$N\left(\sum_{i=0}^{n} v_i\right) = \sum_{i=0}^{n} H_i, n = 0, 1, 2, ... \tag{3.2.18}$$

Proof. The proof of this theorem will be done by induction.

For $n = 0$, from (3.2.17), we have

$$\begin{aligned}
S_1 &= -L^{-1}N(S_0 + v_0) - u_0 + L^{-1}(g(t)) \\
&= -L^{-1}(H_0) - u_0 + L^{-1}(g(t)).
\end{aligned}$$

So

$$v_1 = -L^{-1}(H_0) - u_0 + L^{-1}(g(t)).$$

For

$$\begin{aligned}
S_2 &= -L^{-1}N(S_1 + v_0) - u_0 + L^{-1}(g(t)) \\
&= -L^{-1}(H_1 + H_0) - u_0 + L^{-1}(g(t)) \\
&= -L^{-1}(H_1) + v_1.
\end{aligned}$$

According to $S_2 = v_1 + v_2$, we get

$$v_2 = -L^{-1}\left(H_1\right).$$

Assume that

$$v_{k+1} = -L^{-1}\left(H_k\right), \text{ for } k = 1, 2, ..., n-1,$$

therefore

$$\begin{aligned} S_{n+1} &= -L^{-1}N\left(S_n + v_0\right) - u_0 + L^{-1}\left(g(t)\right) \\ &= -L^{-1}\left(\sum_{i=0}^{n} H_i\right) - u_0 + L^{-1}\left(g(t)\right) \\ &= -\sum_{i=0}^{n} L^{-1}\left(H_i\right) - u_0 + L^{-1}\left(g(t)\right) \\ &= v_1 + v_2 + ... + v_n - L^{-1}\left(H_n\right). \end{aligned}$$

Then, from (3.2.16) we can find

$$v_{n+1} = -L^{-1}\left(H_n\right).$$

This result is identical to that obtained in (3.2.15) by the HPT. ■

Theorem 3.2.2 *Let B be a Banach space.*

1) $\sum_{i=0}^{\infty} v_i$ converges to $S \in B$, if there exists $0 < \lambda < 1$ such that

$$\left\|v_n\right\| \leq \lambda \left\|v_{n-1}\right\|, \tag{3.2.19}$$

for all $n \in \mathbb{N}$

2) $S = \sum_{i=1}^{\infty} v_i$ satisfies

$$S = -L^{-1}N\left(S + v_0\right) - u_0 + L^{-1}\left(g(t)\right). \tag{3.2.20}$$

Proof. 1) From relation (3.2.19) we have

$$\left\|S_{n+1} - S_n\right\| = \left\|v_{n+1}\right\| \leq \lambda \left\|v_n\right\| \leq \lambda^2 \left\|v_{n-1}\right\| \leq ... \leq \lambda^{n+1} \left\|v_0\right\|.$$

For $n, m \in \mathbb{N}$, $n \geq m$,we have

$$\begin{aligned}
\|S_n - S_m\| &= \|S_n - S_{n-1} + S_{n-1} - S_{n-2} + ... + S_{m+1} - S_m\| \\
&\leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + ... + \|S_{m+1} - S_m\| \\
&\leq \lambda^n \|v_0\| + \lambda^{n-1} \|v_0\| + ... + \lambda^{m+1} \|v_0\| \\
&\leq \left(\lambda^n + \lambda^{n-1} + ... + \lambda^{m+1}\right) \|v_0\| \\
&\leq \left(\lambda^{m+1} + ... + \lambda^n\right) \|v_0\| \\
&\leq \lambda^{m+1} \left(1 + ... + \lambda^{n-m-2} + \lambda^{n-m-1}\right) \|v_0\| \\
&\leq \lambda^{m+1} \left(\frac{1 - \lambda^{n-m}}{1 - \lambda}\right) \|v_0\| .
\end{aligned}$$

Since $0 < \lambda < 1$ and $\|v_0\| < \infty$, we have $1 - \lambda^{n-m} > 0$ and

$$\lim_{n,m \longrightarrow \infty} \|S_n - S_m\| = 0.$$

Thus, $(S_n)_{n \geq 0}$ is a Cauchy sequence in Banach space B and it is convergent, i.e.

$$\exists S \in B : \lim_{n \longrightarrow \infty} S_n = \sum_{n=1}^{\infty} v_n = S.$$

2) According to (3.2.17), we have

$$\begin{aligned}
\lim_{n \longrightarrow \infty} S_{n+1} &= -L^{-1} \lim_{n \longrightarrow \infty} N\left(S_n + v_0\right) - u_0 + L^{-1}\left(g(t)\right) \\
&= -L^{-1} \lim_{n \longrightarrow \infty} N\left(\sum_{i=0}^{n} v_i\right) - u_0 + L^{-1}\left(g(t)\right) \\
S &= -L^{-1} \lim_{n \longrightarrow \infty} \sum_{i=0}^{n} H_i - u_0 + L^{-1}\left(g(t)\right) \\
&= -L^{-1} \sum_{i=0}^{\infty} H_i - u_0 + L^{-1}\left(g(t)\right) .
\end{aligned}$$

By equations (3.2.18) and (3.2.12) for $p = 1$, it comes

$$\sum_{i=0}^{\infty} H_i = N\left(\sum_{i=0}^{\infty} v_i\right).$$

Consequently, we have

$$\begin{aligned}
S &= -L^{-1} N\left(\sum_{i=0}^{\infty} v_i\right) - u_0 + L^{-1}\left(g(t)\right) \\
&= -L^{-1} N\left(S + v_0\right) - u_0 + L^{-1}\left(g(t)\right) .
\end{aligned}$$

■

Lemma 3.2.1 *Equation (3.2.20) is equivalent to*

$$L(u) + N(u) - g(t) = 0. \tag{3.2.21}$$

Proof. Equation (3.2.20) is written as follows

$$S + u_0 = -L^{-1}N\left(S + v_0\right) + L^{-1}\left(g(t)\right). \tag{3.2.22}$$

Applying the operator L to equation (3.2.22) yields

$$L\left(S + u_0\right) = -N\left(S + v_0\right) + g(t).$$

As $u_0 = v_0$, we have

$$L\left(S + v_0\right) = -N\left(S + v_0\right) + g(t).$$

Let

$$u = S + v_0 = \sum_{i=0}^{\infty} v_i.$$

Equation (3.2.21) becomes the original equation and the solution of equation (3.2.20) is the same as that of the solution of $A(u) - g(t) = 0$. ■

Definition 3.2.1 *For all $i \in \mathbb{N}$, we define*

$$\lambda_i = \begin{cases} \frac{\|v_{i+1}\|}{\|v_i\|}, & \|v_i\| \neq 0, \\ 0, & \|v_i\| = 0. \end{cases}$$

In Theorem 3.2.2, the series $\sum_{i=0}^{\infty} v_i$ converges to the exact solution when $0 \leq \lambda_i < 1$.

Corollary 3.2.1 *If v_i and v_i' are obtained by two different homotopies and for each $i \in \mathbb{N}$ we have $\lambda_i < \lambda_i'$, then the convergence rate of $\sum_{i=0}^{\infty} v_i$ is greater than $\sum_{i=0}^{\infty} v_i'$.*

Example 3.2.1 *Let us consider the nonlinear differential equation*

$$u' + u^2 = 0, \tag{3.2.23}$$

subject to the initial condition

$$u(0) = 1. \tag{3.2.24}$$

Here $u = \{u(t), t \in [0,T], T > 0\}$.

According to the HPT, the following homotopy can be constructed: $u : [0,T] \times [0,1] \longrightarrow \mathbb{R}$

$$(1-p)(v' - u_0') + p(v' + v^2) = 0, p \in [0,1], t \in [0,T], \tag{3.2.25}$$

with $u_0 = 1$.

The solutions of equations (3.2.23)-(3.2.24), can be expressed as follows

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{3.2.26}$$

By substituting (3.2.26) in (3.2.25) and identifying the terms with those of the same powers of p*, we get*

$$\begin{aligned} p^0 &: v_0' = u_0', \\ p^1 &: v_1' = -u_0 - v_0^2, v_1(0) = 0, \\ p^2 &: v_2' = -2v_0 v_1, v_2(0) = 0, \\ &\vdots \end{aligned}$$

So, the first terms of the solution are given by

$$\begin{aligned} p^0 &: v_0 = 1, \\ p^1 &: v_1 = -t, \\ p^2 &: v_2 = t^2, \\ &\vdots \end{aligned}$$

Finally, the solution of equations (3.2.23)-(3.2.24) is presented as

$$\begin{aligned} u &= \lim_{p \longrightarrow 1} v = v_0 + v_1 + v_2 + ... = 1 - t + t^2 - ... \\ &= \sum_{n=0}^{\infty} (-t)^n = \frac{1}{1+t}. \end{aligned}$$

Example 3.2.2 *Let us consider the nonlinear differential equation*

$$u' = 2u - u^2 + 1, \tag{3.2.27}$$

subject to the initial condition

$$u(0) = 0. \tag{3.2.28}$$

Here $u = \{u(t), t \in [0,T], T > 0\}$.

According to the HPT, the following homotopy can be constructed: $u : [0,T] \times [0,1] \longrightarrow \mathbb{R}$

$$(1-p)(v' - u_0') + p(v' - 2v + v^2 - 1) = 0, p \in [0,1], t \in [0,T], \tag{3.2.29}$$

with $u_0 = 0$.

The solutions of equations (3.2.27)-(3.2.28), can be expressed as follows

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{3.2.30}$$

By substituting (3.2.30) in (3.2.29) and identifying the terms with those of the same powers of p*, we get*

$$\begin{aligned}
p^0 &: v_0' = u_0', \\
p^1 &: v_1' + u_0' + u_0^2 - 1 = 0, \\
p^2 &: v_2' + 2v_0 v_1 = 0, \\
p^3 &: v_3' + v_1' + 2v_0 v_1 = 0, \\
p^4 &: v_4' + v_1' + 2v_0 v_3 + 2v_1 v_2 = 0, \\
&\vdots
\end{aligned}$$

So, the first terms of the solution are given by

$$\begin{aligned}
p^0 &: v_0 = t, \\
p^1 &: v_1 = \frac{1}{4}\left(-1 + e^t - 2t + 2t^2\right), \\
p^2 &: v_2 = \frac{1}{4}\left(t^2 - t^2 e^t + 2t^3\right), \\
&\vdots
\end{aligned}$$

Finally, the solution of equations (3.2.23)-(3.2.24) is presented as

$$\begin{aligned} u &= \lim_{p \longrightarrow 1} v = v_0 + v_1 + v_2 + ... = 1 - t + t^2 - ... \\ &= t + \frac{1}{4}\left(-1 + e^t - 2t + 2t^2\right) + \frac{1}{4}\left(t^2 - t^2 e^t + 2t^3\right) + ... \end{aligned} \quad (3.2.31)$$

By computing the limited development of e^t at $t = 0$, then the solution in equation (3.2.31) becomes

$$\begin{aligned} u &= t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \frac{71}{315}t^8 + ... \\ &= 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \end{aligned}$$

3.3 Variational iteration technique (VIT)

The VIT was first proposed for solving problems in mechanics by Chinese mathematician Je-Haun-He in the early 1990s **[23]**,**[24]**,**[25]**. The VIT is one of the well-known semi-analytical techniques to solve linear and nonlinear ordinary as well as partial differential equations. The main advantage of the technique is its flexibility and ability to easily solve nonlinear equations with successive approximations that converge rapidly to the exact solution if it exists.

3.3.1 Basic idea of VIT

In order to illustrate the basic concepts of VIT , we consider the following general nonlinear functional equation

$$L(u) + N(u) = h(t), \quad (3.3.1)$$

where L is a linear differential operator, N is a nonlinear operator, $h(t)$ is an inhomogeneous term, and $u = \{u(t), t \in [0, T], T > 0\}$ is an unknown function.

The VIT is based on determining the general Lagrange multiplier optimally through variational theorem **[28]**. The main advantage of the technique is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more accurate approximation at a particular point can be obtained.

According to the VIT , we can construct a correction functional for equation (3.3.1) as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau) \left[L\left(u_n(\tau)\right) + N(\widetilde{u_n}(\tau)) - h(\tau) \right] d\tau, n \geq 0, \tag{3.3.2}$$

where λ is a general Lagrange multiplier. The index n represents the n^{th} approximation and $\widetilde{u_n}(\tau)$ is considered a restricted variation, i.e. $\delta\widetilde{u_n}(\tau) = 0$. For solving equation (3.3.1) by the VIT, it is first necessary to determine the Lagrange multiplier λ which will be optimally identified via integration by parts. Then, the successive approximations u_n of the solution $u(t)$ will be obtained using the Lagrange multiplier and a well-chosen function u_0 which must at least satisfy the initial conditions, therefore, the exact solution will be the limit

$$\lim_{n \longrightarrow \infty} u_n(t) = u(t).$$

3.3.2 Alternative approach to VIT

Here we introduce an alternative approach to VIT. This approach can be implemented reliably and efficiently to solve the following nonlinear functional equation

$$Lu(t) + Nu(t) = h(t), t > 0. \tag{3.3.3}$$

subject to the initial conditions

$$u^{(k)}(0) = u_k, k = 0, 1, ..., m-1, \tag{3.3.4}$$

where L is a linear differential operator defined by $L = \dfrac{d^m}{dt^m}, m \in \mathbb{N}$, N is a nonlinear operator, h is an inhomogeneous term, and u_k are real constants.

According to theVIM, the correction functional formula for (3.3.3) can be constructed as follows

$$u_{k+1}(t) = u_k(t) + \int_0^t \left[\lambda(\tau) \left(Lu_k(\tau) + N\widetilde{u}_k(\tau) - h(\tau) \right) \right] d\tau, \tag{3.3.5}$$

where λ is a general Lagrange multiplier that can be identified optimally via the variational theory. Here we apply restricted variations to the nonlinear term Nu, in this case we can easily determine the multiplier.

Then make the correction functional (3.3.5) stationary by noting that $\delta\widetilde{u_k}(\tau) = 0$. Equation

$$\delta u_{k+1}(t) = \delta u_k(t) + \delta \int_0^t \left[\lambda(\tau)\left(Lu_k(\tau) - h(\tau)\right)\right] d\tau, \tag{3.3.6}$$

provide the following Lagrange multipliers

$$\begin{aligned} \lambda &= -1 \text{ for } m = 1, \\ \lambda &= \tau - t \text{ for } m = 2, \end{aligned}$$

and in general, we have

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \text{ for } m \geq 1. \tag{3.3.7}$$

Thus, by replacing (3.3.7) in the formula (3.3.5), we obtain the following iteration relation

$$u_{k+1}(t) = u_k(t) + \int_0^t \left[\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left(Lu_k(\tau) + Nu_k(\tau) - h(\tau)\right)\right] d\tau, \tag{3.3.8}$$

Now we define the operator $A(u)$ as

$$A(u) = \int_0^t \left[\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}\left(Lu_k(\tau) + Nu_k(\tau) - h(\tau)\right)\right] d\tau, \tag{3.3.9}$$

and we define the components $v_k, k = 0, 1, 2, ...,$ as follows

$$\begin{cases} v_0 = u_0, \\ v_1 = A(v_0), \\ v_2 = A(v_0 + v_1), \\ \vdots \\ v_{k+1} = A(v_0 + v_1 + ... + v_k). \end{cases} \tag{3.3.10}$$

Then, we have

$$u(t) = \lim_{k \longrightarrow \infty} u_k(t) = \sum_{k=0}^{\infty} v_k(t). \tag{3.3.11}$$

Finally, the solution of equations (3.3.3)-(3.3.4) can be deduced using (3.3.9) and (3.3.10) iin the form of a series as

$$u(t) = \sum_{k=0}^{\infty} v_k(t). \tag{3.3.11}$$

The initial approximation $v_0 = u_0$ can be chosen freely if it satisfies the initial conditions of the problem. The success of the technique depends on the correct choice of the initial approximation v_0. In this alternative approach, we choose the initial approximation as follows

$$v_0 = \sum_{k=0}^{m-1} \frac{u_k}{k!} t^k. \tag{3.3.12}$$

3.3.3 Convergence analysis of VIT

Now we discuss the necessary conditions that ensure the convergence of the VIT **[47]**,**[57]**.

Theorem 3.3.1 *Let H be a Hilbert space and $A : H \to H$ an operator given by (3.3.9).*

The series solution (3.3.11) converge if if there exists $0 < \gamma < 1$ such that

$$\|A(v_0 + v_1 + ... + v_{k+1})\| \leq \gamma \|A(v_0 + v_1 + ... + v_k)\|,$$

i.e.

$$\|v_{k+1}\| \leq \gamma \|v_k\|, \forall k \in \mathbb{N} \cup \{0\}.$$

Proof. We take $\{S_n\}_{n\geq 0}$ a sequence given by

$$\begin{cases} S_0 = v_0, \\ S_1 = v_0 + v_1, \\ S_2 = v_0 + v_1 + v_2, \\ \vdots \\ S_n = v_0 + v_1 + v_2 ... + v_n. \end{cases}.$$

For the convergence of sequence $\{S_n\}_{n\geq 0}$ of the partial sums of the series solution (3.3.11), we prove that $\{S_n\}_{n\geq 0}$ is a Cauchy sequence in Hilbert space H. As

$$\|S_{n+1} - S_n\| = \|v_{n+1}\| \leq \gamma \|v_n\| \leq \gamma^2 \|v_{n-1}\| \leq ... \leq \gamma^{n+1} \|v_0\|.$$

Hence, for $n, m \in \mathbb{N}$ such that $n \geq m$, we have

$$\begin{aligned}
\|S_n - S_m\| &= \|S_n - S_{n-1} + S_{n-1} - S_{n-2} + ... + S_{m+1} - S_m\| \\
&\leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + ... + \|S_{m+1} - S_m\| \\
&\leq \gamma^n \|v_0\| + \gamma^{n-1} \|v_0\| + ... + \gamma^{m+1} \|v_0\| \\
&\leq \left(\gamma^n + \gamma^{n-1} + ... + \gamma^{m+1}\right) \|v_0\| \\
&\leq \left(\gamma^{m+1} + ... + \gamma^{n-1} + \gamma^n\right) \|v_0\| \\
&\leq \gamma^{m+1} \left(1 + ... + \gamma^{n-m-2} + \gamma^{n-m-1}\right) \|v_0\| \\
&\leq \gamma^{m+1} \left(\frac{1 - \gamma^{n-m}}{1 - \gamma}\right) \|v_0\| .
\end{aligned}$$

Since $0 < \gamma < 1$, then we have $1 - \gamma^{n-m} > 0$ and

$$\|S_n - S_m\| \leq \frac{\gamma^{m+1}}{1 - \gamma} \|v_0\| .$$

Also v_0 is bounded, therefore

$$\lim_{n,m \longrightarrow \infty} \|S_n - S_m\| = 0.$$

Consequently $\{S_n\}_{n \geq 0}$ is a Cauchy sequence in Hilbert space H and this implies that the series solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ is convergent. ■

Theorem 3.3.2 *If the series solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ is convergent, then it is an exact solution of the nonlinear functional equation (3.3.3).*

Proof. Assume that the series solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ is convergent, then

$$\lim_{k \longrightarrow \infty} v_k = 0,$$

and

$$\sum_{k=0}^{n} (v_{k+1} - v_k) = v_{n+1} - v_0.$$

Therefore, we have

$$\sum_{k=0}^{\infty} (v_{k+1} - v_k) = \lim_{k \longrightarrow \infty} (v_{n+1} - v_0) = -v_0. \tag{3.3.13}$$

Applying the operator $L = \dfrac{d^m}{dt^m}, m \in \mathbb{N}$ to both sides of equation (3.3.13) and using the relation (3.3.12), we get

$$\sum_{k=0}^{\infty} L\left(v_{k+1} - v_k\right) = -L\left(v_0\right) = 0. \tag{3.3.14}$$

On the other hand, according to relation (3.3.10), we have

$$L\left(v_{k+1} - v_k\right) = L\left(A\left(v_0 + v_1 + ... + v_k\right) - A\left(v_0 + v_1 + ... + v_{k-1}\right)\right), k \geq 1.$$

By the definition of the operator $A(u)$ given by (3.3.9), we get

$$\begin{aligned} L\left(v_{k+1} - v_k\right) \quad = \quad & L\left(\int_0^t \left[\frac{(-1)^m}{(m-1)!}\left(\tau - t\right)^{m-1}\left(L\left(v_0 + v_1 + ... + v_k\right)\right.\right.\right. \\ & -L\left(v_0 + v_1 + ... + v_{k-1}\right) + N\left(v_0 + v_1 + ... + v_k\right) \\ & \left.\left.\left.-N\left(v_0 + v_1 + ... + v_{k-1}\right)\right)\right] d\tau\right), k \geq 1. \end{aligned} \tag{3.3.15}$$

Now the operator $A(u)$ given by (3.3.9), gives the integral of the m^{th} times of $Lu(t) + Nu(t) - h(t)$ and since the differential operator $L = \dfrac{d^m}{dt^m}$ of ordre m is the inverse of the integral operator m^{th} times, then equation (3.3.15) becomes

$$L\left(v_{k+1} - v_k\right) = L\left(v_k\right) + N\left(v_0 + v_1 + ... + v_k\right) - N\left(v_0 + v_1 + ... + v_{k-1}\right), k \geq 1.$$

Hence, we have

$$\begin{aligned} \sum_{k=0}^{n} L\left(v_{k+1} - v_k\right) \quad = \quad & L\left(v_0\right) + N\left(v_0\right) - h(t) \\ & +L\left(v_1\right) + N\left(v_0 + v_1\right) - N\left(v_0\right) \\ & +L\left(v_2\right) + N\left(v_0 + v_1 + v_2\right) - N\left(v_0 + v_1\right) \\ & \vdots \\ & +L\left(v_n\right) + N\left(v_0 + v_1 + ... + v_n\right) - N\left(v_0 + v_1 + ... + v_{n-1}\right). \end{aligned}$$

Therefore

$$\sum_{k=0}^{\infty} L\left(v_{k+1} - v_k\right) = L\left(\sum_{k=0}^{\infty} v_k\right) + N\left(\sum_{k=0}^{\infty} v_k\right) - g(t). \tag{3.3.16}$$

From equations (3.3.13) and (3.3.16) we can observe that $u(t) = \sum\limits_{k=0}^{\infty} v_k(t)$ is an exact solution to the nonlinear functional equation (3.3.3). ■

Example 3.3.1 *Consider the nonlinear differential equation subject to the initial condition as follows*

$$\begin{aligned} u'(t) &= u^2(t) + 1, 0 < t \leq 1, \\ u(0) &= 0. \end{aligned} \tag{3.3.17}$$

According to the VIT, the correction functional of equation (3.3.17) is described as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)(u'_n(\tau) - (\widetilde{u_n})^2(\tau) - 1)d\tau.$$

Utilizing relation (3.3.7), the Lagrange multiplier $\lambda(\tau)$ *can be defined as* $\lambda(\tau) = -1$, *so the recurrence formula can be obtained as follows*

$$u_{n+1}(t) = u_n(t) - \int_0^t (u'_n(\tau) - (u_n)^2(\tau) - 1)d\tau. \tag{3.3.18}$$

Employing formula (3.3.18), we find the terms of the approximate solution as

$$\begin{aligned} u_0(t) &= 0, \\ u_1(t) &= t, \\ u_2(t) &= t + \frac{1}{3}t^3, \\ u_3(t) &= t + \frac{1}{3}t^3 + \frac{2}{15}t^6 + \frac{1}{63}t^7, \\ &\vdots \end{aligned}$$

As

$$u(t) = \lim_{n \longrightarrow \infty} u_n(t),$$

therefore, the solution of equation (3.3.17) is given as a series that converges to the exact solution

$$\begin{aligned} u(t) &= t + \frac{1}{3}t^3 + \frac{2}{15}t^6 + \frac{1}{63}t^7 + ... \\ &= \tan(t). \end{aligned}$$

Example 3.3.2 *Consider the nonlinear differential equation subject to the initial conditions as follows*

$$\begin{aligned} u''(t) + u(t) &= 0, 0 < t \leq 1, \\ u(0) &= 1, u'(0) = 0 \end{aligned} \tag{3.3.19}$$

According to the VIT, the correction functional of equation (3.3.19) is described as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)(u_n^{''}(\tau) + u_n(\tau))d\tau.$$

Utilizing relation (3.3.7), the Lagrange multiplier $\lambda(\tau)$ can be defined as $\lambda(\tau) = \tau - t$, so the recurrence formula can be obtained as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t (\tau - t)(u_n^{''}(\tau) + u_n(\tau))d\tau. \tag{3.3.20}$$

Employing formula (3.3.20), we find the terms of the approximate solution as

$$\begin{aligned} u_0(t) &= 1, \\ u_1(t) &= 1 - \frac{1}{2!}t^2, \\ u_2(t) &= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4, \\ u_3(t) &= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6, \\ &\vdots \end{aligned}$$

As

$$u(t) = \lim_{n \longrightarrow \infty} u_n(t),$$

therefore, the solution of equation (3.3.19) is given as a series that converges to the exact solution

$$\begin{aligned} u(t) &= 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + ... \\ &= \lim_{n \longrightarrow \infty} \sum_{k=0}^{n} (-1)^k \frac{t^{2k}}{2k!} = \cos(t). \end{aligned}$$

3.4 Daftardar-Gejji-Jafari technique (DGJT)

In recent years, Daftardar-GejjiandJafari [**17**] suggested a new technique, called the Daftardar-Gejji-Jafari technique (DGJT), for solving evolution equations and nonlinear functional equations. This technique has been successfully employed to solve various mathematical problems and is easily applicable to ordinary and partial differential equations, integral equations, and integro-differential equations, whether linear or nonlinear. It is computationally efficient, avoiding complex calculations such as those involving Adomian polynomials. Furthermore, it converges to the exact solution if it exists, by successive approximations. The DGJT offer improved accuracy and efficiency compared to older techniques and can be applied to solve complex nonlinear systems or generalized singular-value problems, as well as other mathematical problems.

3.4.1 Basic idea of DGJT

Here, we will introduce the basic idea dealing with DGJT, we can write the general nonlinear functional equation as

$$u(t) = N(u(t)) + g(t), \tag{3.4.1}$$

where N is a nonlinear operator in the Banach space $B \to B$ and g is a known function.

A solution $u(t)$ of equation in (3.4.1) is given the following series form

$$u(t) = \sum_{i=0}^{\infty} u_i(t). \tag{3.4.2}$$

The nonlinear operator N can be written as

$$N\left(\sum_{i=0}^{\infty} u_i(t)\right) = N\left(u_0(t)\right) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j(t)\right) - N\left(\sum_{j=0}^{i-1} u_j(t)\right)\right\}. \tag{3.4.3}$$

Putting (3.4.2) and (3.4.3) in (3.4.1), we get

$$\sum_{i=0}^{\infty} u_i(t) = g(t) + N\left(u_0(t)\right) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j(t)\right) - N\left(\sum_{j=0}^{i-1} u_j(t)\right)\right\}. \tag{3.4.4}$$

From (3.4.4), the recurrence relation can be obtained as

$$\begin{aligned} u_0(t) &= g(t), \\ u_1(t) &= N\left(u_0(t)\right), \\ u_{n+1}(t) &= N\left(\sum_{j=0}^{n} u_j(t)\right) - N\left(\sum_{j=0}^{n-1} u_j(t)\right), n = 1, 2, ... \end{aligned} \tag{3.4.5}$$

So we see that

$$u_1(t) + u_2(t) + ... + u_{n+1}(t) = N\left(u_0(t) + u_1(t) + ... + u_n(t)\right), n = 1, 2, ...,$$

and

$$u(t) = \sum_{i=0}^{\infty} u_i(t) = g(t) + N\left(\sum_{i=1}^{\infty} u_i(t)\right).$$

The n-term approximation solution of equation (3.4.1) is given by

$$u(t) = \sum_{i=0}^{n-1} u_i(t) = u_0(t) + u_1(t) + ... + u_{n-1}(t).$$

3.4.2 Convergence analysis of DGJT

Now we analyze the convergence of the DGJT to solve the general nonlinear functional equation (3.4.1). For more details, see **[10]**

Theorem 3.4.1 *If N is a contraction, i.e. there exists $0 < k < 1$ such that*

$$\|N(u) - N(v)\|_B \leq k\|u - v\|_B, \text{ for all } u, v \in B,$$

then the series $\sum_{i=0}^{\infty} u_i(t)$ converges absolutely and uniformly to a solution of equation (3.4.1).

Proof. From (3.4.5), we have

$$\begin{aligned} u_0 &= g, \\ \|u_1\| &= \|N\left(u_0\right)\| \leq k\|u_0\|, \\ \|u_2\| &= \|N\left(u_0 + u_1\right) - N\left(u_0\right)\| \leq k\|u_1\| \leq k^2\|u_0\|, \\ \|u_3\| &= \|N\left(u_0 + u_1 + u_2\right) - N\left(u_0 + u_1\right)\| \leq k\|u_2\| \leq k^3\|u_0\|, \\ &\vdots \\ \|u_{n+1}\| &= \|N\left(u_0 + u_1 + ... + u_n\right) - N\left(u_0 + u_1 + ... + u_{n-1}\right)\|, \\ &\leq k\|u_n\| \leq k^{n+1}\|u_0\|, n = 0, 1, 2, ... \end{aligned}$$

As N is a contraction in the Banach space B, then the series solution $\sum_{i=0}^{\infty} u_i$ converges absolutely and uniformly to a solution of equation (3.4.1). ■

Corollary 3.4.1 *Let*

$$e = u - u^*,$$

where u is the exact solution, u^* is the approximate solution and e is the error in the solution of (3.4.1).

It is clear that e satisfies (3.4.1), i.e.

$$e = N(e) + g,$$

and the recurrence relation (3.4.5) becomes

$$\begin{aligned} e_0 &= g, \\ e_1 &= N(e_0), \\ e_{n+1} &= N\left(\sum_{j=0}^{n} e_j\right) - N\left(\sum_{j=0}^{n-1} e_j\right), n = 1, 2, ... \end{aligned}$$

If N is a contraction, then

$$\begin{aligned} e_0 &= g, \\ \|e_1\| &= \|N(e_0)\| \leq k\|e_0\|, \\ \|e_2\| &= \|N(e_0 + e_1) - N(e_0)\| \leq k\|e_1\| \leq k^2\|e_0\|, \\ \|e_3\| &= \|N(e_0 + e_1 + e_2) - N(e_0 + e_1)\| \leq k\|e_2\| \leq k^3\|e_0\|, \\ &\vdots \\ \|e_{n+1}\| &= \|N(e_0 + e_1 + ... + e_n) - N(e_0 + e_1 + ... + e_{n-1})\|, \\ &\leq k\|e_n\| \leq k^{n+1}\|e_0\|, n = 0, 1, 2, ... \end{aligned}$$

Therefore $e_{n+1} \longrightarrow 0$ when $n \longrightarrow \infty$, which proves the convergence of the DGJM to solve the general nonlinear functional equation (3.4.1).

Example 3.4.1 *Consider the following nonlinear Riccati equation*

$$u'(t) + u^2(t) = 1, t > 0, \tag{3.4.6}$$

subject to the initial condition

$$u(0) = 0. \tag{3.4.7}$$

Integrating equation (3.4.6) from 0 *to* t *and using the initial condition (3.4.7), we obtain*

$$u(t) = t - \int_0^t u^2(\tau)d\tau. \tag{3.4.8}$$

Putting

$$\begin{aligned} f(t) &= t, \\ N(u(t)) &= -\int_0^t u^2(\tau)d\tau, \end{aligned}$$

then equation (3.4.8) can be written in the form of a functional equation as

$$u(t) = f(t) + N(u(t)).$$

Applying the DGJT, the following first approximations values are given as

$$\begin{aligned} u_0(t) &= t, \\ u_1(t) &= N(u_0(t)) = -\frac{t^3}{3}, \\ u_2(t) &= N(u_1(t) + u_0(t)) - N(u_0(t)) = \frac{2}{15}t^5 - \frac{1}{63}t^7, \\ &\vdots \end{aligned}$$

As

$$u(t) = \sum_{i=0}^{\infty} u_i(t),$$

then the approximate analytical solution for proposed equation is expressed as

$$\begin{aligned} u(t) &= t - \frac{t^3}{3} + \frac{2}{15}t^5 - \frac{1}{63}t^7 + ... \\ &= \frac{e^{2t} - 1}{e^{2t} + 1}. \end{aligned}$$

Example 3.4.2 *Consider the following nonlinear differential equation*

$$u''(t) + 2u'(t) + u(t) + 8u^3(t) = 1 - 3t, \tag{3.4.8}$$

subject to the initial conditions

$$u(0) = \frac{1}{2}, u'(0) = \frac{1}{2}. \tag{3.4.9}$$

Integrating equation (3.4.8) from 0 *to* t *and using the initial condition (3.4.9), we obtain*

$$\begin{aligned} u(t) &= \frac{1}{2} + \frac{t}{2} + \frac{t^2}{2} - \frac{t^3}{3} - 2\int_0^t u(\tau)d\tau - \int_0^t\int_0^t \left(u(\tau) + 8u^3(\tau)\right) d\tau d\tau \\ &= \frac{1}{2} + \frac{t}{2} + \frac{t^2}{2} - \frac{t^3}{3} - 2\int_0^t u(\tau)d\tau - \int_0^t (t-\tau)\left(u(\tau) + 8u^3(\tau)\right) d\tau \\ &= f(t) + N(u(t)). \end{aligned}$$

Putting

$$\begin{aligned} f(t) &= \frac{1}{2} + \frac{t}{2} + \frac{t^2}{2} - \frac{t^3}{3}, \\ N(u(t)) &= -2\int_0^t u(\tau)d\tau - \int_0^t (t-\tau)\left(u(\tau) + 8u^3(\tau)\right) d\tau, \end{aligned}$$

then equation (3.4.8) can be written in the form of a functional equation as

$$u(t) = f(t) + N(u(t)).$$

Applying the DGJT, the following first approximations values are given as

$$\begin{aligned} u_0(t) &= \frac{1}{2} + \frac{t}{2} + \frac{t^2}{2} - \frac{t^3}{3}, \\ u_1(t) &= N(u_0(t)) = -t - \frac{5t^2}{4} - \frac{11t^3}{12} - \frac{7t^4}{24} + \frac{7t^5}{40}, \\ u_2(t) &= N(u_1(t) + u_0(t)) - N(u_0(t)) = t^2 + 2t^3 + \frac{19t^4}{16} + \frac{31t^5}{80} - \frac{371t^6}{720}, \\ &\vdots \end{aligned}$$

As

$$u(t) = \sum_{i=0}^{\infty} u_i(t),$$

then the approximate analytical solution for proposed equation is expressed as

$$\begin{aligned} u(t) &= \frac{1}{2}\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - ...\right) \\ &= \frac{1}{2}e^{-t}. \end{aligned}$$

Chapter 4

Solving nonlinear Caputo time-fractional partial differential equations

In this chapter, we illustrate the practicability of ADT, HPT, VIT, and DGJT for solving nonlinear time-fractional partial differential equations (NTFPDEs) described by the Caputo time-fractional operator. The effectiveness and validity of these techniques are demonstrated by solving different examples of NTFPDEs.

Consider the general nonlinear time-fractional partial differential equation

$$ {}^{C}D_t^{\alpha}u(X,t) = Lu(X,t) + Nu(X,t) + h(X,t), \tag{4.1.1} $$

subject to the initial conditions

$$ u^{(k)}(X,0) = u_k(X), k = 0,1,2,...,n-1, \tag{4.1.2} $$

where $X = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}^*$, L and N respectively are the linear and nonlinear operators, $h(X,t)$ is the source term, and ${}^{C}D_t^{\alpha}$ is the Caputo time-fractional derivative operator of the function $u(X,t)$ of order $n-1 < \alpha \leq n$ defined by

$$ {}^{C}D_t^{\alpha}u(X,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \dfrac{\partial^n u(X,\tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n, \\ \dfrac{\partial^n u(X,t)}{\partial t^n}, & \alpha = n. \end{cases} $$

4.1 Analysis of the ADT

To solve the NTFPDE expressed in (4.1.1) and (4.1.2) via the ADT, we start by operating the Riemann-Liouville fractional integral operator I_t^α on both sides of equation (4.1.1) to obtain the following

$$I_t^\alpha \left[{}^C D_t^\alpha u(X,t) \right] = I_t^\alpha \left[Lu(X,t) + Nu(X,t) + h(X,t) \right].$$

Furthermore, using relation (1.4.19) with the prescribed initial conditions, we obtain

$$u(X,t) = \sum_{k=0}^{n-1} \frac{u^{(k)}(X,0)}{k!} t^k + I_t^\alpha \left[h(X,t) \right] + I_t^\alpha \left[Lu(X,t) + Nu(X,t) \right]. \tag{4.1.3}$$

The ADT procedure expresses the solution $u(X,t)$ by the following infinite series

$$u(X,t) = \sum_{n=0}^{\infty} u_n(X,t), \tag{4.1.4}$$

while the nonlinear term $Nu(X,t)$ is expressed by the series of Adomian polynomials A_n as follows

$$Nu(X,t) = \sum_{n=0}^{\infty} A_n, \tag{4.1.5}$$

where A_n are the Adomian polynomials of $u_0, u_1, u_2, .., u_n$, expressed by the formula (3.1.11).

Therefore, with the above decomposed series, equation (4.1.3) is thus rewritten as follows

$$\sum_{n=0}^{\infty} u_n(X,t) = \sum_{k=0}^{n-1} u_k(X) \frac{t^k}{k!} + I_t^\alpha \left[h(X,t) \right] + I_t^\alpha \left[L \sum_{n=0}^{\infty} u_n(X,t) + \sum_{n=0}^{\infty} A_n \right]. \tag{4.1.6}$$

The recurrence relation in (4.1.6) yields

$$\begin{aligned}
u_0(X,t) &= \sum_{k=0}^{n-1} u_k(X) \frac{t^k}{k!} + I_t^\alpha \left[h(X,t) \right], \\
u_1(X,t) &= I_t^\alpha \left[Lu_0(X,t) + A_0 \right], \\
u_2(X,t) &= I_t^\alpha \left[Lu_1(X,t) + A_1 \right], \\
u_3(X,t) &= I_t^\alpha \left[Lu_2(X,t) + A_2 \right], \\
&\vdots \\
u_n(X,t) &= I_t^\alpha \left[Lu_{n-1}(X,t) + A_{n-1} \right].
\end{aligned}$$

Thus, the solution $u(X,t)$ can be approximated as an infinite series

$$\begin{aligned} u(X,t) &= \sum_{n=0}^{\infty} u_n(X,t) \\ &= u_0(X,t) + u_1(X,t) + u_2(X,t) + ... \end{aligned}$$

Example 4.1.1 *Assume the nonlinear Caputo time-fractional Burger equation*

$$^{C}D_t^{\alpha} u + uu_x - u_{xx} = 0, 0 < \alpha \leq 1, \tag{4.1.7}$$

subject to the initial condition

$$u(x,0) = x, \tag{4.1.8}$$

where $u = \{u(x,t), (x,t) \in \mathbb{R} \times \mathbb{R}^*\}$ *is an unknown function.*

Using the same step of the ADT procedure provided in Part 4.1, we get

$$\sum_{n=0}^{\infty} u_n(x,t) = x + I_t^{\alpha} \left[u_{nxx} - \sum_{n=0}^{\infty} A_n \right], \tag{4.1.9}$$

where A_n *are the Adomian polynomials representing the nonlinear term* uu_x.

The first Adomian polynomials are described as

$$\begin{aligned} A_0 &= u_0 u_{0x}, \\ A_1 &= u_0 u_{1x} + u_1 u_{0x}, \\ A_2 &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \\ A_3 &= u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} + u_3 u_{0x}, \\ &\vdots \end{aligned}$$

Comparing both side series terms in equation (4.1.9) yields

$$\begin{aligned} u_0(x,t) &= x, \\ u_1(x,t) &= I_t^{\alpha} \left[u_{0xx} - A_0 \right] = -x \frac{1}{\Gamma(\alpha+1)} t^{\alpha}, \\ u_2(x,t) &= I_t^{\alpha} \left[u_{1xx} - A_1 \right] = x \frac{2}{\Gamma(2\alpha+1)} t^{2\alpha}, \\ u_3(x,t) &= I_t^{\alpha} \left[u_{2xx} - A_2 \right] = -x \frac{4\Gamma^2(\alpha+1) + \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha}, \\ &\vdots \end{aligned}$$

and so on.

Provided that the solution in series form is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Therefore, the approximate series solution of equations (4.1.7)-(4.1.8) is

$$u(x,t) = x\left(1 - \frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \frac{2}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha} + ...\right). \quad (4.1.10)$$

We take $\alpha = 1$ in equation (4.1.10), we get the solution of classical nonlinear Burger equation as

$$u(x,t) = x\left(1 - t + t^2 - t^3 + ...\right),$$

which converges very fast to the exact solution

$$u(x,t) = \frac{x}{1+t}, |t| < 1.$$

Example 4.1.2 *Assume the two-dimensional nonlinear Caputo time-fractional wave-like equation with variable coefficients*

$$D_t^{\alpha}u = \frac{\partial^2}{\partial x \partial y}(u_{xx}u_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyu_x u_y) - u, \ 1 < \alpha \leq 2, \quad (4.1.11)$$

subject to the initial conditions

$$u(x,y,0) = e^{xy}, \ u_t(x,y,0) = e^{xy}, \quad (4.1.12)$$

where $u = \{u(x,y,t), (x,y,t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+\}$ is an unknown function.

Using the same step of the ADT procedure provided in Part 4.1, we get

$$\sum_{n=0}^{\infty} u_n(x,y,t) = e^{xy} + te^{xy} + I_t^{\alpha}\left[\frac{\partial^2}{\partial x \partial y}\left(\sum_{n=0}^{\infty} A_n\right) - \frac{\partial^2}{\partial x \partial y}\left(xy\sum_{n=0}^{\infty} B_n\right) - \sum_{n=0}^{\infty} u_n\right], \quad (4.1.13)$$

where A_n anb B_n are the Adomian polynomials representing the nonlinear terms $u_{xx}u_{yy}$ and $u_x u_y$, respectively.

The first Adomian polynomials are described as

$$\begin{aligned}
A_0 &= (u_0)_{xx}(u_0)_{yy}, \\
A_1 &= (u_1)_{xx}(u_0)_{yy} + (u_0)_{xx}(u_1)_{yy}, \\
A_2 &= (u_2)_{xx}(u_0)_{yy} + (u_1)_{xx}(u_1)_{yy} + (u_0)_{xx}(u_2)_{yy}, \\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
B_0 &= (u_0)_x (u_0)_y\,, \\
B_1 &= (u_1)_x (u_0)_y + (u_0)_x (u_1)_y\,, \\
B_2 &= (u_2)_x (u_0)_y + (u_1)_x (u_1)_y + (u_0)_x (u_2)_y\,, \\
&\vdots
\end{aligned}$$

Comparing both side series terms in equation (4.1.13) yields

$$\begin{aligned}
u_0(x,y,t) &= e^{xy} + te^{xy} = (1+t)\,e^{xy}, \\
u_1(x,y,t) &= I_t^{\alpha}\left[\frac{\partial^2}{\partial x \partial y}(A_0) - \frac{\partial^2}{\partial x \partial y}(xyB_0) - u_0\right] = -\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) e^{xy}, \\
u_2(x,y,t) &= I_t^{\alpha}\left[\frac{\partial^2}{\partial x \partial y}(A_1) - \frac{\partial^2}{\partial x \partial y}(xyB_1) - u_1\right] = \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) e^{xy}, \\
u_3(x,y,t) &= I_t^{\alpha}\left[\frac{\partial^2}{\partial x \partial y}(A_2) - \frac{\partial^2}{\partial x \partial y}(xyB_2) - u_2\right] = -\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) e^{xy}, \\
&\vdots
\end{aligned}$$

and so on.

Provided that the solution in series form is

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t),$$

therefore, the approximate series solution of equations (4.1.7)-(4.1.8) is

$$\begin{aligned}
u(x,y,t) &= \left(1 + t - \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right. \\
&\quad \left. - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \ldots\right) e^{xy}. \qquad (4.1.14)
\end{aligned}$$

We take $\alpha = 2$ in equation (4.1.14), we get the solution of classical 2-dimensional nonlinear wave-like equation with variable coefficients as

$$u(x,y,t) = \left(1 + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \frac{t^6}{6!} - \frac{t^7}{7!} + \ldots\right) e^{xy},$$

which converges very fast to the exact solution

$$u(x,y,t) = (\sin t + \cos t)\, e^{xy}.$$

4.2 Analysis of the HPT

In order to better understand the basic concept of the HPT algorithm, we consider the NTFPDE (4.1.1) subject to the initial conditions (4.1.2).

Executing the Riemann-Liouville fractional integral operator I_t^{α} on both sides of equation (4.1.1) and using relation (1.4.19) with the descripted initial conditions, we obtain

$$u(X,t) = \sum_{k=0}^{n-1} \frac{u^{(k)}(X,0)}{k!} t^k + I_t^{\alpha} \left[h(X,t) \right] + I_t^{\alpha} \left[Lu(X,t) + Nu(X,t) \right]. \tag{4.2.1}$$

In the context of HPT, the solution $u(X,t)$ is expressed in the form

$$u(X,t) = \sum_{n=0}^{\infty} p^n u_n(X,t), \tag{4.2.2}$$

where the homotopy parameter $p \in [0,1]$.

Here, we calculate the nonlinear term as

$$Nu(X,t) = \sum_{n=0}^{\infty} p^n H_n(u). \tag{4.2.3}$$

To obtain the values of He's polynomials $H_n(u)$, we use relation (3.2.11).

Substituting (4.2.2) and (4.2.3) in (4.2.1), we obtain

$$\sum_{n=0}^{\infty} p^n u_n(X,t) = \sum_{k=0}^{n-1} u_k(X) \frac{t^k}{k!} + I_t^{\alpha} \left[h(X,t) \right] + I_t^{\alpha} \left[L \sum_{n=0}^{\infty} p^n u_n(X,t) + \sum_{n=0}^{\infty} p^n H_n(u) \right], \tag{4.2.4}$$

Equating the similar components of p in both sides of equation (4.2.4), we obtain

$$\begin{aligned}
p^0 &: \quad u_0(X,t) = \sum_{k=0}^{n-1} u_k(X) \frac{t^k}{k!} + I^{\alpha} \left[h(X,t) \right], \\
p^1 &: \quad u_1(X,t) = I_t^{\alpha} \left[Lu_0(X,t) + H_0(u) \right], \\
p^2 &: \quad u_2(X,t) = I_t^{\alpha} \left[Lu_1(X,t) + H_1(u) \right], \\
p^3 &: \quad u_3(X,t) = I_t^{\alpha} \left[Lu_2(X,t) + H_2(u) \right], \\
& \quad \vdots \\
p^n &: \quad u_n(X,t) = I_t^{\alpha} \left[Lu_{n-1}(X,t) + H_{n-1}(u) \right].
\end{aligned}$$

Thus, the approximate solution of equations (4.1.1) and (4.1.2) is computed as

$$u(X,t) = \lim_{p \longrightarrow 1} \sum_{n=0}^{\infty} p^n u_n(X,t) = \sum_{n=0}^{\infty} u_n(X,t).$$

Example 4.2.1 *Assuming the nonlinear Caputo time-fractional KDV equation*

$$ {}^{C}D_t^{\alpha} u - 3\left(u^2\right)_x + u_{xxx} = 0, 0 < \alpha \leq 1, \tag{4.2.5}$$

subject to the initial condition

$$u(x,0) = 6x, \tag{4.2.6}$$

where $u = \{u(x,t), (x,t) \in \mathbb{R} \times \mathbb{R}^{+}\}$ is an unknown function.

Applying the HPT algorithm introduced in Part 4.2 leads to

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 6x + I_t^{\alpha}\left[3\sum_{n=0}^{\infty} p^n \left(H_n(u)\right)_x - \sum_{n=0}^{\infty} p^n u_{nxxx}\right]. \tag{4.2.7}$$

Using relation (3.2.11), the first terms of the He's polynomials $H_n(u)$ of the nonlinear term u^2, are calculated as

$$\begin{aligned} H_0 &= u_0^2, \\ H_1 &= 2u_0u_1, \\ H_2 &= 2u_0u_2 + u_1^2, \\ H_3 &= 2u_0u_3 + 2u_1u_2, \\ &\vdots \end{aligned}$$

Equating the similar components of p in both sides of equation (4.2.7), we get

$$\begin{aligned} p^0 &: u_0(x,t) = 6x, \\ p^1 &: u_1(x,t) = I_t^{\alpha}\left[3\left(H_0(u)\right)_x - u_{0xxx}\right] = 6x(36)\frac{1}{\Gamma(\alpha+1)}t^{\alpha}, \\ p^2 &: u_2(x,t) = I_t^{\alpha}\left[3\left(H_1(u)\right)_x - u_{1xxx}\right] = 6x(36)^2\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}, \\ p^3 &: u_3(x,t) = I_t^{\alpha}\left[3\left(H_2(u)\right)_x - u_{2xxx}\right] = 6x(36)^3\frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}, \\ p^4 &: u_4(x,t) = I_t^{\alpha}\left[3\left(H_3(u)\right)_x - u_{3xxx}\right] = \\ & \quad 6x(36)^4\left[\frac{8\Gamma^2(\alpha+1)+2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right] \times \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha}, \\ & \quad \vdots \end{aligned}$$

and so on.

Thus, the approximate solution of equations (4.2.5)-(4.2.6) is computed as

$$\begin{aligned} u(x,t) &= 6x\left(1+(36)\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+(36)^2\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}+(36)^3\frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}\right.\\ &\left.+(36)^4\left[\frac{8\Gamma^2(\alpha+1)+2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]\times\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha}+....\right). \end{aligned}$$

For $\alpha=1$, the solution obtained above becomes

$$u(x,t)=6x\left(1+36t+(36)^2t^2+(36)^3t^3+(36)^4t^4+...\right),$$

which converges rapidly to

$$u(x,t)=\frac{6x}{1-36t}, |36t|<1.$$

This is the exact solution of the classical nonlinear KDV equation.

Example 4.2.2 *Assuming the one-dimensional nonlinear Caputo time-fractional wave-like equation with variable coefficients*

$${}^{C}D_t^{\alpha}u=u^2\frac{\partial^2}{\partial x^2}(u_xu_{xx}u_{xxx})+u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3)-18u^5+u,\ 1<\alpha\leq 2, \tag{4.2.8}$$

subject to the initial conditions

$$u(x,0)=e^x, u_t(x,0)=e^x, \tag{4.2.9}$$

where $u=\{u(x,t),(x,t)\in\mathbb{R}\times\mathbb{R}^+\}$ is an unknown function.

Applying the HPT algorithm introduced in Part 4.2 leads to

$$\sum_{n=0}^{\infty}p^nu_n(x,t)=e^x+te^x+I_t^{\alpha}\left[\sum_{n=0}^{\infty}p^nH_n^1(u)+\sum_{n=0}^{\infty}p^nH_n^2(u)-18\sum_{n=0}^{\infty}p^nH_n^3(u)+\sum_{n=0}^{\infty}p^nu_n\right]. \tag{4.2.10}$$

Using relation (3.2.11), the first terms of the He's polynomials $H_n^1(u), H_n^2(u)$ and $H_n^3(u)$ of the nonlinear terms $u^2\frac{\partial^2}{\partial x^2}(u_xu_{xx}u_{xxx}), u_x^2\frac{\partial^2}{\partial x^2}(u_{xx}^3)$ and u^5, respectively, are calculated as

$$\begin{aligned} H_0^1(u) &= u_0^2\frac{\partial^2}{\partial x^2}\left((u_0)_x(u_0)_{xx}(u_0)_{xxx}\right),\\ H_1^1(u) &= 2u_{0u1}\frac{\partial^2}{\partial x^2}\left((u_0)_x(u_0)_{xx}(u_0)_{xxx}\right)+u_0^2\frac{\partial^2}{\partial x^2}\left((u_1)_x(u_0)_{xx}(u_0)_{xxx}\right.\\ &\quad\left.+(u_0)_x(u_1)_{xx}(u_0)_{xxx}+(u_0)_x(u_0)_{xx}(u_1)_{xxx}\right),\\ &\ \ \vdots \end{aligned}$$

$$
\begin{aligned}
H_0^2(u) &= (u_0)_x^2 \frac{\partial^2}{\partial x^2}\left((u_0)_{xx}^3\right), \\
H_1^2(u) &= 2(u_0)_x (u_1)_x \frac{\partial^2}{\partial x^2}\left((u_0)_{xx}^3\right) + (v_0)_x^2 \frac{\partial^2}{\partial x^2}\left(3(u_0)_{xx}^2 (u_1)_{xx}\right), \\
&\vdots
\end{aligned}
$$

and

$$
\begin{aligned}
H_0^3(u) &= u_0^5, \\
H_1^3(u) &= 5u_0^4 u_1, \\
&\vdots
\end{aligned}
$$

Equating the similar components of p in both sides of equation (4.2.10), we get

$$
\begin{aligned}
p^0 &: u_0(x,t) = e^x + te^x = (1+t)e^x, \\
p^1 &: u_1(x,t) = I_t^\alpha\left[H_0^1(u) + H_0^2(u) - 18H_0^3(u) + u_0\right] = \left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^x, \\
p^2 &: u_2(x,t) = I_t^\alpha\left[H_1^1(u) + H_1^2(u) - 18H_1^3(u) + u_0\right] = \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^x, \\
&\vdots
\end{aligned}
$$

and so on.

Thus, the approximate solution of equations (4.2.8)-(4.2.9) is computed as

$$u(x,t) = \left(1 + t + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + ...\right)e^x.$$

For $\alpha = 2$, *the solution obtained above becomes*

$$u(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + ...\right)e^x,$$

which converges rapidly to

$$u(x,t) = e^{t+x}.$$

This is the exact solution of the classical one-dimensional nonlinear wave-like equation with variable coefficients.

4.3 Analysis of the VIT

In order to explore the VIT methodology, we assume NTFPDE (4.1.1) by means of the initial conditions (4.1.2).

Operating the Riemann-Liouville fractional integral operator I_t^α on both sides of equation (4.1.1) and by relation (1.4.19) with the proposed initial conditions, we get

$$u(X,t)=\sum_{k=0}^{n-1}u_k(X)\frac{t^k}{k!}+I_t^\alpha\left[Lu(X,t)+Nu(X,t)+h(X,t)\right]. \tag{4.3.1}$$

Differentiating (4.3.1) with respect to t, to get

$$\frac{\partial u(X,t)}{\partial t}=\frac{\partial}{dt}\left(\sum_{k=0}^{n-1}u_k(X)\frac{t^k}{k!}+I_t^\alpha\left[Lu(X,t)+Nu(X,t)+h(X,t)\right]\right).$$

The VIT is adopted to design the correction functional for (4.3.1) as follows

$$u_{n+1}(X,t)=u_n(X,t)+\int_0^t\lambda(\tau)\left[\frac{\partial}{dt}\left(\sum_{k=0}^{n-1}u_k(X)\frac{t^k}{k!}+I_t^\alpha\left[Lu(X,t)+Nu(X,t)+h(X,t)\right]\right)\right]d\tau. \tag{4.3.2}$$

By variation theory, λ for (4.3.2) can be obtained as

$$1+\lambda(\tau)=0.$$

Therefore

$$\lambda(\tau)=-1.$$

Substituting the value of $\lambda(\tau)$ into (4.3.2), we get the iterative formula for $n=0,1,2,...,$ as follows

$$u_{n+1}(X,t)=u_n(X,t)-\int_0^t\left[\frac{\partial}{dt}\left(\sum_{k=0}^{n-1}u_k(X)\frac{t^k}{k!}+I_t^\alpha\left[Lu(X,t)+Nu(X,t)+h(X,t)\right]\right)\right]d\tau.$$

Or alternatively

$$u_{n+1}(X,t)=\sum_{k=0}^{n-1}u_k(X)\frac{t^k}{k!}+I_t^\alpha\left[Lu_n(X,t)+Nu_n(X,t)+h(X,t)\right]. \tag{4.3.4}$$

Start with the initial iteration

$$u_0(X,t)=\sum_{k=0}^{n-1}u_k(X)\frac{t^k}{k!}.$$

The successive approximations $u_n(X,t), n = 0,1,2,...$ converge rapidly to the exact solution of equation (4.1.1)-(4.1.2) when $n \to \infty$,i.e.

$$u(X,t) = \lim_{n \longrightarrow \infty} u_n(X,t).$$

Example 4.3.1 *Considering the nonlinear Caputo time-fractional gas-dynamic equation*

$$^cD_t^\alpha u + \frac{1}{2}\left(u^2\right)_x - u(1-u) = 0, 0 < \alpha \leq 1, \tag{4.3.5}$$

subject to the initial condition

$$u(x,0) = e^{-x}, \tag{4.3.6}$$

where $u = \{u(x,t), (x,t) \in \mathbb{R} \times \mathbb{R}^+\}$ is an unknown function.

Employing relation (4.3.4), we obtain the following iteration formula

$$u_{n+1}(x,t) = e^{-x} - I_t^\alpha \left[\frac{1}{2}\left(u_n^2\right)_x - u_n(1-u_n)\right], \tag{4.3.7}$$

where the initial iteration $u_0(x,t)$ is given by

$$u_0(x,t) = e^{-x}. \tag{4.3.8}$$

Using (4.3.7) and (4.3.8), we arrive at the following iterations

$$\begin{aligned}
u_0(x,t) &= e^{-x}, \\
u_1(x,t) &= e^{-x} + e^{-x}\frac{t^\alpha}{\Gamma(\alpha+1)}, \\
u_2(x,t) &= e^{-x} + e^{-x}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
u_3(x,t) &= e^{-x} + e^{-x}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{-x}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\
&\vdots
\end{aligned}$$

and so on.

Then, the successive approximations can be expressed as

$$\begin{aligned}
u(x,t) &= e^{-x}\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right) \\
&= e^{-x}\sum_{k=0}^{n}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.
\end{aligned}$$

Therefore, the solution of equations (4.3.5)-(4.3.6) is given by

$$\begin{aligned} u(x,t) &= \lim_{n\longrightarrow\infty} u_n(x,t) \\ &= e^{-x}\lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \\ &= e^{-x}\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \end{aligned} \tag{4.3.9}$$

When $\alpha = 1$ in equation (4.3.9), we obtain the exact solution of the standard nonlinear gas-dynamic equation as

$$u(x,t) = e^{-x+t}.$$

Example 4.3.2 *Considering the nonlinear Caputo time-fractional hyperbolic-like equation with variable coefficients*

$${}^{c}D_t^{\alpha}u = x^2\frac{\partial}{\partial x}(u_x u_{xx}) - x^2(u_{xx})^2 - u, t > 0, 1 < \alpha \leq 2, \tag{4.3.10}$$

subject to the initial conditions

$$u(x,0) = 0, u_t(x,0) = x^2, \tag{4.3.11}$$

where $u = \{u(x,t), (x,t) \in \mathbb{R}\times\mathbb{R}^+\}$ is an unknown function.

Employing relation (4.3.4), we obtain the following iteration formula

$$u_{n+1}(x,t) = x^2 + I_t^{\alpha}\left[x^2\frac{\partial}{\partial x}(u_{nx}u_{nxx}) - x^2(u_{nxx})^2 - u_n\right], \tag{4.3.12}$$

where the initial iteration $u_0(x,t)$ is given by

$$u_0(x,t) = tx^2. \tag{4.3.13}$$

Using (4.3.12) and (4.3.13), we arrive at the following iterations

$$\begin{aligned} u_0(x,t) &= tx^2, \\ u_1(x,t) &= tx^2 - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2, \\ u_2(x,t) &= tx^2 - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2 + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2, \\ u_3(x,t) &= tx^2 - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2 + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2 - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}x^2, \\ &\vdots \end{aligned}$$

Then, the successive approximations can be expressed as

$$\begin{aligned} u(x,t) &= x^2\left(t-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+...+\frac{(-1)^n\,t^{n\alpha+1}}{\Gamma(n\alpha+2)}\right) \\ &= x^2\left(\sum_{k=0}^{n}\frac{(-1)^k\,t^{k\alpha+1}}{\Gamma(k\alpha+2)}\right). \end{aligned}$$

Therefore, the solution of equations (4.3.10)-(4.3.11) is given by

$$\begin{aligned} u(x,t) &= \lim_{n\longrightarrow\infty} u_n(x,t) \\ &= x^2\lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\frac{(-1)^k\,t^{k\alpha+1}}{\Gamma(k\alpha+2)} \\ &= x^2\sum_{k=0}^{\infty}\frac{(-1)^k\,t^{k\alpha+1}}{\Gamma(k\alpha+2)}. \end{aligned} \tag{4.3.14}$$

When $\alpha = 2$ in equation (4.3.14), we obtain the exact solution of the standard nonlinear hyperbolic-like equation with variable coefficients

$$u(x,t) = x^2 \sin t.$$

4.4 Analysis of the DGJT

To give the fundamental description of the DGJT, let us examine the NTFPDE (4.1.1) under the initial conditions (4.1.2).

Implementing the Riemann-Liouville fractional integral operator I_t^α on equation (4.1.1) and utilize relation (1.4.19) with the initial conditions (4.1.2), we obtain

$$\begin{aligned} u(X,t) &= \sum_{k=0}^{n-1} u_k(X)\frac{t^k}{k!}+I_t^\alpha\left[h(X,t)\right]+I_t^\alpha\left[Lu(X,t)+Nu(X,t)\right] \\ &= H(X,t)+I_t^\alpha\left[Lu(X,t)\right]+I_t^\alpha\left[Nu(X,t)\right], \end{aligned} \tag{4.4.1}$$

where $H(X,t)$ is the term resulting from the initial conditions and the source term.

Putting

$$\begin{aligned} g(X,t) &= H(X,t), \\ R(u(X,t)) &= I_t^\alpha\left[Lu(X,t)\right], \\ M(u(X,t)) &= I_t^\alpha\left[Nu(X,t)\right]. \end{aligned}$$

Then equation (4.4.1) can be represented as follows

$$u(X,t) = g(X,t) + R(u(X,t)) + M(u(X,t)), \tag{4.4.2}$$

where R and M are the linear and nonlinear operators and g is a source function.

Through the DGJT, the solution to equation (4.4.2) is written as an infinite series

$$u(X,t) = \sum_{i=0}^{\infty} u_i(X,t). \tag{4.4.3}$$

Substituting equation (4.4.3) into equation (4.4.2), we have the following

$$\sum_{i=0}^{\infty} u_i(X,t) = g(X,t) + R\left(\sum_{i=0}^{\infty} u_i(X,t)\right) + M\left(\sum_{i=0}^{\infty} u_i(X,t)\right). \tag{4.4.4}$$

Further, the linear operator R is defined as

$$R\left(\sum_{i=0}^{\infty} u_i(X,t)\right) = \sum_{i=0}^{\infty} R\left(u_i(X,t)\right), \tag{4.4.5}$$

and the nonlinear operator M is decomposed as

$$M\left(\sum_{i=0}^{\infty} u_i(X,t)\right) = M\left(u_0(X,t)\right) + \sum_{i=1}^{\infty}\left\{M\left(\sum_{j=0}^{i} u_j(X,t)\right) - M\left(\sum_{j=0}^{i-1} u_j(X,t)\right)\right\}. \tag{4.4.6}$$

Replacing equations (4.4.5) and (4.4.6) into equation (4.4.4) gives

$$\begin{aligned}\sum_{i=0}^{\infty} u_i(X,t) &= g(X,t) + \sum_{i=0}^{\infty} R\left(u_i(X,t)\right) + M\left(u_0(X,t)\right) \\ &\quad + \sum_{i=1}^{\infty}\left\{M\left(\sum_{j=0}^{i} u_j(X,t)\right) - M\left(\sum_{j=0}^{i-1} u_j(X,t)\right)\right\}.\end{aligned} \tag{4.4.7}$$

Utilizing the recursive relation (4.4.7), we define the following iterations.

$$\begin{aligned}u_0(X,t) &= g(X,t), \\ u_1(X,t) &= R\left(u_0(X,t)\right) + M\left(u_0(X,t)\right), \\ u_{n+1}(X,t) &= R\left(u_n(X,t)\right) + M\left(\sum_{j=0}^{n} u_j(X,t)\right) - M\left(\sum_{j=0}^{n-1} u_j(X,t)\right), n = 1, 2, ...\end{aligned}$$

After summing all the components, the n^{th} approximate series solution to the NTFPDE (4.1.1) under the initial conditions (4.1.2) using DGJT is given as follows

$$u(X,t) = \sum_{i=0}^{n} u_i(X,t) = u_0(X,t) + u_1(X,t) + u_2(X,t) + ... + u_n(X,t).$$

When $n \to \infty$, the n^{th} approximate series results in the exact solution.

Example 4.4.1 *Let us consider the nonlinear Caputo time-fractional reaction-diffusion-convection equation*

$$^{C}D_t^{\alpha} u = u_{xx} - u_x + uu_x - u^2 + u, 0 < \alpha \leq 1, \tag{4.4.8}$$

subject to the initial condition

$$u(x,0) = e^x, \tag{4.4.9}$$

where $u = \{u(x,t), (x,t) \in \mathbb{R} \times \mathbb{R}^+\}$ *is an unknown function.*

Taking the Riemann-Liouville fractional integral operator I_t^{α} *on both sides of (4.4.8), utilize relation (1.4.19) and the initial condition (4.4.9) to get*

$$u(x,t) = e^x + I_t^{\alpha}\left[u_{xx} - u_x + u\right] + I_t^{\alpha}\left[uu_x - u^2\right].$$

Putting

$$\begin{aligned} g(x,t) &= e^x, \\ R(u(x,t)) &= I_t^{\alpha}\left[u_{xx} - u_x + u\right], \\ M(u(x,t)) &= I_t^{\alpha}\left[uu_x - u^2\right]. \end{aligned}$$

By employing the DGJT described in Part 4.3, we obtain the following

$$\begin{aligned} u_0(x,t) &= e^x, \\ u_1(x,t) &= e^x \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ u_2(x,t) &= e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3(x,t) &= e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ &\vdots \\ u_n(x,t) &= e^x \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{aligned}$$

Therefore, the n^{th} approximate series solution to equations (4.4.8)-(4.4.9) is given as follows

$$\begin{aligned} u(x,t) &= e^x\left(1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...+\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right) \\ &= e^x\sum_{i=0}^{n}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}. \end{aligned}$$

If $n\to\infty$, the result is

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} e^x\sum_{i=0}^{n}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} \\ &= e^x\sum_{i=0}^{\infty}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}. \end{aligned}$$

When $\alpha=1$, the exact solution of the standard nonlinear reaction-diffusion-convection equation is as follows

$$u(x,t)=e^{x+t}.$$

Example 4.4.2 *Let us consider the nonlinear Caputo time-fractional Newell-Whitehead-Segel equation*

$$^{C}D_t^\alpha u = u_{xx}+2u-3u^2, 0<\alpha\leq 1, \tag{4.4.10}$$

subject to the initial condition

$$u(x,0)=\lambda. \tag{4.4.11}$$

Taking the Riemann-Liouville fractional integral operator I_t^α on both sides of (4.4.10), utilize relation (1.4.19) and the initial condition (4.4.11) to get

$$u(x,t)=\lambda+I_t^\alpha\left[u_{xx}+2u\right]-3I_t^\alpha\left[u^2\right].$$

Putting

$$\begin{aligned} g(x,t) &= \lambda, \\ R(u(x,t)) &= I_t^\alpha\left[u_{xx}+2u\right], \\ M(u(x,t)) &= I_t^\alpha\left[u^2\right]. \end{aligned}$$

By employing the DGJT described in Part 4.3, we obtain the following

$$\begin{aligned}
u_0(x,t) &= \lambda, \\
u_1(x,t) &= \frac{2\lambda - 3\lambda^2}{\Gamma(\alpha+1)} t^{\alpha}, \\
u_2(x,t) &= \frac{2\lambda(2-3\lambda)(1-3\lambda)}{\Gamma(2\alpha+1)} t^{2\alpha}, \\
u_3(x,t) &= \frac{2\lambda(2-3\lambda)(2-18\lambda+27\lambda^2)}{\Gamma(3\alpha+1)} t^{3\alpha}, \\
&\vdots
\end{aligned}$$

and so on.

Therefore, the approximate series solution to equations (4.4.10)-(4.4.11) is given as follows

$$u(x,t) = \lambda + \frac{\left(2\lambda - 3\lambda^2\right)}{\Gamma(\alpha+1)} t^{\alpha} + \frac{2\lambda(2-3\lambda)(1-3\lambda)}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{2\lambda(2-3\lambda)(2-18\lambda+27\lambda^2)}{\Gamma(3\alpha+1)} t^{3\alpha} + ...$$

When $\alpha = 1$, the exact solution of the standard nonlinear Newell-Whitehead-Segel equation is as follows

$$\begin{aligned}
u(x,t) &= \lambda + \left(2\lambda - 3\lambda^2\right) t + 2\lambda(2-3\lambda)(1-3\lambda)\frac{t^2}{2!} \\
&\quad + 2\lambda(2-3\lambda)(2-18\lambda+27\lambda^2)\frac{t^3}{3!} + ... \\
&= \frac{-2\lambda e^{2t}}{-2+3\lambda(1-e^{2t})}.
\end{aligned}$$

Chapter 5

A novel hybrid technique to resolve the nonlinear time-fractional biological population model

In this chapter, we develop a novel hybrid technique to resolve a special case of nonlinear time-fractional partial differential equations, which is called nonlinear time-fractional biological population model with the Caputo fractional derivative. This technique is called Khalouta reduced differential transform method (KHRDTM) that is a mixture of two effective methods: Khalouta transform method (KHTM) and reduced differential transform method (RDTM). We will start by presenting an overview of the time-fractional biological population model and the mathematical formulation of the problem, then we provide the definition and fundamental characteristics of the KHT and the RDTM, then present the methodology of the proposed method as well as various numerical applications of the nonlinear time-fractional biological population equation.

5.1 Mathematical formulation of the nonlinear time-fractional biological population model

According to the belief of biologists, the population of the species can be regulated by dispersal or emigration. The diffusion of a biological species in a region D is defined by three functions of the position $\overrightarrow{X} = (x, y)$ and the time t namely the population density $u = u(\overrightarrow{X}, t)$, the diffusion velocity $v = v(\overrightarrow{X}, t)$ and the population supply $F = F(\overrightarrow{X}, t)$ **[32]**. The population density $u(\overrightarrow{X}, t)$ indicates the number of individuals per unit volume at a given position $\overrightarrow{X}$ and at a given time t. At time t, its integral over any subregion Ω of region D gives the total population of Ω in that subregion. The entity $F(\overrightarrow{X}, t)$ represents the rate at which individuals are added to the population at position $\overrightarrow{X}$ per unit volume through births and deaths. The diffusion velocity $v(\overrightarrow{X}, t)$ is the average velocity of all individuals occupying position $\overrightarrow{X}$ at time t, and it describes the flow of a population from point to point. The entities u, v and F must be compatible with the following population equilibrium law for any regular subregion Ω of D and for any time t

$$\frac{d^\alpha}{dt}\int_\Omega u dV + \int_{\partial\Omega} u\overrightarrow{v}.\widehat{n} dA = \int_\Omega F dV, \tag{5.1.1}$$

where $\widehat{n}$ is the exterior unit normal to the boundary $\partial\Omega$ of Ω. In equation (5.1.1), the derivative has been taken in the sense of Caputo's fractional derivative. It follows from equation (5.1.1), that the rate of population change in Ω plus the rate at which individuals depart Ω over its boundary must equal the rate at which individuals are supplied directly to Ω. Based on the assumptions **[42]**, we can see that

$$F = F(u) \text{ and } \overrightarrow{v} = -w(u)\nabla u,$$

where ∇ is the Laplace operator and $w(u) > 0$ for $u > 0$. For the population density u, the following two-dimensional nonlinear degenerate parabolic partial differential equation can be obtain and which is given as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 G(u)}{\partial x^2} + \frac{\partial^2 G(u)}{\partial y^2} + F(u). \tag{5.1.2}$$

The model described in equation (5.1.2) is called the nonlinear time-fractional biological population model. For modeling of the animal population, Gurney and Nisbet **[31]** used $h(u)$ as a special case. This model leads to equation (5.1.2) with $G(u) = u^2$, to the following equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + F(u), \tag{5.1.3}$$

subject to the initial condition $u(x, y, 0)$.

For $\alpha = 1$, equation (5.1.3) reduces to the classical nonlinear biological population model

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + F(u).$$

Here are three laws of constitutive equations for $F(u)$ that can be expressed as follow

(i) $F(u) = cu$ where c is a constant (Malthusian law **[32]**),

(ii) $F(u) = c_1 u - c_2 u^2$ where c_1, c_2 are positive constants (Verhulst law **[42]**),

(iii) $F(u) = cu^{\varphi}$ where $c > 0, 0 < \varphi < 1$ (Porous media **[9]**).

In this chapter, we consider a more general form of $F(u)$ such that $F(u) = hu^{\mu}\left(1 - qu^{\delta}\right)$ so that equation (5.1.3) becomes

$$^{C}D_t^{\alpha} u = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^{\mu}\left(1 - qu^{\delta}\right), \tag{5.1.4}$$

subject to the initial condition

$$u(x, y, 0) = u_0(x, y), \tag{5.1.5}$$

where $^{C}D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ represents the time-fractional derivative operator in the Caputo's sense of the function u of order α with $0 < \alpha \leq 1$ and $u = \left\{u(x, y, t), (x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{+}\right\}$ represents the population density and h, μ, q, δ are real numbers. If $h = c, \mu = 1, q = 0$ and $h = c_1, \mu = \delta = 1, q = c_2/c_1$ where $c_1 > 0, c_2 > 0$, c are constants, then equation (5.1.4) leads to Malthusian law and Verhulst law, respectively.

5.2 Khalouta transform (KHT)

Recently, Ali Khalouta [36] defined and developed a new integral transform called Khalouta transform (KHT) to solve various types of differential equations. This new integral transform is a generalization of many integral transforms such as: Laplace transform, Fourier transform, Sumudu transform, Elzaki transform, Aboodh transform, Shehu transform, ZZ transform, and ZMA transform. In this part, we mention the definition and important properties of KHT that need in this chapter.

Definition 5.2.1 *[36] The KHT of a continuous function $u(x, y, t)$ is described as*

$$\mathbb{KH}\left[u(x, y, t)\right] = \mathcal{K}(x, y, s, \gamma, \eta) = \frac{s}{\gamma\eta}\int_0^{\infty} \exp\left(-\frac{st}{\gamma\eta}\right) u(x, y, t)dt,$$

where $s, \gamma, \eta > 0$ are the KHT variables.

Theorem 5.2.1 *[36] The inverse KHT of the function $u(x,y,t)$ is described as*

$$\mathbb{KH}^{-1}\left[\mathcal{K}(x,y,s,\eta,\gamma)\right] = u(x,y,t), \text{ for } t \geq 0.$$

This is equivalent to

$$u(x,y,t) = \mathbb{KH}^{-1}\left[\mathcal{K}(x,y,s,\eta,\gamma)\right] = \frac{1}{2\pi i}\int_{\varphi-i\infty}^{\varphi+i\infty} \frac{1}{s}\exp\left(\frac{st}{\gamma\eta}\right)\mathcal{K}(x,y,s,\eta,\gamma)ds,$$

where φ is a real constant and the integral is taken along $s = \varphi$ in the complex plane $s = u + iv$.

Proof. For more details on the proof of this theorem, see [36]. ■

Theorem 5.2.2 *[36] The fundamental properties of KHT are:*

(1) Linearity property.

$$\mathbb{KH}\left[au(x,y,t) + bv(x,y,t)\right] = a\mathbb{KH}\left[u(x,y,t)\right] + b\mathbb{KH}\left[v(x,y,t)\right], a, b \in \mathbb{R}$$

(2) Differentiation property.

$$\mathbb{KH}\left[u^{(n)}(x,y,t)\right] = \frac{s^n}{\gamma^n\eta^n}\mathbb{KH}\left[u(x,y,t)\right] - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}u^{(k)}(x,y,0), n \geq 1,$$

where $u^{(n)}(x,y,t)$ is the n^{th} derivative of $u(x,y,t)$.

(3) Convolution property.

$$\mathbb{KH}\left[(u * v)(x,y,t)\right] = \frac{\gamma\eta}{s}\mathbb{KH}\left[u(x,y,t)\right]\mathbb{KH}\left[v(x,y,t)\right],$$

where $\mathbb{KH}\left[(u * v)(x,y,t)\right]$ is the Khalouta convolution of the functions $u(x,y,t)$ and $v(x,y,t)$.

(4) The KHTs of the usual functions.

$$\begin{aligned}
\mathbb{KH}\left[1\right] &= 1, \\
\mathbb{KH}\left[t\right] &= \frac{\gamma\eta}{s}, \\
\mathbb{KH}\left[\frac{t^n}{n!}\right] &= \frac{\gamma^n\eta^n}{s^n}, n \geq 0, \\
\mathbb{KH}\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] &= \frac{\gamma^\alpha\eta^\alpha}{s^\alpha}, \alpha > -1.
\end{aligned}$$

Proof. For more details on the proof of this theorem, see [36]. ■

Below, we present the main results related to the KHT for the Riemann-Liouville fractional integral and the Caputo time-fractional derivative [39].

Theorem 5.2.3 *[39] The KHT of the Riemann-Liouville fractional integral of order $\alpha > 0$ is described as*

$$\mathbb{KH}\left[I_t^{\alpha} u(x,y,t)\right] = \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}} \mathbb{KH}\left[u(x,y,t)\right].$$

Proof. Taking the KHT to both sides of equation (1.4.2), we get

$$\begin{aligned}
\mathbb{KH}\left[I_t^{\alpha} u(x,y,t)\right] &= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(x,y,\tau) d\tau\right] \\
&= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * u(x,y,t)\right].
\end{aligned}$$

Using Theorem 5.2.2, we get

$$\begin{aligned}
\mathbb{KH}\left[I_t^{\alpha} u(x,y,t)\right] &= \frac{\gamma\eta}{s} \mathbb{KH}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \mathbb{KH}\left[u(x,y,t)\right] \\
&= \frac{\gamma\eta}{s} \frac{\gamma^{\alpha-1}\eta^{\alpha-1}}{s^{\alpha-1}} \mathbb{KH}\left[u(x,y,t)\right] \\
&= \frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}} \mathbb{KH}\left[u(x,y,t)\right].
\end{aligned}$$

The proof is complete. ■

Theorem 5.2.4 *[39] The KHT of the Caputo fractional derivative of order $n-1 < \alpha \leq n$ with $n \geq 1$ is described as*

$$\mathbb{KH}\left[{}^{C}D_t^{\alpha} u(x,y,t)\right] = \frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}} \mathbb{KH}\left[u(x,y,t)\right] - \sum_{k=0}^{n-1} \left(\frac{s}{\gamma\eta}\right)^{\alpha-k} u^{(k)}(x,y,0).$$

Proof. Taking

$$v(x,y,t) = u^{(n)}(x,y,t). \tag{5.2.1}$$

Then, equation (1.4.14) becomes

$$\begin{aligned}
{}^{C}D_t^{\alpha} u(x,y,t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(x,y,\tau) d\tau \\
&= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} v(x,y,\tau) d\tau \\
&= I_t^{n-\alpha} v(x,y,t). \qquad (5.2.2)
\end{aligned}$$

Applying the KHT on both sides of equation (5.2.2) and using Theorem 5.2.3, we get

$$\mathbb{KH}\left[{}^{C}D_t^{\alpha}u(x,y,t)\right] = \mathbb{KH}\left[I_t^{n-\alpha}v(x,y,t)\right] = \frac{\gamma^{n-\alpha}\eta^{n-\alpha}}{s^{n-\alpha}}\mathcal{V}(x,y,s,\gamma,\eta), \tag{5.2.3}$$

where $\mathcal{V}(x,y,s,\gamma,\eta)$ is the KHT of the function $v(x,y,t)$.

Implementing the KHT on both sides of equation (5.2.1) and Theorem 5.2.2, gives

$$\begin{aligned} \mathbb{KH}\left[v(x,y,t)\right] &= \mathbb{KH}\left[u^{(n)}(x,y,t)\right], \\ \mathcal{V}(x,y,s,\gamma,\eta) &= \frac{s^n}{\gamma^n\eta^n}\mathbb{KH}\left[u(x,y,t)\right] - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}u^{(k)}(x,y,0). \end{aligned} \tag{5.2.4}$$

Replacing equation (5.2.4) into equation (5.2.3), we get

$$\begin{aligned} \mathbb{KH}\left[{}^{C}D_t^{\alpha}u(x,y,t)\right] &= \frac{\gamma^{n-\alpha}\eta^{n-\alpha}}{s^{n-\alpha}}\left(\frac{s^n}{\gamma^n\eta^n}\mathbb{KH}\left[u(x,y,t)\right] - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}u^{(k)}(x,y,0)\right) \\ &= \frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathbb{KH}\left[u(x,y,t)\right] - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}u^{(k)}(x,y,0). \end{aligned}$$

The proof is complete. ■

5.3 Reduced differential transform method (RDTM)

In this part, we mention the basic definitions and fundamental theorems of the RDTM which are defined and proved in [34].

Consider a function of three variables $u(x,y,t)$ and assume that it can be represented as a product of three single-variable functions, i.e., $u(x,y,t) = f(x)g(y)h(t)$. The function $u(x,y,t)$ can be represented due to the properties of $(2+1)-$dimensional differential transform as follows

$$\begin{aligned} u(x,y,t) &= \left(\sum_{i=0}^{\infty}F(i)x^i\right)\left(\sum_{j=0}^{\infty}G(j)y^j\right)\left(\sum_{l=0}^{\infty}H(l)t^l\right) \\ &= \sum_{k=0}^{\infty}U_k(x,y)t^k, \end{aligned}$$

where $U_k(x,y)$ is called $t-$dimensional spectrum function of the original function $u(x,y,t)$.

Definition 5.3.1 *[34] The RDT function of the function $u(x, y, t)$ is given by*

$$U_k(x, y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x, y, t)\right]_{t=0}, \tag{5.3.1}$$

where $u(x, y, t)$ is the original function which is analytic and differentiated continuously function with regard to time t and space in the domain of interest and $U_k(x, y)$ is the transformed function

Definition 5.3.2 *[34] The RDT inverse of the function $U_k(x, y)$ is given by*

$$u(x, y, t) = \sum_{k=0}^{\infty} U_k(x, y)t^k. \tag{5.3.2}$$

Combining equations (5.3.1) and (5.3.2) we find that

$$u(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x, y, t)\right]_{t=0} t^k.$$

Theorem 5.3.1 *[34] Suppose that $U_k(x, y), V_k(x, y)$ and $W_k(x, y)$ be the RDTs of the functions $u(x, y, t), v(x, y, t)$ and $w(x, y, t)$ respectively.*

(1) If

$$w(x, y, t) = \lambda u(x, y, t) + \mu v(x, y, t),$$

then

$$W_k(x, y) = \lambda U_k(x, y) + \mu V_k(x, y), \lambda, \mu \in \mathbb{R}.$$

(2) If

$$w(x, y, t) = u(x, y, t)v(x, y, t),$$

then

$$W_k(x, y) = \sum_{r=0}^{k} U_r(x, y)V_{k-r}(x, y).$$

(3) If

$$w(x, y, t) = u^1(x, y, t)u^2(x, y, t) \times ... \times u^{n-1}(x, y, t)u^n(x, y, t),$$

then

$$W_k(x, y) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U^1_{k_1}(x, y)U^2_{k_2-k_1}(x, y) \times ... \times U^{n-1}_{k_{n-1}-k_{n-2}}(x, y)U^n_{k-k_{n-1}}(x, y).$$

5.4 Khalouta reduced differential transform method (KHRDTM)

In this part, we propose a novel hybrid technique that uses KHT and RDTM. We call this developed method KHRDTM. To illustrate the basic idea of this method, we assume the following general nonlinear time-fractional partial differential equation

$$^{C}D_t^{\alpha}u(x,y,t) + Lu(x,y,t) + Nu(x,y,t) = f(x,y,t), \tag{5.4.1}$$

subject to the initial conditions

$$u^{(k)}(x,y,0) = u_k(x,y), k = 0,1,2,...,n-1, \tag{5.4.2}$$

where $^{C}D_t^{\alpha}$ is the Caputo time-fractional derivative operator of the function $u(x,y,t)$ of order α with $n-1 < \alpha \leq n$, L represents the linear differential operator, N represents a nonlinear differential operator, and $f(x,y,t)$ represents the non-homogeneous.

Theorem 5.4.1 *[19] According to the KHRDTM, the solution of equations (5.4.1) and (5.4.2) is described as an infinite series which rapidly converge to the exact solution as follows*

$$u(x,y,t) = \sum_{r=0}^{\infty} U_r(x,y), \tag{5.4.3}$$

where $U_r(x,y)$ is the RDT of the function $u(x,y,t)$.

Proof. To begin the proof of this theorem, let us consider the general nonlinear time-fractional partial differential equations (5.4.1) subjet to the initial conditions (5.4.2).

Taking the KHT to equation (5.4.1) and using the linearity property of the KHT (See. Theorem 5.2.2), we have

$$\mathbb{KH}\left[^{C}D_t^{\alpha}u(x,y,t)\right] + \mathbb{KH}\left[Lu(x,y,t) + Nu(x,y,t)\right] = \mathbb{KH}\left[f(x,y,t)\right].$$

Using Theorem 5.2.4 we obtain

$$\frac{s^{\alpha}}{\gamma^{\alpha}\eta^{\alpha}}\mathbb{KH}\left[u(x,y,t)\right] - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}u^{(k)}(x,y,0) = \mathbb{KH}\left[f(x,y,t)\right] - \mathbb{KH}\left[\begin{array}{c} Lu(x,y,t) \\ +Nu(x,y,t) \end{array}\right]. \tag{5.4.4}$$

Replacing the initial conditions (5.4.2) and simple calculate, equation (5.4.4) can be rewritten as

$$\mathbb{KH}\left[u(x,y,t)\right]=\sum_{k=0}^{n-1}\left(\frac{\gamma\eta}{s}\right)^{k}u_k(x,y)+\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathbb{KH}\left[f(x,y,t)\right]-\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathbb{KH}\left[\begin{array}{c}Lu(x,y,t)\\+Nu(x,y,t)\end{array}\right]. \tag{5.4.5}$$

Application of the inverse KHT of equation (5.4.5), gives

$$u(x,y,t)=F(x,y,t)-\mathbb{KH}^{-1}\left(\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathbb{KH}\left[Lu(x,y,t)+Nu(x,y,t)\right]\right). \tag{5.4.6}$$

where $F(x,y,t)$ represents the term resulting from the initial conditions and the non-homogeneous term.

Now, by executing the RDTM on equation (5.4.6), we get

$$\begin{aligned}U_0(x,y) &= F(x,y,t),\\ U_{k+1}(x,y) &= -\mathbb{KH}^{-1}\left(\frac{\gamma^{\alpha}\eta^{\alpha}}{s^{\alpha}}\mathbb{KH}\left[LU_k(x,y)+R_k(x,y)\right]\right), k\geq 0,\end{aligned} \tag{5.4.7}$$

where $R_k(x,y)$ is the RDT of the nonlinear term $Nu(x,y,t)$.

Note that the recurrence formula (5.4.7) to the iterative terms of equations (5.4.1) and (5.4.2) is denoted KHRDTM, and and its $k^{th}-$order approximate series solution is given as

$$\mathcal{P}_k(x,y)=\sum_{r=0}^{k}U_r(x,y). \tag{5.4.8}$$

Thus, in the following theorem, we prove that the approximate series solution (5.4.8) converges to the exact solution if $k\to\infty$, that is

$$u(x,y,t)=\lim_{k\to\infty}\mathcal{P}_k(x,y)=\sum_{r=0}^{\infty}U_r(x,y).$$

The proof is complete. ■

5.5 Convergence analysis of the KHRDTM

In this part, we discusse the necessary condition for convergence of the solution. We follow these theorems applied to the series solutions obtained by KHRDTM.

Theorem 5.5.1 *[19] Let $\mathcal{B} = (C(\Omega), \|.\|)$ be a Banach space of all continuous mappings defined on $\Omega \subset \mathbb{R}^2 \times \mathbb{R}^+$ having the norm $\|u(x,y,t)\| = \sup_{(x,y,t)\in\Omega} |u(x,y,t)|$.*

If $\exists 0 < \varpi < 1$ such that $\|U_r(x,y)\| \leq \varpi \|U_{r-1}(x,y)\|, \forall r \in \mathbb{Z}^+$, then the approximate series solution $\sum_{r=0}^{\infty} U_r(x,y)$ which is obtained by KHRDTM will uniquely converge to $u(x,y,t)$.

Proof. Let $\mathcal{P}_k(x,y)$ be the k^{th} partial sums given by the recurrence formula (5.4.7)

$$\mathcal{P}_k(x,y) = \sum_{r=0}^{k} U_r(x,y).$$

We have to show that $\{\mathcal{P}_k(x,y)\}_{k=0}^{\infty}$ is a Cauchy sequence in Banach space $\mathcal{B}$.

Therefore, we consider

$$\begin{aligned}
\|\mathcal{P}_{k+1}(x,y) - \mathcal{P}_k(x,y)\| &\leq \|U_{r+1}(x,y)\| \leq \varpi \|U_r(x,y)\| \\
&\leq \varpi^2 \|U_{r-1}(x,y)\| \leq ... \leq \varpi^{n+1} \|U_0(x,y)\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{P}_n(x,y) - \mathcal{P}_m(x,y)\| &= \left\| \begin{array}{c} \mathcal{P}_n(x,y) - \mathcal{P}_{n-1}(x,y) + \mathcal{P}_{n-1}(x,y) - \mathcal{P}_{n-2}(x,y) \\ +... + \mathcal{P}_{m+1}(x,y) - \mathcal{P}_m(x,y) \end{array} \right\| \\
&\leq \|\mathcal{P}_n(x,y) - \mathcal{P}_{n-1}(x,y)\| + \|\mathcal{P}_{n-1}(x,y) - \mathcal{P}_{n-2}(x,y)\| \\
&\quad +... + \|\mathcal{P}_{m+1}(x,y) - \mathcal{P}_m(x,y)\| \\
&\leq \varpi^n \|U_0(x,y)\| + \varpi^{n-1} \|U_0(x,y)\| + ... + \varpi^{m+1} \|U_0(x,y)\| \\
&= \varpi^{m+1} \left(1 + \varpi + ... + \varpi^{n-m-1}\right) \|U_0(x,y)\| \\
&\leq \varpi^{m+1} \left(\frac{1-\varpi^{n-m}}{1-\varpi}\right) \|U_0(x,y)\|, \forall n, m \in \mathbb{N}. \qquad (5.5.1)
\end{aligned}$$

Since $0 < \varpi < 1$, we have $1 - \varpi^{n-m} < 1$, therefore the inequality equation (5.5.1) can be reduced to

$$\|\mathcal{P}_n(x,y) - \mathcal{P}_m(x,y)\| \leq \frac{\varpi^{m+1}}{1-\varpi} \|U_0(x,y)\|. \qquad (5.5.2)$$

As $U_0(x,y)$ is bounded, then $\|\mathcal{P}_n(x,y) - \mathcal{P}_m(x,y)\| \to 0$ when $n, m \to \infty$.

Thus $\{\mathcal{P}_k(x,y)\}_{k=0}^{\infty}$ is a Cauchy sequence in Banach space and consequently it is converges to $u(x,y,t) \in \mathcal{B}$ such that

$$\lim_{k\to\infty} \mathcal{P}_k(x,y) = \sum_{r=0}^{\infty} U_r(x,y) = u(x,y,t).$$

Now, suppose that the sequence $\{\mathcal{P}_k(x,y)\}_{k\geq 0}$ converges to two functions of $u(x,y,t), v(x,y,t) \in \mathcal{B}$, that is

$$\lim_{k\to\infty} \mathcal{P}_k(x,y) = u(x,y,t) \text{ and } \lim_{k\to\infty} \mathcal{P}_k(x,y) = v(x,y,t).$$

Using the triangle inequality (5.5.2), we get

$$\|u(x,y,t) - v(x,y,t)\| \leq \|u(x,y,t) - \mathcal{P}_k(x,y)\| + \|\mathcal{P}_k(x,y) - v(x,y,t)\| = 0,$$

when $k \to \infty$.

Hence we conclude that $u(x,y,t) = v(x,y,t)$.

The proof is complete. ■

Corollary 5.5.1 *If the series* $\sum_{r=0}^{\infty} U_r(x,y)$ *converges, then it is an exact solution of the nonlinear time-fractional partial differential equation (5.4.1).*

Theorem 5.5.2 *[19] The obtained solution (5.4.3) of equations (5.4.1)-(5.4.2) is estimated to have a maximum absolute truncation error of the form*

$$\left\| u(x,y,t) - \sum_{l=0}^{N} U_l(x,y) \right\| \leq \frac{\varpi^{N+1}}{1-\varpi} \|U_0(x,y)\|.$$

Proof. Using Theorem 5.5.1 and the inequality equation (5.5.2), we get

$$\|\mathcal{P}_k(x,y) - \mathcal{P}_N(x,y)\| \leq \frac{\varpi^{N+1}}{1-\varpi} \|U_0(x,y)\|. \tag{5.5.3}$$

We suppose that $\mathcal{P}_k(x,y) = \sum_{l=0}^{k} U_l(x,y)$ and since $k \to +\infty$, we get $\mathcal{P}_k(x,y) \to u(x,y,t)$, so the inequality equation (5.5.3) can be rewritten as

$$\|u(x,y,t) - \mathcal{P}_N(x,y)\| = \left\| u(x,y,t) - \sum_{l=0}^{N} U_l(x,y) \right\| \leq \frac{\varpi^{N+1}}{1-\varpi} \|U_0(x,y)\|.$$

The proof is complete. ■

5.6 Applications and numerical results

we demonstrate the validity and efficiency of the results obtained using the present method to determine the exact solutions for three special cases of the nonlinear time-fractional biological population model (5.1.4) subject to the initial condition (5.1.5).

Example 5.6.1 *Let us assume that the nonlinear time-fractional biological population model described in the sense of Caputo*

$$ {}^{C}D_t^{\alpha} u = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u, 0 < \alpha \leq 1, \tag{5.6.1} $$

having the following initial condition

$$ u(x, y, 0) = \sqrt{\sin x \sinh y}. \tag{5.6.2} $$

Following the KHRDTM methodology presented in Part 5.4, we get the series solution of our model

$$ u(x, y, t) = \sum_{r=0}^{\infty} U_r(x, y), $$

where in this case, we set

$$ Lu(x, y, t) = u \text{ and } Nu(x, y, t) = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2}. $$

For the convenience of the reader, the first few RDT of the nonlinear term $Nu(x, y, t)$ are as follows

$$ \begin{aligned} R_0(x, y) &= \frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2}, \\ R_1(x, y) &= 2\frac{\partial^2 u_0 u_1}{\partial x^2} + 2\frac{\partial^2 u_0 u_1}{\partial y^2}, \\ R_2(x, y) &= 2\frac{\partial^2 (u_0 u_2)}{\partial x^2} + \frac{\partial^2 u_1^2}{\partial x^2} + 2\frac{\partial^2 (u_0 u_2)}{\partial y^2} + \frac{\partial^2 u_1^2}{\partial y^2}. \end{aligned} $$

According to the recurrence formula (5.4.7), we get

$$ \begin{aligned} U_0(x, y) &= \sqrt{\sin x \sinh y}, \\ U_1(x, y) &= \sqrt{\sin x \sinh y}\frac{t^{\alpha}}{\Gamma(\alpha + 1)}, \\ U_2(x, y) &= \sqrt{\sin x \sinh y}\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}, \\ U_3(x, y) &= \sqrt{\sin x \sinh y}\frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}, \\ &\vdots \\ U_k(x, y) &= \sqrt{\sin x \sinh y}\frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}. \end{aligned} $$

Thus, the $k^{th}-$order approximate series solution is given by

$$\begin{aligned}
\mathcal{P}_k(x,y) &= U_0(x,y)+U_1(x,y)+U_2(x,y)+...+U_k(x,y)\\
&= \sqrt{\sin x\sinh y}+\sqrt{\sin x\sinh y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\sqrt{\sin x\sinh y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad+\sqrt{\sin x\sinh y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...+\sqrt{\sin x\sinh y}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\\
&= \sqrt{\sin x\sinh y}\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...+\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\right)\\
&= \sqrt{\sin x\sinh y}\sum_{r=0}^{k}\frac{t^{r\alpha}}{\Gamma(r\alpha+1)}.
\end{aligned}$$

So, the $k^{th}-$order approximate series solution converges to the exact solution as $k\to\infty$

$$\begin{aligned}
u(x,y,t) &= \lim_{k\to\infty}\mathcal{P}_k(x,y)\\
&= \sqrt{\sin x\sinh y}\lim_{k\to\infty}\sum_{r=0}^{k}\frac{(t^{\alpha})^r}{\Gamma(r\alpha+1)}\\
&= \sqrt{\sin x\sinh y}E_{\alpha}(t^{\alpha}),
\end{aligned}\tag{5.6.3}$$

where $E_{\alpha}(t^{\alpha})$ is the Mittag Leffler function defined by equation (1.3.5).

As $\alpha=1$ in equation (5.6.3), we have

$$\begin{aligned}
u(x,y,t) &= \sqrt{\sin x\sinh y}\sum_{r=0}^{\infty}\frac{t^r}{r!}\\
&= \sqrt{\sin x\sinh y}\exp(t),
\end{aligned}$$

which is the same exact solution found in the literature [38].

Figures 1 and 2 illustrate the 2D and 3D graphs of KHRDRM in solving equations (5.6.1)-(5.6.2). Moreover, the comparison of the $6^{th}-$order approximate and exact solutions of KHRDRM and their associated absolute errors for equations (5.6.1)-(5.6.2) are presented in Table 1. The numerical simulation of the proposed method indicates that the numerical solutions of the method can approximate the exact solution well.

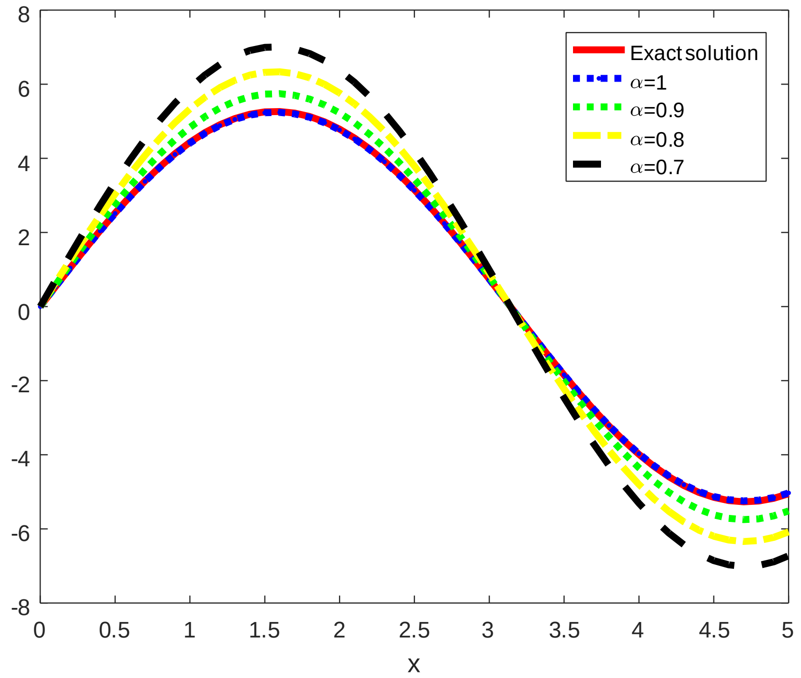

Figure 1: 2D graphs of the 6^{th}-order approximate series solution and exact solution with $y = 1$ and $t = 1.5$.

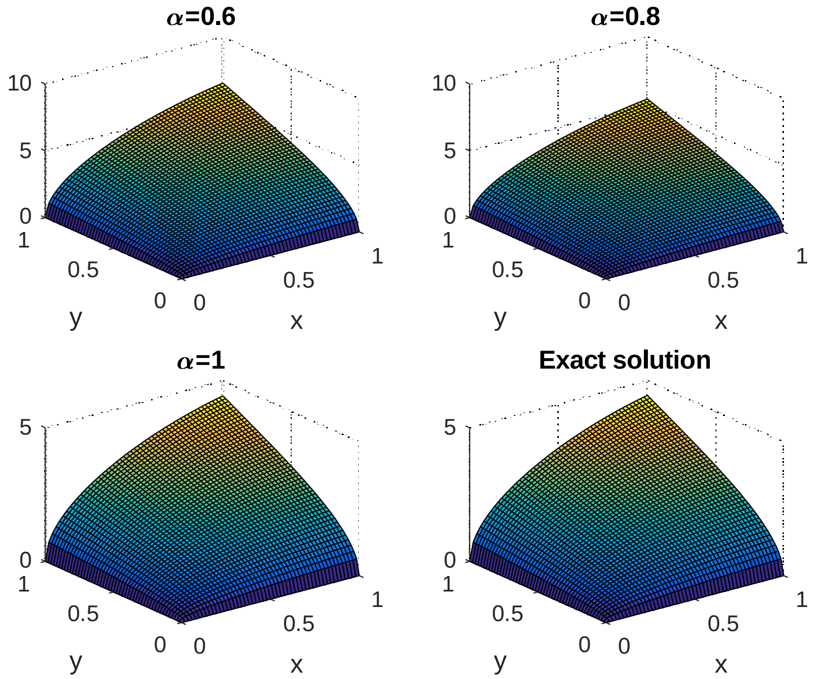

Figure 2: 3D graphs of the 6^{th}-order approximate series solution and exact solution with $t = 1.5$.

	u_{KHRDTM}	u_{KHRDTM}	u_{KHRDTM}	u_{KHRDTM}	u_{Exact}	$\lvert u_{Exact} - u_{KHRDTM} \rvert$
$(x, y)/\alpha$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	*Exact solution*	*Absolute errors*
0.1	0.12556	0.11892	0.11408	0.11052	0.11052	1.4090×10^{-10}
0.3	0.37667	0.35673	0.34224	0.33154	0.33154	4.2268×10^{-10}
0.5	0.62759	0.59437	0.57023	0.55239	0.55239	7.0425×10^{-10}
0.7	0.87776	0.83130	0.79753	0.77259	0.77259	9.8497×10^{-10}
0.9	1.12590	1. 06630	1.02300	0.99102	0.99102	1.2635×10^{-9}

Table 1: Comparison of the 6^{th}-order approximate series solutions and exact solution and their associated absolute errors for different values of α.

Example 5.6.2 *Let us assume that the nonlinear time-fractional biological population model described in the sense of Caputo*

$$ {}^{C}D_t^{\alpha} u = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu\,(1 - qu)\,, 0 < \alpha \leq 1, \tag{5.6.4}$$

subject to the initial condition

$$ u(x, y, 0) = \exp\left(\sqrt{\frac{hq}{8}}(x + y)\right). \tag{5.6.5}$$

Following the KHRDTM methodology presented in Part 5.4, we get the series solution of our model

$$ u(x, y, t) = \sum_{r=0}^{\infty} U_r(x, y), $$

where in this case, we set

$$ Lu(x, y, t) = hu \text{ and } Nu(x, y, t) = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hqu^2. $$

For the convenience of the reader, the first few RDT of the nonlinear term $Nu(x, y, t)$ are as follows

$$ \begin{aligned} R_0(x, y) &= \frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + hqu_0^2, \\ R_1(x, y) &= 2\frac{\partial^2 u_0 u_1}{\partial x^2} + 2\frac{\partial^2 u_0 u_1}{\partial y^2} + 2hqu_0 u_1, \\ R_2(x, y) &= 2\frac{\partial^2 (u_0 u_2)}{\partial x^2} + \frac{\partial^2 u_1^2}{\partial x^2} + 2\frac{\partial^2 (u_0 u_2)}{\partial y^2} + \frac{\partial^2 u_1^2}{\partial y^2} + 2hqu_0 u_2 + hqu_1^2. \end{aligned} $$

According to the recurrence formula (5.4.7), we get

$$\begin{aligned}
U_0(x,y) &= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right),\\
U_1(x,y) &= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\frac{ht^{\alpha}}{\Gamma(\alpha+1)},\\
U_2(x,y) &= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)},\\
U_3(x,y) &= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\frac{h^3t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\vdots\\
U_k(x,y) &= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\frac{h^kt^{k\alpha}}{\Gamma(k\alpha+1)}.
\end{aligned}$$

Thus, the $k^{th}-$order approximate series solution is given by

$$\begin{aligned}
\mathcal{P}_k(x,y) &= U_0(x,y)+U_1(x,y)+U_2(x,y)+...+U_k(x,y)\\
&= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)+\exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\frac{ht^{\alpha}}{\Gamma(\alpha+1)}+\exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\frac{(ht^{\alpha})^2}{\Gamma(2\alpha+1)}\\
&\quad+\exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\frac{(ht^{\alpha})^3}{\Gamma(3\alpha+1)}+...+\exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\frac{(ht^{\alpha})^k}{\Gamma(k\alpha+1)}\\
&= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\left(1+\frac{ht^{\alpha}}{\Gamma(\alpha+1)}+\frac{(ht^{\alpha})^2}{\Gamma(2\alpha+1)}+\frac{(ht^{\alpha})^3}{\Gamma(3\alpha+1)}+...+\frac{(ht^{\alpha})^k}{\Gamma(k\alpha+1)}\right)\\
&= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\sum_{r=0}^{k}\frac{(ht^{\alpha})^r}{\Gamma(r\alpha+1)}.
\end{aligned}$$

So, the $k^{th}-$order approximate series solution converges to the exact solution as $k\to\infty$

$$\begin{aligned}
u(x,y,t) &= \lim_{k\to\infty}\mathcal{P}_k(x,y)\\
&= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)\lim_{k\to\infty}\sum_{r=0}^{k}\frac{(ht^{\alpha})^r}{\Gamma(r\alpha+1)}\\
&= \exp\left(\sqrt{\frac{hq}{8}}(x+y)\right)E_{\alpha}(ht^{\alpha}), \qquad (5.6.6)
\end{aligned}$$

where $E_{\alpha}(ht^{\alpha})$ is the Mittag Leffler function defined by equation (1.3.5).

As $\alpha = 1$ in equation (5.6.6), we have

$$\begin{aligned} u(x,y,t) &= \exp\left(\frac{1}{2}\sqrt{\frac{hq}{2}}(x+y)\right)\sum_{r=0}^{\infty}\frac{(ht)^r}{r!} \\ &= \exp\left(\frac{1}{2}\sqrt{\frac{hq}{2}}(x+y)+ht\right), \end{aligned}$$

which is the same exact solution found in the literature [38].

Figures 3 and 4 illustrate the 2D and 3D graphs of KHRDRM in solving equations (5.6.4)-(5.6.5). Moreover, the comparison of the $6^{th}-$order approximate and exact solutions of KHRDRM and their associated absolute errors for equations (5.6.4)-(5.6.5) are presented in Table 2. The numerical simulation of the proposed method indicates that the numerical solutions of the method can approximate the exact solution well.

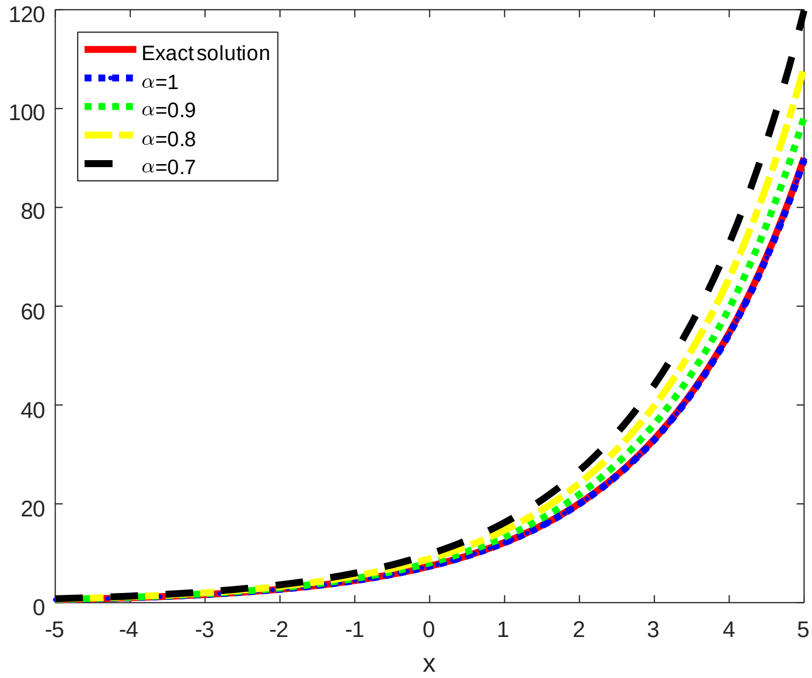

Figure 3: 2D graphs of the 6^{th}-order approximate series solution and exact solution with $y = 1$ and $t = 1.5$.

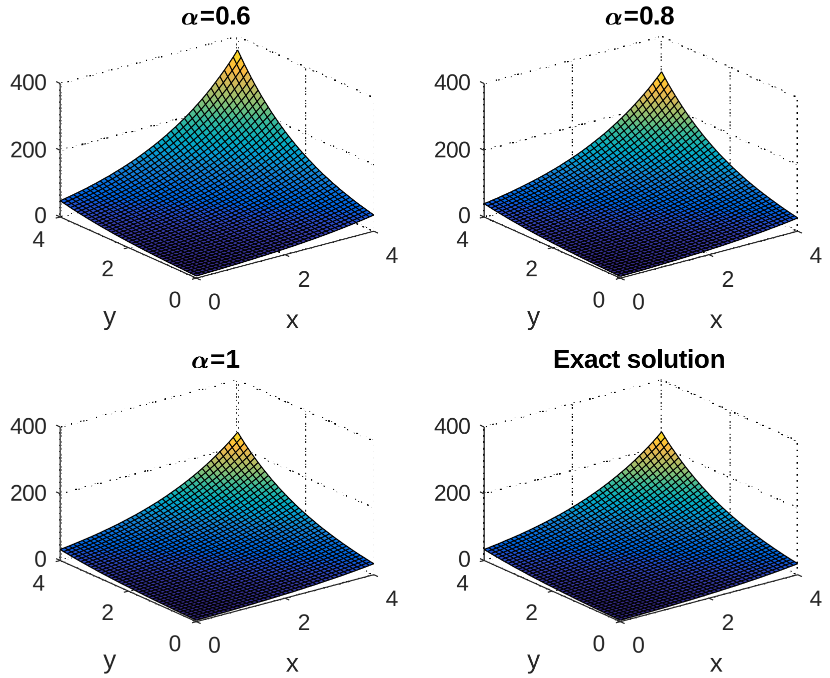

Figure 4: 3D graphs of the 6^{th}-order approximate series solution and exact solution with $h = 1$ and $t = 1.5$.

	u_{KHRDTM}	u_{KHRDTM}	u_{KHRDTM}	u_{KHRDTM}	u_{Exact}	$\lvert u_{Exact} - u_{KHRDTM}\rvert$
$(x, y)/\alpha$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	*Exact solution*	*Absolute errors*
0.1	1.3877	1.3142	1.2608	1.2214	1.2214	1.5572×10^{-9}
0.3	1.6949	1.6052	1.5400	1.4918	1.4918	1.9019×10^{-9}
0.5	2.0702	1.9606	1.8809	1.8221	1.8221	2.3230×10^{-9}
0.7	2.5285	2.3947	2.2974	2.2255	2.2255	2.8373×10^{-9}
0.9	3.0883	2.9249	2.8060	2.7183	2.7183	3.4655×10^{-9}

Table 2: Comparison of the 6^{th}-order approximate series solutions and exact solution and their associated absolute errors for different values of α.

Example 5.6.3 *Let us assume that the nonlinear time-fractional biological population model described in the sense of Caputo*

$$^{C}D_t^{\alpha}u = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^{-1}\left(1 - qu\right), 0 < \alpha \leq 1, \tag{5.6.7}$$

subject to the initial condition

$$u(x,y,0)=\sqrt{\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5}, \tag{5.6.8}$$

Following the KHRDTM methodology presented in Part 5.4, we get the series solution of our model

$$u(x,y,t)=\sum_{r=0}^{\infty}U_r(x,y),$$

where in this case, we set

$$Lu(x,y,t)=hq \text{ and } Nu(x,y,t)=\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}+hu^{-1}.$$

For the convenience of the reader, the first few RDT of the nonlinear term $Nu(x,y,t)$ are as follows

$$\begin{aligned}
R_0(x,y) &= \frac{\partial^2 u_0^2}{\partial x^2}+\frac{\partial^2 u_0^2}{\partial y^2}+hu_0^{-1},\\
R_1(x,y) &= 2\frac{\partial^2 u_0u_1}{\partial x^2}+2\frac{\partial^2 u_0u_1}{\partial y^2}-hu_0^{-2}u_1,\\
R_2(x,y) &= 2\frac{\partial^2 (u_0u_2)}{\partial x^2}+\frac{\partial^2 u_1^2}{\partial x^2}+2\frac{\partial^2 (u_0u_2)}{\partial y^2}+\frac{\partial^2 u_1^2}{\partial y^2}-hu_0^{-2}u_2+hu_0^{-3}u_1^2.
\end{aligned}$$

According to the recurrence formula (5.4.7), we get

$$\begin{aligned}
U_0(x,y) &= \sqrt{\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5},\\
U_1(x,y) &= h\left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{-1/2}\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
U_2(x,y) &= -2h^2\left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{-3/2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
U_3(x,y) &= 3h^3\left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{-5/2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\vdots\\
U_k(x,y) &= (-1)^{r-1}rh^r\left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{1/2-r}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.
\end{aligned}$$

Thus, the $k^{th}-$order approximate series solution is given by

$$\begin{aligned}
\mathcal{P}_k(x,y) &= U_0(x,y)+U_1(x,y)+U_2(x,y)+...+U_k(x,y)\\
&= \left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{1/2}+h\left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{-1/2}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\quad -2h^2\left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{-3/2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad +3h^3\left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{-5/2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&\quad +...+(-1)^k\,(r+1)\,h^{k+1}\left(\frac{hq}{4}x^2+\frac{hq}{4}y^2+y+5\right)^{1/2-(k+1)}\frac{t^{(k+1)\alpha}}{\Gamma((k+1)\,\alpha+1)}\\
&= U_0(x,y)+\frac{ht^{\alpha}}{U_0(x,y)}\sum_{r=0}^{k}\frac{r+1}{\Gamma((r+1)\,\alpha+1)}\left(\frac{-ht^{\alpha}}{U_0^2(x,y)}\right)^r.
\end{aligned}$$

So, the $k^{th}-$order approximate series solution converges to the exact solution as $k\to\infty$

$$\begin{aligned}
u(x,y,t) &= \lim_{k\to\infty}\mathcal{P}_k(x,y)\\
&= U_0(x,y)+\frac{ht^{\alpha}}{U_0(x,y)}\lim_{k\to\infty}\sum_{r=0}^{k}\frac{r+1}{\Gamma((r+1)\,\alpha+1)}\left(\frac{-ht^{\alpha}}{U_0^2(x,y)}\right)^r\\
&= U_0(x,y)+\frac{ht^{\alpha}}{U_0(x,y)}\sum_{r=0}^{\infty}\frac{r+1}{\Gamma((r+1)\,\alpha+1)}\left(\frac{-ht^{\alpha}}{U_0^2(x,y)}\right)^r, \qquad (5.6.9)
\end{aligned}$$

As $\alpha=1$ in equation (5.6.9), we have

$$\begin{aligned}
u(x,y,t) &= U_0(x,y)+\frac{ht}{U_0(x,y)}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{-ht}{U_0^2(x,y)}\right)^r\\
&= U_0(x,y)+\frac{ht}{U_0(x,y)}\exp\left(\frac{-ht}{U_0^2(x,y)}\right).
\end{aligned}$$

which is the same exact solution found in the literature [38].

Figures 5 and 6 illustrate the 2D and 3D graphs of KHRDRM in solving equations (5.6.7)-(5.6.8). Moreover, the comparison of the $6^{th}-$order approximate and exact solutions of KHRDRM and their associated absolute errors for equations (5.6.7)-(5.6.8) are presented in Table 2. The numerical simulation of the proposed method indicates that the numerical solutions of the method can approximate the exact solution well.

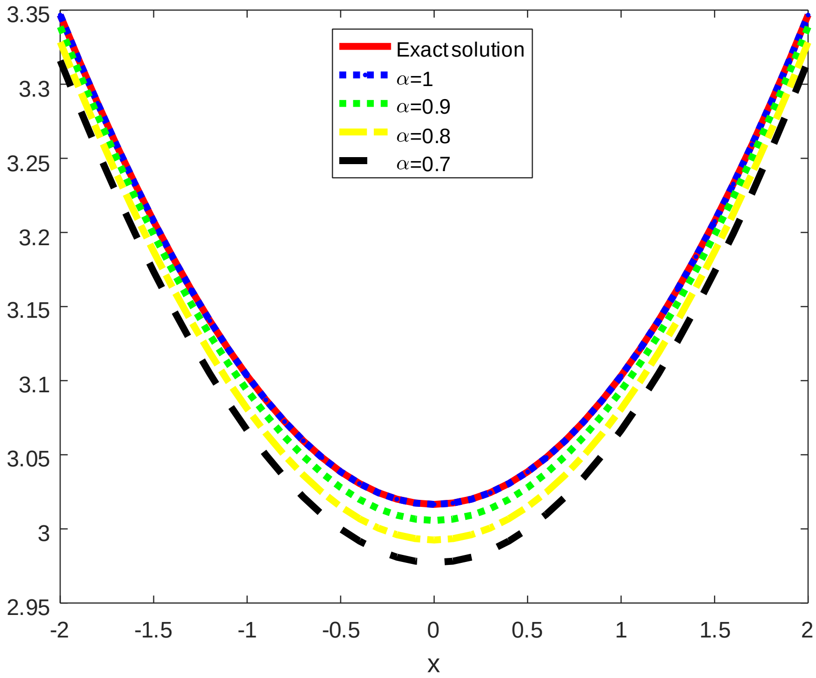

Figure 5: 2D graphs of the 6^{th}-order approximate series solution and exact solution with $h = y = 1$ and $t = 1.5$.

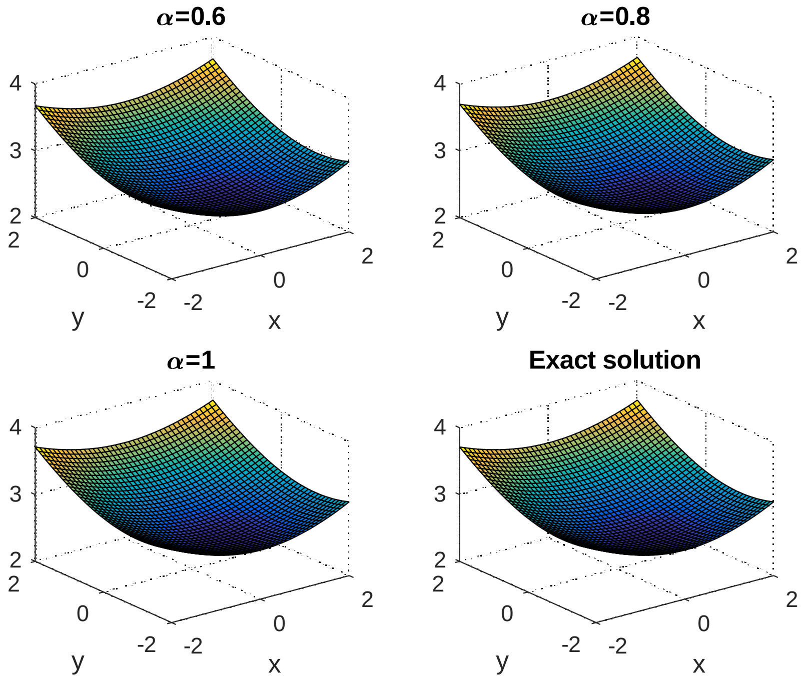

Figure 6: 3D graphs of the 6^{th}-order approximate series solution and exact solution with $h = 1$ and $t = 1.5$.

	u_{KHRDTM}	u_{KHRDTM}	u_{KHRDTM}	u_{KHRDTM}	u_{Exact}	$\lvert u_{Exact} - u_{KHRDTM}\rvert$
$(x,y)/\alpha$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	*Exact solution*	*Absolute errors*
0.1	2.3523	2.3328	2.3168	2.3039	2.3039	2.6928×10^{-10}
0.3	2.4113	2.3922	2.3765	2.3639	2.3639	2.1185×10^{-10}
0.5	2.4850	2.4664	2.4511	2.4389	2.4389	1.5841×10^{-10}
0.7	2.5755	2.5567	2.5413	2.5275	2.5275	1.1370×10^{-10}
0.9	2.6715	2.6541	2.6398	2.6284	2.6284	7.9112×10^{-11}

Table 3: Comparison of the 6^{th}-order approximate series solutions and exact solution and their associated absolute errors for different values of α.

Conclusions and future work

In this thesis, we suggested a novel hybrid technique called Khalouta reduced differential transform method (KHRDTM) to find exact solutions for nonlinear fractional partial differential equations (NFPDEs). This method is used directly without using any linearization, perturbation, polynomials or restrictive assumptions in comparison with other existing methods. Also, the method gives more realistic series solutions that converge very rapidly in nonlinear fractional problems. KHRDTM is applied to determine the exact solutions of some special cases of nonlinear fractional biological population equation. The exact solution is described in terms of the Mittag-Leffler functions. The proposed method has proven to be an accurate and appropriate analytical technique to solve NFPDEs. Thus, we conclude that KHRDTM can be considered as an effective method for solving linear and nonlinear fractional problems arising in mathematical sciences.

Fractional partial differential equations and fractional calculation are very important fields in modeling many natural phenomena. Thus, our future work can be summarized in:

1- Resolve the nonlinear fractional biological population equation which involves other fractional derivatives such as: Riemann-Liouville, conformable, Caputo-Fabrizio, Hadamard, and Atangana-Baleanu, using the proposed method.

2- Suggest new analytical methods to solve other nonlinear fractional partial differential equations arising in the application of nonlinear phenomena.

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ملخص:

تُستخدم المعادلات التفاضلية الجزئية الكسرية كتعميمات للمعادلات التفاضلية الجزئية الصحيحة الكلاسيكية لنمذجة العديد من المشكلات في العالم الحقيقي. الدافع الرئيسي لهذه الأطروحة هو اقتراح تقنية تحليلية جديدة لحل نموذج السكان البيولوجي الكسري غير الخطي. تستخدم التقنية المقترحة تحويل خلوطة والتحويل التفاضلي المختزل لبناء الحل التحليلي التقريبي للنموذج المقترح. يتم تقديم ثلاثة تطبيقات عددية لتوضيح كفاءة ودقة الطريقة المقترحة.

كلمات مفتاحية: المعادلات التفاضلية الجزئية الكسرية ، المشتقة الكسرية لكابوتو، تحويل خلوطة.

Résumé:

Les équations aux dérivées partielles fractionnaires comme généralisations des équations aux dérivées partielles entières classiques sont utilisées pour modéliser de nombreux problèmes du monde réel. La motivation principale de cette thèse est de proposer une nouvelle technique analytique pour résoudre le modèle de population biologique fractionnaire non linéaire en temps. La technique proposée utilise la transformée de Khalouta et la transformée différentielle réduite pour construire la solution analytique approximative du modèle proposé. Trois applications numériques sont présentées pour illustrer l'efficacité et la précision de la méthode proposée.

Mots clés: Équations aux dérivées partielles fractionnaires, Dérivée fractionnaire de Caputo, Transformation de Khalouta.

Abstract :

Fractional partial differential equations as generalizations of classical integer partial differential equations are used to model many real-world problems. The main motivation of this thesis is to propose a novel analytical technique for solving the nonlinear time-fractional biological population model The proposed technique utilizes the Khalouta transform and the reduced differential transform to construct the approximate analytical solution of the proposed model. Three numerical applications are presented to illustrate the efficiency and accuracy of the proposed method.

Key words: Fractional partial differential equations, Caputo fractional derivative, Khalouta transform.