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HANDOUT FOR EVALUATION

Titled

Mathematics 2: Summaries and Exercises

Author : Djabi Abdelmoumene

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Contents

Preface

Introduction

This handout is designed for first-year university students, focusing specifically on Linear Algebra and Matrices. These mathematical concepts are essential for economic analysis, enabling the modeling of complex systems and informed decision-making through data analysis.

The handout emphasizes exercises and solutions to reinforce understanding of key topics, including matrices, determinants, the inverse of a matrix, systems of linear equations, eigenvalues, eigenvectors, and diagonalization. Each chapter contains practice exercises that encourage students to apply their knowledge, with corresponding solutions provided for self-assessment.

Structure

This handout centers on Matrices and Linear Algebra, which are crucial instruments for economic analysis. These concepts provide the mathematical foundation necessary for solving systems of equations, grasping optimization, and modeling various economic scenarios. The content is organized into four chapters, each delivering an in-depth examination of essential topics.

1. Chapter 1: Introduction to Matrices
   This chapter establishes the basics of matrix algebra, covering operations, properties, and fundamental transformations.

2. Chapter 2: Determinants and Inverses of Matrices
   In this chapter, we examine the concepts of determinants and inverses of matrices. The determinant acts as a scaling factor for matrix transformations and offers insights into the matrix's properties. A solid grasp of determinants is vital for assessing matrix invertibility and singularity. Various techniques for computing determinants and determining matrix inverses are discussed.

3. Chapter 3: Solving Systems of Linear Equations
   This chapter focuses on techniques for solving systems of linear equations through the use of matrices. Mastering these methods is key to analyzing economic models and estimating parameters within economics.

4. Chapter 4: Eigenvalues and Eigenvectors
   The final chapter addresses eigenvalues and eigenvectors, which are crucial for economic analysis. A thorough understanding of these concepts is important for evaluating stability in economic models and tackling dynamic optimization problems.

By engaging with the content of this handout, readers will build a solid understanding of Matrices and Linear Algebra, equipping them to confidently address intricate economic challenges. Wishing you a productive learning experience!

Notations

Miscellaneous symbols

$=$ is equal to

$\neq$ is not equal to

$<$ is less than

$\leq$ is less than or equal to

$>$ is greater than

$\geq$ is greater than or equal to

∞ infinity

$\Rightarrow$ implies

$\Leftarrow$ is implied by

$\Leftrightarrow$ implies and is implied by (is equivalent to)

Operations

$a+b$ a plus b

$a-b$ a minus b

$a \times b, ab$ a multiplied by b (or a times b)

$\frac{a}{b}$ a divided by b

$\sqrt{a}$ the non-negative square root of a

Matrices

$A_{m\times n}$ Matrix with m rows and n columns.

$A=[a_{ij}$ $]Matrix$A with elements a_{ij}.

$A_{n\times n}$ Square Matrix

$A_{m\times 1}$ Column Matrix

$A_{1\times n}$ Row Matrix

$\mathrm{O}_{m\times n}$	Zero Matrix (all elements are zero)
$\mathrm{I}_{n\times n}$	Identity Matrix (diagonal elements are 1, others are 0)
A^{T}	The transpose of a matrix A
$tr(\mathrm{A})$	The trace of a square matrix A
$\det(\mathrm{A}), \lvert\mathrm{A}\rvert$	Determinant of a square matrix A
$\mathrm{adj}(\mathrm{A})$	The adjoint matrix of A
A^{-1}	The inverse of a square matrix A

Chapter 1

Matrices

Introduction

Matrices serve as essential mathematical instruments used in various fields. They play a crucial role in linear algebra and are extensively applied for solving systems of equations and conducting data analysis. In this chapter, we offer a concise overview of matrix algebra, which includes key operations, properties, and fundamental transformations, accompanied by exercises designed to reinforce understanding and practice the concepts.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

1.1 Introduction to Matrices

A matrix is a rectangular array of numbers arranged in rows and columns. Matrices are denoted by capital letters such as A, B, and C.

Definition

A matrix A with m rows and n columns is written as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} represents the element in the i-th row and j-th column.

1.1.1 Types of Matrices

- **Square matrix:** A matrix is called square if $m = n$.
- **Diagonal matrix:** A square matrix where all off-diagonal elements are zero, i.e., $a_{ij} = 0$ for $i \neq j$.
- **Identity matrix:** A diagonal matrix with all diagonal elements equal to 1. It is denoted by I.
- **Zero matrix:** A matrix where all elements are zero.
- **Upper Triangular Matrix:** A square matrix U is upper triangular if all its entries below the main diagonal are zero.
- **Lower Triangular Matrix:** A square matrix L is lower triangular if all its entries above the main diagonal are zero.
- **Symmetric Matrix:** A square matrix A is symmetric if $A^T = A$.
- **Skew-Symmetric Matrix:**A square matrix B is skew-symmetric if $B^T = -B$.

Property 1.1.

- *The product of any matrix* A *and the identity matrix* I *is the matrix* A *itself:* $AI = IA = A$.

Property 1.2.

- *For any matrix* A, $A+O=O+A=A$, *where* O *is the zero matrix.*
- *The product of any matrix* A *and the zero matrix* O *is the zero matrix:* $AO=OA=O$.
- *For any matrix* A, $A-A=O$, *where* O *is the zero matrix.*

Property 1.3.

- *The transpose of a symmetric matrix is equal to itself:* $A^T=A$.
- *The sum of two symmetric matrices is symmetric: If* A *and* B *are symmetric matrices, then* $A+B$ *is symmetric.*
- *The product of a symmetric matrix with a scalar is symmetric: If* A *is a symmetric matrix and* k *is a scalar, then* kA *is symmetric.*
- *The product of two symmetric matrices need not be symmetric in general.*
- *The transpose of a skew-symmetric matrix is equal to the negative of the matrix itself:* $A^T=-A$.
- *The sum of two skew-symmetric matrices is skew-symmetric: If* A *and* B *are skew-symmetric matrices, then* $A+B$ *is skew-symmetric.*
- *The product of a skew-symmetric matrix with a scalar is skew-symmetric: If* A *is a skew-symmetric matrix and* k *is a scalar, then* kA *is skew-symmetric.*
- *The product of two skew-symmetric matrices is symmetric: If* A *and* B *are skew-symmetric matrices, then* AB *is symmetric.*

1.1.2 Matrix Operations

1. **Addition:** If A and B are two matrices of the same size, their sum is

$$A+B=(a_{ij}+b_{ij})$$

.

2. **Scalar Multiplication:** If k is a scalar and A is a matrix, then

$$kA = (k \cdot a_{ij})$$

3. **Matrix Multiplication:** If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix given by:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

4. **Matrix Power:** Let A be a square matrix of size $n \times n$. The *power* of matrix A to the kth exponent, denoted as A^k, is defined as the result of multiplying matrix A by itself k times.

5. **Trace:**The trace of a $n \times n$ matrix A is the sum of the entries on A's diagonal.

6. **Transpose:** For an $m \times n$ matrix A, the transpose A^T is the $n \times m$ matrix obtained by interchanging rows and columns, so that each row i of A becomes column i in A^T.

Property 1.4.

- *The sum of two matrices of* ***different order*** *is not possible.*
- *The addition is Commutative in matrices* $A + B = B + A$.
- *The addition is associative in matrices* $A + (B + C) = (A + B) + C$.
- ***Associativity:*** $(AB)C = A(BC)$
- ***Distributivity over Addition:*** $A(B + C) = AB + AC$
- ***Scalar Multiplication:*** $(kA)B = k(AB) = A(kB)$
- ***Identity Matrix:*** $IA = AI = A$

Property 1.5.

- *The trace of a matrix* A *is denoted as* $tr(\mathrm{A})$.
- *The trace of a scalar* k *is equal to the scalar itself:* $tr(k) = k$.
- *The trace of the sum of two matrices is equal to the sum of their traces:* $tr(\mathrm{A}+\mathrm{B}) = tr(\mathrm{A}) + tr(\mathrm{B})$.

Property 1.6.

- *The transpose of a matrix* A*, denoted as* A^{T}*, has the same elements but with rows and columns interchanged.*
- *The transpose of the transpose of a matrix is the original matrix:* $(\mathrm{A}^{\mathrm{T}})^{\mathrm{T}} = \mathrm{A}$.
- *The transpose of the sum of two matrices is equal to the sum of their transposes:* $(\mathrm{A}+\mathrm{B})^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}} + \mathrm{B}^{\mathrm{T}}$.
- *The transpose of the product of two matrices is equal to the product of their transposes in reverse order:* $(\mathrm{AB})^{\mathrm{T}} = \mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}$.

1.2 Exercises

Exercise 1.1. Given the matrix

$$\mathrm{A} = \begin{pmatrix} 1 & 3 & 0 & -4 \\ 3 & 2 & 5 & 7 \\ -4 & 8 & 9 & -9 \end{pmatrix},$$

provide a_{23}, a_{31} and a_{34}

Exercise 1.2. Square matrix A is said to be **symmetric** iff $a_{ij} = a_{ji}$ for all $1 \le i, j \le n$. It is **antisymmetric** iff $a_{ij} = -a_{ji}$ for all $1 \le i, j \le n$. For each of the following matrices, determine if it is symmetric, antisymmetric both, or neither.

1. the identity matrix I_n
2. the $n \times n$ zero matrix
3. the matrix for which $a_{ij} = i + j$
4. the matrix for which $a_{ij} = i - j$

Exercise 1.3. Suppose A and B are 4×5 matrices, and C, D, and E are 5×2, 4×2, and 5×4 matrices, respectively. Which of the following matrix operations are well-defined? For those that are well defined, provide the dimension of the resulting matrix.

1. BA
2. AC + D
3. AE + B
4. AB + B
5. E(A + B)
6. E(AC)

Exercise 1.4. Given the matrices

$$\mathrm{A} = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 2 & 3 \\ 3 & 1 & -4 \end{pmatrix},$$

and

$$\mathrm{B} = \begin{pmatrix} 3 & -1 \\ 2 & 1 \\ -1 & 2 \end{pmatrix},$$

compute entries c_{11}, c_{21}, and c_{32} of matrix $\mathrm{C} = \mathrm{AB}$.

Exercise 1.5. Given the matrices

$$A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix},$$

Compute AB.

Exercise 1.6. Given the matrices

$$A = \begin{pmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{pmatrix},$$

Compute AB and BA.

Exercise 1.7. Given the matrices

$$A = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} B = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix} C = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix},$$

verify that $(AB)C = A(BC)$ and $A(B + C) = AB + AC$.

Exercise 1.8. If row i of matrix A is all zeros, then prove that row i of AB is also all zeros, where B is any matrix for which AB is well-defined.

Exercise 1.9. Given matrices

$$A=\begin{pmatrix}3&1\\5&2\end{pmatrix} \text{ and } B=\begin{pmatrix}2&-3\\4&4\end{pmatrix},$$

verify that

$$A^{-1}=\begin{pmatrix}2&-1\\-5&3\end{pmatrix} \text{ and } B^{-1}=\begin{pmatrix}1/5&3/20\\-1/5&1/10\end{pmatrix}.$$

Also, determine $(AB)^{-1}$.

Exercise 1.10. Let $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$, $B=\begin{pmatrix}4&-3\\-2&1\end{pmatrix}$, $C=\begin{pmatrix}2&2&9\\-1&0&8\end{pmatrix}$, and $D=\begin{pmatrix}1&2&3\\5&2&3\end{pmatrix}$. Compute each of the following AB, BA, CD, BC, CB, if possible. If a computation is not possible, explain why it is not.

Exercise 1.11. Let A, B and C be three matrices, and let $\alpha=3, \beta=\frac{1}{2}$, such that:

$$A=\begin{pmatrix}1&-2&3&5\\0&4&-3&2\end{pmatrix}, B=\begin{pmatrix}1&6&4&-2\\0&-4&5&3\end{pmatrix}, C=\begin{pmatrix}\frac{1}{2}&5\\3&4\\2&\frac{1}{4}\\-1&0\end{pmatrix}$$

1. Perform the following operations if it is possible if it is not explain why.
$$A+C, B^T+C, \alpha A-\beta B, A+B-C^T$$
2. Find the matrix D that satisfies $2A+\alpha B-\frac{1}{\beta}D=O$.

Exercise 1.12. Let A, B and C be three matrices, and let α, β be two real numbers, such that: : :

$$A = \begin{pmatrix} 1 & \alpha & -3 & 6 \\ 0 & 2 & \beta & 3 \end{pmatrix}, B = \begin{pmatrix} \alpha & 1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ 0 & \beta & 4 & -1 \\ 3 & 3 & 2 & \beta \end{pmatrix}, C = \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ 3 & -1 \\ -2 & 4 \end{pmatrix}$$

1. Perform the following operations if it is possible if it is not explain why.

$$AB, (AB)^T, CB, CA, tr(CA)A, C^TB, BA^T, A^2, B^2$$

1.3 Solutions

1. $a_{23} = 5$, $a_{23} = -4$, and $a_{34} = -9$.

2. For each of the following matrices,

   a. the identity matrix I_n is symmetric, but not antisymmetric since $a_{11} \neq -a_{11}$.

   b. the $n \times n$ zero matrix is both symmetric and antisymmetric.

   c. the matrix for which $a_{ij} = i + j$ is symmetric only.

   d. the matrix for which $a_{ij} = i - j$ is antisymmetric, and symmetric if $n = 1$.

3. Suppose A and B are 4×5 matrices, and C, D, and E are 5×2, 4×2, and 5×4 matrices, respectively. Which of the following matrix operations are well-defined?

   a. BA is not since the number of columns of B (5) does not equal the number of rows of A (4).

   b. AC + D is a 4×2 matrix.

   c. AE + B is not since AE is a 4×4, but B is a 4×5 matrix.

   d. AB + B is not since the number of columns of A (5) does not equal the number of rows of B (4).

   e. E(A + B) is a 5×5 matrix.

   f. E(AC) is a 5×2 matrix.

4. The entries are

$$c_{11} = a_{1,} \cdot b_{,1} = (1)(3) + (0)(2) + (-2)(-1) = 5,$$

$$c_{21} = a_{2,} \cdot b_{,1} = (-1)(3) + (2)(2) + (3)(-1) = -2,$$

and

$$c_{32} = a_{3,} \cdot b_{,2} = (3)(-1) + (1)(1) + (-4)(2) = -10.$$

5. The product of A and B is

$$\mathrm{AB} = \begin{pmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{pmatrix}.$$

6. The product of A and B is

$$\mathrm{AB}=\begin{pmatrix} 67 & 41 & 41 \\ 64 & 21 & 59 \\ 63 & 67 & 57 \end{pmatrix},$$

while

$$\mathrm{BA}=\begin{pmatrix} 6 & -6 & 70 \\ 6 & 17 & 31 \\ 63 & 41 & 122 \end{pmatrix}.$$

7. We have

$$\mathrm{AB}=\begin{pmatrix} -9 & -1 \\ 2 & 2 \end{pmatrix} \text{ and } (\mathrm{AB})\mathrm{C}=\begin{pmatrix} -8 & 7 \\ 0 & 2 \end{pmatrix}.$$

while

$$\mathrm{BC}=\begin{pmatrix} -2 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \mathrm{A}(\mathrm{BC})=\begin{pmatrix} -8 & 7 \\ 0 & 2 \end{pmatrix}.$$

Also,

$$\mathrm{B}+\mathrm{C}=\begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix} \text{ and } \mathrm{A}(\mathrm{B}+\mathrm{C})=\begin{pmatrix} -4 & -7 \\ 0 & 6 \end{pmatrix},$$

while

$$\mathrm{AC}=\begin{pmatrix} 5 & -6 \\ -2 & 4 \end{pmatrix} \text{ and } \mathrm{AB}+\mathrm{AC}=\begin{pmatrix} -4 & -7 \\ 0 & 6 \end{pmatrix}.$$

8. Entry (i, j) of AB is $a_{i,}\cdot b_{,j}$. But $a_{i,}$ is the zero vector, and the zero vector dotted with any other vector equals zero. Hence, entry (i, j) equals zero, and, since j was arbitrary, row i of AB must be all zeros.

9. Matrix

$$(\mathrm{AB})^{-1}=\mathrm{B}^{-1}\mathrm{A}^{-1}=\begin{pmatrix} -7/20 & 1/4 \\ -9/10 & 1/2 \end{pmatrix}.$$

Verify that this is the inverse of

$$\mathrm{AB}=\begin{pmatrix} 10 & -5 \\ 18 & -7 \end{pmatrix}.$$

10. AB.
This computation is possible, Since the size of both matrices is 2×2, *the*

number of columns of the first is the same as the number of rows of the second. Note that the size of the product is 2×2*. Here we go:*

$$\mathrm{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1^{st} \textit{ row of } \mathrm{A} \textit{ times } 1^{st} \textit{ column of} \mathrm{B} & 1^{st} \textit{ row of } \mathrm{A} \textit{ times } 2^{nd} \textit{ column of} \mathrm{B} \\ 2^{nd} \textit{ row of } \mathrm{A} \textit{ times } 1^{st} \textit{ column of} \mathrm{B} & 2^{nd} \textit{ row of } \mathrm{A} \textit{ times } 2^{nd} \textit{ column of} \mathrm{B} \end{pmatrix}$$

$$= \begin{pmatrix} 1(4)+2(-2) & 1(-3)+2(1) \\ 3(4)+4(-2) & 3(-3)+4(1) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 4 & -5 \end{pmatrix}.$$

BA.
Again, the size of both matrices is 2×2*, so this computation is possible and the size of the product is* 2×2*. Here we go again:*

$$\mathrm{BA} = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1^{st} \textit{ row of } \mathrm{B} \textit{ times } 1^{st} \textit{ column of} \mathrm{A} & 1^{st} \textit{ row of } \mathrm{B} \textit{ times } 2^{nd} \textit{ column of} \mathrm{A} \\ 2^{nd} \textit{ row of } \mathrm{B} \textit{ times } 1^{st} \textit{ column of} \mathrm{A} & 2^{nd} \textit{ row of } \mathrm{B} \textit{ times } 2^{nd} \textit{ column of} \mathrm{A} \end{pmatrix}$$

$$= \begin{pmatrix} 4(1)+(-3)(3) & 4(2)+(-3)(4) \\ -2(1)+1(3) & -2(2)+1(4) \end{pmatrix} = \begin{pmatrix} -5 & -4 \\ 1 & 0 \end{pmatrix}.$$

Did you notice what just happened? We have that $\mathrm{AB} \neq \mathrm{BA}$*! Yes, it's true: Matrix multiplication is not commutative.*

CD.
No can do. The size of the first matrix is 2×3 *and the size of the second is also* 2×3*. The number of columns of the first, 3, is not the same as the number of rows of the second, 2.*

BC.
This computation is possible. The size of the first matrix is 2×2 *and the size of the second is* 2×3*. The number of columns of the first, 2, is the same as the number of rows of the second, 2. Then the size of this product is* 2×3*. Here we go:*

$$\begin{aligned} \mathrm{BC} &= \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 & 9 \\ -1 & 0 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 4(2)+(-3)(-1) & 4(2)+(-3)(0) & 4(9)+(-3)(8) \\ -2(2)+1(-1) & -2(2)+1(0) & -2(9)+1(8) \end{pmatrix} \\ &= \begin{pmatrix} 11 & 8 & 12 \\ -5 & -4 & -10 \end{pmatrix}. \end{aligned}$$

CB.
No can do. The size of the first matrix is 2×3 *and the size of the second is* 2×2. *The number of columns of the first, 3, is not the same as the number of rows of the second, 2.*

Did you notice what just happened here? We were able to find the product BC, *but not the product* CB. *It's not that* $BC \neq CB$, *it's that* CB *isn't even possible. This is what makes matrix multiplication sooooo not commutative.*

11. $A+C$: *We can't calculate, because* A *and* C *are* ***not*** *with the same order.*

$$B^T + C = \begin{pmatrix} 1 & 0 \\ 6 & -4 \\ 4 & 5 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 5 \\ 3 & 4 \\ 2 & \frac{1}{4} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 5 \\ 9 & 0 \\ 6 & \frac{21}{4} \\ -3 & 3 \end{pmatrix}$$

$$\begin{aligned} \alpha A - \beta B &= 3\begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 4 & -3 & 2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 6 & 4 & -2 \\ 0 & -4 & 5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{2} & -9 & 7 & 16 \\ 0 & 14 & -\frac{23}{2} & \frac{9}{2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A + B - C^T &= \begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 4 & -3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 6 & 4 & -2 \\ 0 & -4 & 5 & 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 3 & 2 & -1 \\ 5 & 4 & \frac{1}{4} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} & 1 & 5 & 4 \\ -5 & -4 & \frac{7}{4} & 5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} D &= 2\beta A + \alpha\beta B \\ &= 2\times\frac{1}{2}\times\begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 4 & -3 & 2 \end{pmatrix} + 3\times\frac{1}{2}\times\begin{pmatrix} 1 & 6 & 4 & -2 \\ 0 & -4 & 5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{2} & 7 & 9 & 2 \\ 0 & -2 & \frac{9}{2} & \frac{13}{2} \end{pmatrix} \end{aligned}$$

12.

$$\begin{aligned} AB &= \begin{pmatrix} 1 & \alpha & -3 & 6 \\ 0 & 2 & \beta & 3 \end{pmatrix}\begin{pmatrix} \alpha & 1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ 0 & \beta & 4 & -1 \\ 3 & 3 & 2 & \beta \end{pmatrix} \\ &= \begin{pmatrix} 2\alpha+18 & 2\alpha-3\beta+19 & 3-\alpha & 6\beta+5 \\ 11 & \beta^2+13 & 4\beta+4 & 2\beta \end{pmatrix} \end{aligned}$$

$$(AB)^T = \begin{pmatrix} 2\alpha+18 & 11 \\ 2\alpha-3\beta+19 & \beta^2+13 \\ 3-\alpha & 4\beta+4 \\ 6\beta+5 & 2\beta \end{pmatrix}$$

CB : *We can't calculate because the number of columns of matrix* C *doesn't match the number of rows of matrix* B

$$\begin{aligned}
CA &= \begin{pmatrix} -1 & 2 \\ 0 & 3 \\ 3 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & \alpha & -3 & 6 \\ 0 & 2 & \beta & 3 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 4-\alpha & 2\beta+3 & 0 \\ 0 & 6 & 3\beta & 9 \\ 3 & 3\alpha-2 & -\beta-9 & 15 \\ -2 & 8-2\alpha & 4\beta+6 & 0 \end{pmatrix} \\
tr\,(CA)\,A &= \left(-\beta-4\right) \begin{pmatrix} 1 & \alpha & -3 & 6 \\ 0 & 2 & \beta & 3 \end{pmatrix} \\
&= \begin{pmatrix} -\beta-4 & -\alpha\left(\beta+4\right) & 3\beta+12 & -6\beta-24 \\ 0 & -2\beta-8 & -\beta\left(\beta+4\right) & -3\beta-12 \end{pmatrix} \\
C^T B &= \begin{pmatrix} -1 & 0 & 3 & -2 \\ 2 & 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} \alpha & 1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ 0 & \beta & 4 & -1 \\ 3 & 3 & 2 & \beta \end{pmatrix} \\
&= \begin{pmatrix} -\alpha-6 & 3\beta-7 & 5 & -2\beta-5 \\ 2\alpha+15 & 20-\beta & 7 & 4\beta+5 \end{pmatrix} \\
BA^T &= \begin{pmatrix} \alpha & 1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ 0 & \beta & 4 & -1 \\ 3 & 3 & 2 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 2 \\ -3 & \beta \\ 6 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 2\alpha+3 & 3\beta+8 \\ 2\alpha+4 & 4-\beta \\ \alpha\beta-18 & 6\beta-3 \\ 3\alpha+6\beta-3 & 5\beta+6 \end{pmatrix}
\end{aligned}$$

A^2 : *We can't calculate, because* A *is not a square matrix.*

$$\begin{aligned}
B^2 &= \begin{pmatrix} \alpha & 1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ 0 & \beta & 4 & -1 \\ 3 & 3 & 2 & \beta \end{pmatrix} \begin{pmatrix} \alpha & 1 & 3 & 2 \\ 1 & 2 & -1 & 0 \\ 0 & \beta & 4 & -1 \\ 3 & 3 & 2 & \beta \end{pmatrix} \\
&= \begin{pmatrix} \alpha^2+7 & \alpha+3\beta+8 & 3\alpha+15 & 2\alpha+2\beta-3 \\ \alpha+2 & 5-\beta & -3 & 3 \\ \beta-3 & 6\beta-3 & 14-\beta & -\beta-4 \\ 3\alpha+3\beta+3 & 5\beta+9 & 2\beta+14 & \beta^2+4 \end{pmatrix}
\end{aligned}$$

Chapter 2

Determinant and Inverse of Matrix

Introduction

This chapter provides a brief overview of the determinant and inverse. Along with the summary, exercises are provided to help students practice different methods for calculating determinants and finding inverses, reinforcing their understanding of these important topics.

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$A^{-1} = \tfrac{1}{\det(A)} adj(A)$$

2.1 Determinant Calculation Methods

1.1 Sarrus' Rule (for 3×3 Matrices)

Sarrus' rule is a shortcut for finding the determinant of a 3×3 matrix. For a matrix A:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The determinant is given by:

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

2.1.1 Cofactor Expansion

For an $n \times n$ matrix A, the determinant can be computed using cofactor expansion along any row or column. The cofactor expansion along the first row is:

$$\det(\mathrm{A}) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(\mathrm{A}_{1j})$$

where A_{1j} is the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and the j-th column.

2.1.2 Gaussian Elimination (Row Reduction)

To compute the determinant using Gaussian elimination:

- Transform the matrix into an upper triangular form using row operations.
- The determinant is the product of the diagonal elements of the upper triangular matrix.
- Each row swap changes the sign of the determinant.

Example: Calculate the determinant using Sarrus' Rule

Given the matrix:

$$\mathrm{A} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix}$$

The determinant using Sarrus' rule is:

$$\det(\mathrm{A}) = 2\cdot0\cdot3+3\cdot4\cdot5+1\cdot1\cdot2-(1\cdot0\cdot5+2\cdot4\cdot3+3\cdot1\cdot2) = 0+60+2-(0+24+6) = 62-30 = 32$$

2.2 Inverse Calculation Methods

2.2.1 Gauss-Jordan Elimination

To find the inverse of a matrix A using the Gauss-Jordan method:

- Write the augmented matrix $[\mathrm{A}|\mathrm{I}]$, where I is the identity matrix.
- Use row operations to transform A into the identity matrix. Simultaneously apply the same operations to I.
- Once A is reduced to I, the matrix on the right will be A^{-1}.

2.2.2 Cofactor Method (Adjugate Method)

To find the inverse of an $n \times n$ matrix A:

- Compute the cofactor matrix of A.
- Transpose the cofactor matrix to get the adjugate matrix, $\mathrm{adj}(A)$.
- The inverse of A is given by:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

provided that $\det(A) \neq 0$.

2.2.3 Deduction Method

Property 2.1. *For two matrices* A *and* B *satisfying* $\det(A) \neq 0$ *and* $\det(B) \neq 0$*, if*

$$AB = BA = I,$$

then $B = A^{-1}$ *and* $A^{-1} = B$.

For a 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

provided that $\det(A) = ad - bc \neq 0$.

Property 2.2. *Let* A, B, *and* C *be square matrices of the same size* $n \times n$, *and let* k *be a scalar.*

1. ***Multiplicative Property:*** $\det(AB) = \det(A)\det(B)$.
2. ***Scalar Multiplication:*** $\det(kA) = k^n \det(A)$.
3. ***Transpose:*** $\det(A^T) = \det(A)$.
4. ***Inverse:*** $\det(A^{-1}) = \dfrac{1}{\det(A)}$.
5. ***Product of Diagonal Matrices:*** *For diagonal matrices* D_1 *and* D_2,
$$\det(D_1 D_2) = \det(D_1)\det(D_2)$$
.
6. ***Determinant of Upper Triangular Matrix:*** *The determinant of an upper triangular matrix is the product of its diagonal entries.*
7. ***Determinant of Lower Triangular Matrix:*** *The determinant of a lower triangular matrix is the product of its diagonal entries.*
8. ***Row Operations:*** *Performing elementary row operations on* A *does not change its determinant. Specifically:*
   - *If two rows of* A *are interchanged, the determinant changes sign.*
   - *If one row of* A *is multiplied by a scalar* k, *the determinant is multiplied by* k.
9. ***Row or Column of Zeros:*** *If* A *has a row or column of zeros, then* $\det(A) = 0$.
10. ***Two Identical Rows or Columns:*** *If* A *has two identical rows or columns, then* $\det(A) = 0$.
11. ***Sum of Determinants:*** *For matrices* A *and* B, $\det(A+B) \neq \det(A) + \det(B)$ *in general.*

Example: Calculate the inverse using Gauss-Jordan Elimination

Given the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

Step 1: Augment the matrix with the identity matrix:

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right)$$

Step 2: Use row operations to transform A into the identity matrix. Start by making the leading entry of the first row a 1:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 5 & 3 & 0 & 1 \end{array}\right)$$

Step 3: Subtract 5 times the first row from the second row:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right)$$

Step 4: Multiply the second row by 2:

$$\left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -5 & 2 \end{array}\right)$$

Step 5: Subtract $\frac{1}{2}$ times the second row from the first row:

$$\left(\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right)$$

Thus, the inverse is:

$$A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

2.3 Exercises

Exercise 2.1. Calculate the determinant of the matrix:

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 3 & 5 & 7 \\ 8 & 6 & 9 \end{pmatrix}$$

using Sarrus' Rule.

Exercise 2.2. Find the determinant of the matrix:

$$B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

Exercise 2.3. Determine the determinant of the following upper triangular matrix:

$$C = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 5 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

Exercise 2.4. Use cofactor expansion along the first row to compute the determinant of:

$$D = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}$$

Exercise 2.5. Using row reduction, calculate the determinant of the matrix:

$$E = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 5 \\ 7 & 0 & 6 \end{pmatrix}$$

Exercise 2.6. Compute the inverse of the matrix:

$$F = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

using the cofactor method.

Exercise 2.7. Find the inverse of the matrix using Gauss-Jordan elimination:

$$G = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Exercise 2.8. Find the inverse of the following matrix if it exists:

$$H = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$$

Exercise 2.9. Show that the matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$$

is invertible and find its inverse.

Exercise 2.10. Find the determinant of the matrix:

$$J = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{pmatrix}$$

Exercise 2.11. Using Sarrus' rule, calculate the determinant of:

$$K = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & 5 \end{pmatrix}$$

Exercise 2.12. Determine whether the following matrix is invertible by calculating its determinant:

$$L = \begin{pmatrix} 6 & 2 \\ 3 & 5 \end{pmatrix}$$

Exercise 2.13. Find the determinant of the matrix:

$$M = \begin{pmatrix} 4 & 3 \\ 7 & 2 \end{pmatrix}$$

using cofactor expansion.

Exercise 2.14. Calculate the determinant using Gaussian elimination of the matrix:

$$N = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 5 & 2 \end{pmatrix}$$

Exercise 2.15. For the matrix:

$$O = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

calculate the determinant using the cofactor method.

Exercise 2.16. Verify the multiplicative property of determinants for the matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

by calculating $\det(A)$, $\det(B)$, and $\det(AB)$.

Exercise 2.17. Verify that the matrix:

$$W = \begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{pmatrix}$$

is singular.

Exercise 2.18. Let A, B and C be three matrices, and let α, β, γ be real numbers, such that: : :

$$A = \begin{pmatrix} 2 & \alpha & 0 \\ 4 & 3 & -2 \\ 0 & 5 & \alpha \end{pmatrix}, B = \begin{pmatrix} -3\beta & 4 & 7 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 2 & 8 & -1 \\ 3 & 6 & 2 & \beta \end{pmatrix}, C = \begin{pmatrix} 2\gamma & -\gamma & -2 & 0 & 0 \\ -2 & 6 & 7 & 3 & 1 \\ 4 & 4 & 5 & 7 & 6 \\ 1 & 0 & -3 & -2 & 5 \\ 0 & 3 & 1 & 2 & 3 \end{pmatrix}$$

1. Perform the following operations

$$\det(A), A^2, \det\left(A^2\right), \det(B), \det(C),$$

Exercise 2.19. Let A and B two matrices, α and β are two real numbers, such that:

$$A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}, B = \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ \beta & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$$

1. Calculate $\det(A)$, $\det(B)$, $A \times B$ and $B \times A$.
2. Find the values of α and β for which the matrices A and B are invertible.
3. For $\alpha = 0$ and $\beta = 1$, deduce A^{-1} and B^{-1}.

Exercise 2.20. Let A be the following matrix :

$$A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}$$

1. Find A^2 matrix then prove the existence of two real numbers α and β such that: $A^2 = \alpha A + \beta I_3$.
2. Calculate $\det(A)$ and $\det(\frac{1}{2}(A + I_3))$
3. Deduce that the matrix A is invertible, then write A^{-1} in terms of A and I_3.

2.4 Solutions

1. For Exercise 1, using Sarrus' rule:

$$\det(A) = 4\cdot5\cdot9 + 2\cdot7\cdot8 + 1\cdot3\cdot6 - (1\cdot5\cdot8 + 4\cdot7\cdot6 + 2\cdot3\cdot9)$$

$$= 180 + 112 + 18 - (40 + 168 + 54) = 310 - 262 = 48$$

2. For Exercise 2:

$$\det(B) = 2\cdot5 - 3\cdot4 = 10 - 12 = -2$$

3. For Exercise 3: The determinant of an upper triangular matrix is the product of its diagonal entries. Therefore:

$$\det(C) = 1\cdot5\cdot3 = 15$$

4. For Exercise 4, using cofactor expansion along the first row:

$$\det(D) = 1\cdot\det\begin{pmatrix}4 & 5\\ 0 & 6\end{pmatrix} - 2\cdot\det\begin{pmatrix}0 & 5\\ 1 & 6\end{pmatrix} + 3\cdot\det\begin{pmatrix}0 & 4\\ 1 & 0\end{pmatrix}$$

$$= 1\cdot(4\cdot6) - 2\cdot(0\cdot6 - 1\cdot5) + 3\cdot(0\cdot0 - 1\cdot4) = 24 - 2(-5) + 3(-4) = 24 + 10 - 12 = 22$$

5. For Exercise 5, using row reduction:

$$E = \begin{pmatrix}3 & 1 & 2\\ 0 & 4 & 5\\ 7 & 0 & 6\end{pmatrix}$$

Step 1: Apply row operations to form an upper triangular matrix. Subtract $\frac{7}{3}$ times the first row from the third row:

$$\begin{pmatrix}3 & 1 & 2\\ 0 & 4 & 5\\ 0 & -\frac{7}{3} & \frac{4}{3}\end{pmatrix}$$

Step 2: Add $\frac{7}{12}$ times the second row from the third row:

$$\begin{pmatrix}3 & 1 & 2\\ 0 & 4 & 5\\ 0 & 0 & \frac{17}{4}\end{pmatrix}$$

Step 3: Multiply the diagonal elements and adjust for row operations:

$$\det(E) = 3\cdot4\cdot\frac{17}{4} = 51$$

6. For Exercise 6, using the cofactor method, the matrix is:

$$F = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

The determinant is:

$$\det(F) = 2 \cdot 4 - 3 \cdot 1 = 8 - 3 = 5$$

The cofactor matrix is:

$$\text{Cof}(F) = \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$$

The inverse is:

$$F^{-1} = \frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

7. For Exercise 7, using Gauss-Jordan elimination, the matrix is:

$$G = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Step 1: Augment the matrix with the identity matrix:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

Step 2: Subtract 3 times the first row from the second row:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

Step 3: Multiply the second row by $-\frac{1}{2}$:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

Step 4: Subtract 2 times the second row from the first row:

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

Thus, the inverse is:

$$G^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

8. For Exercise 8, the matrix is:

$$H = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$$

The determinant is:

$$\det(H) = 5 \cdot 3 - 2 \cdot 7 = 15 - 14 = 1$$

Since the determinant is not zero, the inverse exists. Using the cofactor method, we have:

$$H^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$$

9. For Exercise 9, the matrix is:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix}$$

We will use row reduction to find the inverse. Augment with the identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right)$$

Step 1: Subtract 2 times the first row from the second row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right)$$

Step 2: Subtract 3 times the first row from the third row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & 1 & -3 & 0 & 1 \end{array}\right)$$

Step 3: Subtract 4 times the second row from the third row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right)$$

Thus, the inverse is:

$$I^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}$$

10. For Exercise 10, the matrix is:

$$J = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{pmatrix}$$

Since it is an upper triangular matrix, the determinant is the product of the diagonal entries:

$$\det(J) = 4 \cdot 5 \cdot 6 = 120$$

11. For Exercise 11, the matrix is:

$$K = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & 5 \end{pmatrix}$$

We will use Sarrus' Rule for this 3×3 matrix:

$$\det(K) = 2(0 \cdot 5 - 1 \cdot 2) - 1(1 \cdot 5 - 2 \cdot 4) + 3(1 \cdot 1 - 0 \cdot 4)$$

$$\det(K) = 2(0-2) - 1(5-8) + 3(1-0)$$

$$\det(K) = 2(-2) - 1(-3) + 3(1) = -4 + 3 + 3 = 2$$

12. For Exercise 12, the matrix is:

$$L = \begin{pmatrix} 6 & 2 \\ 3 & 5 \end{pmatrix}$$

The determinant of this 2×2 matrix is:

$$\det(L) = 6 \cdot 5 - 2 \cdot 3 = 30 - 6 = 24$$

Since $\det(L) \neq 0$, we can find the inverse:

$$L^{-1} = \frac{1}{24} \begin{pmatrix} 5 & -2 \\ -3 & 6 \end{pmatrix}$$

$$L^{-1} = \begin{pmatrix} \frac{5}{24} & -\frac{1}{12} \\ -\frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

13. For Exercise 13, using the cofactor method, the matrix is:

$$M = \begin{pmatrix} 4 & 3 \\ 7 & 2 \end{pmatrix}$$

The determinant is:

$$\det(M) = 4 \cdot 2 - 3 \cdot 7 = 8 - 21 = -13$$

The inverse of the matrix is given by:

$$M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} 2 & -3 \\ -7 & 4 \end{pmatrix}$$

$$M^{-1} = -\frac{1}{13} \begin{pmatrix} 2 & -3 \\ -7 & 4 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} -\frac{2}{13} & \frac{3}{13} \\ \frac{7}{13} & -\frac{4}{13} \end{pmatrix}$$

14. For Exercise 14, using row reduction, the matrix is:

$$N = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 5 & 2 \end{pmatrix}$$

Step 1: Subtract 3 times the first row from the second row:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 5 & 2 \end{pmatrix}$$

Step 2: Add $\frac{5}{2}$ times the second row to the third row:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{9}{2} \end{pmatrix}$$

The determinant is the product of the diagonal entries:

$$\det(N) = 1 \cdot (-2) \cdot \frac{9}{2} = -9$$

15. For Exercise 15, using the cofactor method, the matrix is:

$$O = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

We will expand along the first row:

$$\det(O) = 1 \cdot \det\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + 2 \cdot \det\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$$

$$\det(O) = 1 \cdot (1 - 0) + 2 \cdot (12 - 0) = 1 + 24 = 25$$

16. For Exercise 16, the matrices are:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

The determinants are:

$$\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2, \det(B) = 5 \cdot 8 - 6 \cdot 7 = -2$$

The product AB is:

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Then

$$\det(AB) = 19 \cdot 50 - 22 \cdot 43 = 4 = det(A) \cdot det(B)$$

17. For Exercise 17, using row reduction, the matrix is:

$$W = \begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{pmatrix}$$

We will attempt to row reduce this matrix. Subtract 2 times the first row from the second row:

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 9 & 8 & 7 \end{pmatrix}$$

Subtract 3 times the first row from the third row:

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

Subtract 2 times the second row from the third row:

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the third row is all zeros, the determinant is zero:

$$\det(W) = 0$$

Thus, the matrix is singular and has no inverse.

18.

$$\begin{vmatrix} 2 & \alpha & 0 \\ 4 & 3 & -2 \\ 0 & 5 & \alpha \end{vmatrix} = -4\alpha^2 + 6\alpha + 20$$

$$A^2 = \begin{pmatrix} 9\beta^2+4 & 22-12\beta & 52-21\beta & -7 \\ 2-3\beta & 6 & -3 & 1 \\ -1 & 14 & 60 & -\beta-8 \\ 6-6\beta & 6\beta+28 & 2\beta+31 & \beta^2-2 \end{pmatrix}$$

$$\det\left(A^2\right) = 2916\beta^4 + 5184\beta^3 + 4464\beta^2 + 1920\beta + 400$$

$$\begin{vmatrix} -3\beta & 4 & 7 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 2 & 8 & -1 \\ 3 & 6 & 2 & \beta \end{vmatrix} = -54\beta^2 - 48\beta - 20$$

$$\begin{vmatrix} 2\gamma & -\gamma & -2 & 0 & 0 \\ -2 & 6 & 7 & 3 & 1 \\ 4 & 4 & 5 & 7 & 6 \\ 1 & 0 & -3 & -2 & 5 \\ 0 & 3 & 1 & 2 & 3 \end{vmatrix} = -399\gamma - 170$$

19.

$$\det(A) = 4 - 4\alpha, \det(B) = \frac{1}{4}\beta$$

$$A \times B = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ \beta & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \beta-\alpha & -\frac{3}{4}\alpha & \frac{1}{2}\alpha \\ 2-2\beta & 1 & 0 \\ 1-\beta & 0 & 1 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ \beta & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \times \begin{pmatrix} \alpha & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1-\alpha & 0 & 0 \\ \alpha\beta & \beta & 0 \\ \alpha & 0 & 1 \end{pmatrix}$$

A is an invertible matrix for $\det(A) \neq 0 \Longleftrightarrow 4 - 4\alpha \neq 0$ this implies: $\alpha \neq 1$

$$\alpha \in \mathbb{R} - \{1\}$$

B is an invertible matrix for $\det(B) \neq 0 \Longleftrightarrow \frac{1}{4}\beta \neq 0$ this implies: $\beta \neq 0$

$$\beta \in \mathbb{R} - \{0\}$$

For $\alpha = 0$ and $\beta = 1$,we have:

$$A \times B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then:

$$A^{-1} = B = \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \text{ and } B^{-1} = A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

20.

$$A^2 = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix}$$

$$\begin{aligned} \alpha A + \beta I_3 &= \alpha \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha+\beta & -3\alpha & 6\alpha \\ 6\alpha & \beta - 8\alpha & 12\alpha \\ 3\alpha & -3\alpha & 4\alpha+\beta \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix} = \begin{pmatrix} \alpha+\beta & -3\alpha & 6\alpha \\ 6\alpha & \beta - 8\alpha & 12\alpha \\ 3\alpha & -3\alpha & 4\alpha+\beta \end{pmatrix} \Longleftrightarrow \alpha = -1; \beta = 2$$

$$\det(A) = \begin{vmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{vmatrix} = 4$$

$$\frac{1}{2}(A + I_3) = \frac{1}{2}\left(\begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$$

$$\det\left(\frac{1}{2}(A + I_3)\right) = \begin{vmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{vmatrix} = \frac{1}{4}$$

$$A^2 + A = 2I_3 \Longleftrightarrow A\underbrace{\left(\frac{1}{2}(A + I_3)\right)}_{A^{-1}} = I_3$$

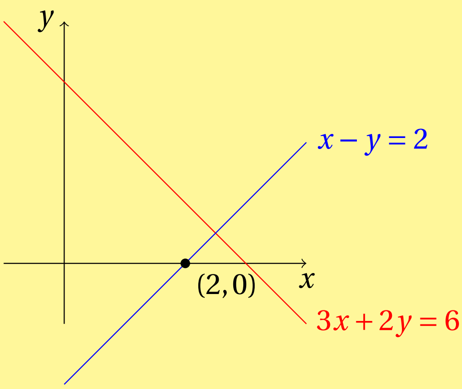
$$A^{-1} = \frac{1}{2}(A + I_3)$$

Chapter 3

Systems of Linear Equations

Introduction

Systems of linear equations are fundamental in mathematics and have wide-ranging applications. In this chapter, we provide a brief overview of how to solve these systems using matrices, building on the concepts of determinants and matrix inverses from the previous chapter. The exercises included will help students apply these methods to find solutions for systems of equations, reinforcing their understanding through practice.



3.1 Introduction to Systems of Linear Equations

A system of linear equations consists of multiple linear equations that share common variables. A general system with n equations and n unknowns can be written as:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

3.1.1 Homogeneous and Non-Homogeneous Systems

- **Homogeneous systemt** : All constants $b_1, b_2, \dots, b_n = 0$. The system has the form:

$$\mathrm{A}\mathbf{x} = \mathbf{0}$$

- **Non-Homogeneous system**: At least one of the constants $b_1, b_2, \dots, b_n \neq 0$. The system has the form:

$$\mathrm{A}\mathbf{x} = \mathbf{b}$$

3.1.2 Cramer's and Non-Cramer's Systems

- **Cramer's System**: A system where the number of equations equals the number of unknowns (n equations and n unknowns), and the determinant of the coefficient matrix A is non-zero ($\det(\mathrm{A}) \neq 0$).
- **Non-Cramer's System**: A system where either the number of equations does not match the number of unknowns, or $\det(\mathrm{A}) = 0$.
Non-Cramer's systems may have no solution, a unique solution, or infinitely many solutions depending on the rank of the matrix.

3.2 Solving Methods

3.2.1 Cramer's Rule

Cramer's rule is used to solve a system of linear equations when the determinant of the coefficient matrix A is non-zero. For a system $\mathrm{A}\mathbf{x} = \mathbf{b}$, the solution is given by:

$$x_i = \frac{\det(\mathrm{A}_i)}{\det(\mathrm{A})}$$

where A_i is the matrix obtained by replacing the i-th column of A with the vector **b**.

Example: Solve the System using Cramer's Rule

Consider the system:

$$\begin{cases} 2x + 3y = 5 \\ 4x - y = 1 \end{cases}$$

The coefficient matrix is:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Compute the determinant of A:

$$\det(A) = 2 \cdot (-1) - 3 \cdot 4 = -2 - 12 = -14$$

Now compute the determinants for x and y:

$$A_x = \begin{pmatrix} 5 & 3 \\ 1 & -1 \end{pmatrix}, \quad A_y = \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}$$

$$\det(A_x) = 5 \cdot (-1) - 3 \cdot 1 = -5 - 3 = -8, \quad \det(A_y) = 2 \cdot 1 - 5 \cdot 4 = 2 - 20 = -18$$

Thus, the solution is:

$$x = \frac{-8}{-14} = \frac{4}{7}, \quad y = \frac{-18}{-14} = \frac{9}{7}$$

3.2.2 Rank Method (Rouché-Capelli Theorem)

The rank method is used for both Cramer's and non-Cramer's systems. The rank of a matrix is the number of independent rows or columns. The Rouché-Capelli theorem states that:

- If the rank of the augmented matrix $[A|\mathbf{b}]$ equals the rank of the coefficient matrix A, the system is consistent (has at least one solution).
- If $\text{rank}(A) < \text{rank}([A|\mathbf{b}])$, the system is inconsistent (no solution).
- If $\text{rank}(A) = \text{rank}([A|\mathbf{b}]) = n$, the system has a unique solution.
- If $\text{rank}(A) = \text{rank}([A|\mathbf{b}]) < n$, the system has infinitely many solutions.

Example: Solving with the Rank Method

Consider the system:

$$\begin{cases} x + 2y - z = 1 \\ 2x + y + z = 3 \\ 3x + 3y = 4 \end{cases}$$

Form the augmented matrix:

$$[\mathrm{A}|\mathbf{b}] = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 3 \\ 3 & 3 & 0 & 4 \end{pmatrix}$$

Use row reduction to find the rank of A and $[\mathrm{A}|\mathbf{b}]$.

3.2.3 Gaussian Elimination

Gaussian elimination transforms the system into an upper triangular form, making it easier to solve by back substitution. Steps:
1. Write the augmented matrix $[\mathrm{A}|\mathbf{b}]$.
2. Perform row operations to get the matrix into upper triangular form.
3. Solve the system using back substitution.

Example: Solving with Gaussian Elimination

Consider the system:

$$\begin{cases} x + y + z = 6 \\ 2x + 3y + 5z = 16 \\ 4x + y + 2z = 11 \end{cases}$$

Form the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 5 & 16 \\ 4 & 1 & 2 & 11 \end{array}\right)$$

Use Gaussian elimination to transform the matrix into upper triangular form:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 7 & -1 \end{array}\right)$$

Solve by back substitution:

$$z = -\frac{1}{7}, \quad y = \frac{31}{7}, \quad x = \frac{12}{7}$$

3.3 Exercises

Exercise 3.1. Solve the following system using Cramer's Rule:

$$\begin{cases} 3x + 2y = 5 \\ 4x - y = 2 \end{cases}$$

Exercise 3.2. Solve the system using Gaussian elimination:

$$\begin{cases} x + y + z = 6 \\ 2x + 3y + 5z = 16 \\ 4x + y + 2z = 11 \end{cases}$$

Exercise 3.3. Determine whether the following system is homogeneous or non-homogeneous:

$$\begin{cases} 2x + y = 0 \\ x - y = 0 \end{cases}$$

Exercise 3.4. Use the Rank Method (Rouché-Capelli Theorem) to determine if the following system is consistent and if it has a unique solution, no solution, or infinitely many solutions:

$$\begin{cases} x + 2y - z = 1 \\ 2x + y + z = 3 \\ 3x + 3y = 4 \end{cases}$$

Exercise 3.5. Find the solutions of the system using the inverse method:

$$\begin{cases} 2x + 3y = 1 \\ 4x + 5y = 2 \end{cases}$$

Exercise 3.6. Solve the following system using the Gauss method:

$$\begin{cases} x+y=2 \\ x-y=0 \end{cases}$$

Exercise 3.7. Solve the following system using Gaussian elimination:

$$\begin{cases} 2x+3y+z=7 \\ 4x-y+3z=9 \\ 6x+5y-z=13 \end{cases}$$

Exercise 3.8. Check whether the following matrix is invertible:

$$B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Then conclude if the following sytem is Cramer's system or Non-Cramer's system:

$$\begin{cases} x+2y+3z=7 \\ 4x+5y+6z=5 \\ 7x+8y+9z=6 \end{cases}$$

Exercise 3.9. Use Cramer's Rule to solve:

$$\begin{cases} x+3y=7 \\ 2x+y=5 \end{cases}$$

Exercise 3.10. Solve the following system using Gaussian elimination:

$$\begin{cases} x+2y+3z=9 \\ 2x-y+z=8 \\ 3x+y-2z=3 \end{cases}$$

Exercise 3.11. Solve the system using the Rank Method (Rouché-Capelli Theorem):

$$\begin{cases} x+y=2 \\ x-y=0 \\ 2x+2y=4 \end{cases}$$

Exercise 3.12. Solve the system using Cramer's Rule:

$$\begin{cases} x+y=3 \\ 2x+3y=7 \end{cases}$$

Exercise 3.13. Solve the system using the inverse matrix method:

$$\begin{cases} x+2y=5 \\ 2x+3y=8 \end{cases}$$

Exercise 3.14. Solve the system using Cramer's Rule:

$$\begin{cases} 2x+3y=5 \\ 4x-y=1 \end{cases}$$

Exercise 3.15. Let α be a real number, (S) is a system of linear equations of variables x, y and z, where:

$$(S):\begin{cases}\alpha x+y+0z=1\\ 0x-2y+2z=2\\ 2x-y+3z=3\end{cases}$$

1. Rewrite the system (S) in matrix form.
2. Find the values of α for which the system (S) becomes a Cramer's system.
3. Solve the system (S) for the values of α for which the system (S) is a Cramer system.
4. Solve the system (S) for the values of $\alpha$for which the system (S) is a Non-Cramer system

Exercise 3.16. Let (S) the system of linear equations of variables x, y and z, where:

$$(S):\begin{cases}2x+3y-z=5\\ 4x+7y-3z=10\\ x+2y+z=3\end{cases}$$

1. Rewrite the system (S) in matrix form.
2. Solve the system (S) using Cramer's method
3. Solve the system (S) using Gauss method.
4. Solve the system (S) using Gauss-Jordan method
5. Solve the system (S) using the inverse method

3.4 Solutions

1. For Exercise 1, using Cramer's Rule, the coefficient matrix is:

$$A=\begin{pmatrix}3&2\\4&-1\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}5\\2\end{pmatrix}$$

Compute the determinant of A:

$$\det(A) = 3 \cdot (-1) - 2 \cdot 4 = -3 - 8 = -11$$

Now, compute the determinants for x and y:

$$A_x = \begin{pmatrix} 5 & 2 \\ 2 & -1 \end{pmatrix}, \quad A_y = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$$

$$\det(A_x) = 5 \cdot (-1) - 2 \cdot 2 = -5 - 4 = -9, \quad \det(A_y) = 3 \cdot 2 - 5 \cdot 4 = 6 - 20 = -14$$

Thus, the solution is:

$$x = \frac{-9}{-11} = \frac{9}{11}, \quad y = \frac{-14}{-11} = \frac{14}{11}$$

2. For Exercise 2, solve the system using Gaussian elimination:

$$\begin{cases} x + y + z = 6 \\ 2x + 3y + 5z = 16 \\ 4x + y + 2z = 11 \end{cases}$$

Write the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 5 & 16 \\ 4 & 1 & 2 & 11 \end{array}\right)$$

Perform row operations to reduce the matrix to upper triangular form. First, subtract 2 times row 1 from row 2 and 4 times row 1 from row 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & -3 & -2 & -13 \end{array}\right)$$

Next, add 3 times row 2 to row 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 7 & -1 \end{array}\right)$$

Now solve by back substitution:

$$z = \frac{-1}{7}, \quad y = \frac{31}{7}, \quad x = \frac{12}{7}$$

Thus, the solution is:

$$x = \frac{12}{7}, \quad y = \frac{31}{7}, \quad z = -\frac{1}{7}$$

3. For Exercise 3, the system is:

$$\begin{cases} 2x + y = 0 \\ x - y = 0 \end{cases}$$

Since the right-hand side constants are both zero, this is a homogeneous system. Solving the system by substitution or Gaussian elimination gives:

$$x = 0, \quad y = 0$$

Thus, the solution is $x = 0$, $y = 0$.

4. For Exercise 4, use the Rank Method (Rouché-Capelli Theorem) to solve:

$$\begin{cases} x + 2y - z = 1 \\ 2x + y + z = 3 \\ 3x + 3y = 4 \end{cases}$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 3 \\ 3 & 3 & 0 & 4 \end{array}\right)$$

Perform row operations to find the rank:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the rank of A (the coefficient matrix) is 2 and the rank of the augmented matrix [A|**b**] is also 2, the system is consistent and has infinitely many solutions.
The equivalent system is:

$$\begin{cases} x + 2y = 1 + z \\ 0x - 3y = 1 - 3z \end{cases}$$

The matriciel form is :

$$\left(\begin{array}{cc|c} 1 & 2 & 1+z \\ 0 & -3 & 1-3z \end{array}\right)$$

The solutions are:

$$x = \frac{5}{3} - z, \quad y = z - \frac{1}{3}$$

5. For Exercise 5, The matriciel form:

$$AX = b \implies \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

   Find the inverse of the matrix:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

   The determinant of A is:

$$\det(A) = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2$$

   The inverse of A is:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$

   Thus, the inverse is:

$$A^{-1} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ \frac{4}{2} & -1 \end{pmatrix}$$

   Then the solutions of the system are:

$$X = A^{-1} b = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ \frac{4}{2} & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$x = \frac{1}{2}, \quad y = 0$$

6. For Exercise 6, solve the system using the Gauss Method:

$$\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$$

   The augmented matrix is:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array} \right)$$

   By row reduction, we get:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array} \right)$$

   Solving gives $y = 1$ and substituting into the first equation gives $x = 1$. Therefore, the solution is:

$$x = 1, \quad y = 1$$

7. For Exercise 7, solve the system using Gaussian elimination:

$$\begin{cases} 2x+3y+z=7 \\ 4x-y+3z=9 \\ 6x+5y-z=13 \end{cases}$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 4 & -1 & 3 & 9 \\ 6 & 5 & -1 & 13 \end{array}\right)$$

Perform row operations to reduce to upper triangular form. Subtract 2 times row 1 from row 2, and 3 times row 1 from row 3:

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 0 & -7 & 1 & -5 \\ 0 & -4 & -4 & -8 \end{array}\right)$$

Using the relation $\mathrm{R}_3 - \frac{-4}{-7}\mathrm{R}_2$, we get:

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 7 \\ 0 & -7 & 1 & -5 \\ 0 & 0 & -\frac{32}{7} & -\frac{36}{7} \end{array}\right)$$

Now solve by back substitution:

$$z=\frac{9}{8}, \quad y=\frac{7}{8}, \quad x=\frac{13}{8}$$

Thus, the solution is:

$$x=\frac{13}{8}, \quad y=\frac{7}{8}, \quad z=\frac{9}{8}$$

8. For Exercise 8, determine if the following matrix is invertible:

$$\mathrm{B}=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

The determinant of B is:

$$\det(\mathrm{B}) = 1\cdot(5\cdot9-6\cdot8)-2\cdot(4\cdot9-6\cdot7)+3\cdot(4\cdot8-5\cdot7)$$

$$\det(B) = 1\cdot(45-48) - 2\cdot(36-42) + 3\cdot(32-35) = -3+12-9 = 0$$

Since the determinant is zero, the matrix is not invertible. The system

$$\begin{cases} x+2y+3z=7 \\ 4x+5y+6z=5 \\ 7x+8y+9z=6 \end{cases}$$

is Non-Cramer's system (Because $det(B)=0$)

9. For Exercise 9 , using Cramer's Rule, we found that:

$$x=\frac{8}{5},\quad y=\frac{9}{5}$$

Thus, the solution is:

$$x=\frac{8}{5},\quad y=\frac{9}{5}$$

10. For Exercise 10, solve the system using Gaussian elimination:

$$\begin{cases} x+2y+3z=9 \\ 2x-y+z=8 \\ 3x+y-2z=3 \end{cases}$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 1 & -2 & 3 \end{array}\right)$$

First, subtract 2 times row 1 from row 2 and 3 times row 1 from row 3:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -5 & -11 & -24 \end{array}\right)$$

Next, subtract row 2 from row 3:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & 0 & -6 & -14 \end{array}\right)$$

Solve by back substitution:

$$z=\frac{7}{3},\quad y=-\frac{1}{3},\quad x=\frac{8}{3}$$

Thus, the solution is:

$$x=\frac{8}{3},\quad y=\frac{1}{3},\quad z=\frac{7}{3}$$

11. For Exercise 11, solve the system using the Rank Method (Rouché-Capelli Theorem):

$$\begin{cases} x+y=2 \\ x-y=0 \\ 2x+2y=4 \end{cases}$$

The augmented matrix is:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 2 & 4 \end{array}\right)$$

Perform row operations to find the rank:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

The rank of A is 2, and the rank of the augmented matrix $[A|\mathbf{b}]$ is also 2. Since $\text{rank}(A) = \text{rank}([A|\mathbf{b}]) = 2$ and the system has two unknowns, it has a unique solution. Solving gives:

$$x=1, \quad y=1$$

Thus, the solution is:

$$x=1, \quad y=1$$

12. For Exercise 12, solve the system using Cramer's Rule:

$$\begin{cases} x+y=3 \\ 2x+3y=7 \end{cases}$$

The coefficient matrix is:

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

The determinant of A is:

$$\det(A) = 1 \cdot 3 - 1 \cdot 2 = 3 - 2 = 1$$

Now compute the determinants for x and y:

$$A_x = \begin{pmatrix} 3 & 1 \\ 7 & 3 \end{pmatrix}, \quad A_y = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$$

$$\det(A_x) = 3 \cdot 3 - 1 \cdot 7 = 9 - 7 = 2, \quad \det(A_y) = 1 \cdot 7 - 3 \cdot 2 = 7 - 6 = 1$$

Thus, the solution is:

$$x = \frac{2}{1} = 2, \quad y = \frac{1}{1} = 1$$

13. For Exercise 13, solve the system using the inverse matrix method:

$$\begin{cases} x+2y=5 \\ 2x+3y=8 \end{cases}$$

The coefficient matrix is:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

First, calculate the inverse of A. The determinant of A is:

$$\det(A) = 1\cdot 3 - 2\cdot 2 = 3-4 = -1$$

The inverse of A is:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

Now, multiply A^{-1} by $\mathbf{b}$:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} -3\cdot 5 + 2\cdot 8 \\ 2\cdot 5 - 1\cdot 8 \end{pmatrix} = \begin{pmatrix} -15+16 \\ 10-8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus, the solution is:

$$x = 1, \quad y = 2$$

14. For Exercise 14, use Cramer's Rule to solve:

$$\begin{cases} 2x+3y=5 \\ 4x-y=1 \end{cases}$$

The coefficient matrix is:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

The determinant of A is:

$$\det(A) = 2\cdot(-1) - 3\cdot 4 = -2-12 = -14$$

Now compute the determinants for x and y:

$$A_x = \begin{pmatrix} 5 & 3 \\ 1 & -1 \end{pmatrix}, \quad A_y = \begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix}$$

$$\det(A_x) = 5\cdot(-1) - 3\cdot 1 = -5-3 = -8, \quad \det(A_y) = 2\cdot 1 - 5\cdot 4 = 2-20 = -18$$

Thus, the solution is:

$$x = \frac{-8}{-14} = \frac{4}{7}, \quad y = \frac{-18}{-14} = \frac{9}{7}$$

15.
$$AX=b \Longleftrightarrow \begin{pmatrix} \alpha & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

the values of α for which the system (S) be a Cramer's system are $\det(A) \neq 0 \Longleftrightarrow 4-4\alpha \neq 0$ this implies: $\alpha \neq 1 \Longleftrightarrow \alpha \in \mathbb{R}-\{1\}$

Solve the system (S) for the values of α for which the system (S) is Cramer system.

For $\alpha \neq 1$, we get $\left\{\left[x=\frac{1}{\alpha-1}, y=-\frac{1}{\alpha-1}, z=\frac{1}{\alpha-1}(\alpha-2)\right]\right\}$ (by using Cramer, Inverse or Gauss methods). Solve the system (S) for the values of α for which the system (S) is Non-Cramer system.

For $\alpha=1$, we get $S=\emptyset$ (by usnig Gauss method)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} [1] & 1 & 0 & 1 \\ 0 & -2 & 2 & 2 \\ 2 & -1 & 3 & 3 \end{array}\right) \xrightarrow[R_3-2R_1 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & [-2] & 2 & 2 \\ 0 & -3 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & [-2] & 2 & 2 \\ 0 & -3 & 3 & 1 \end{array}\right) \xrightarrow[R_3-\left(\frac{-3}{-2}\right)R_2 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

$$(S'):\begin{cases} x+y+0z=1 \\ -2y+2z=2 \\ 0=-2 \text{ (Contraduction)} \end{cases} \Longleftrightarrow S=\emptyset$$

16. We can represent this system as $AX=B$ where

$$A=\begin{pmatrix} 2 & 3 & -1 \\ 4 & 7 & -3 \\ 1 & 2 & 1 \end{pmatrix}, \quad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{and} \quad B=\begin{pmatrix} 5 \\ 10 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 7 & -3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 10 \\ 3 \end{pmatrix}$$

Cramer's method:

We have:

$$\det(A) = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 7 & -3 \\ 1 & 2 & 1 \end{vmatrix} = 4$$

The solutions are:

$$x = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 5 & 3 & -1 \\ 10 & 7 & -3 \\ 3 & 2 & 1 \end{vmatrix}}{4} = \frac{9}{4}$$

$$y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 2 & 5 & -1 \\ 4 & 10 & -3 \\ 1 & 3 & 1 \end{vmatrix}}{4} = \frac{1}{4}$$

$$y = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 2 & 3 & 5 \\ 4 & 7 & 10 \\ 1 & 2 & 3 \end{vmatrix}}{4} = \frac{1}{4}$$

Gauss method

$$[A|B] = \left(\begin{array}{ccc|c} (2) & 3 & -1 & 5 \\ 4 & 7 & -3 & 10 \\ 1 & 2 & 1 & 3 \end{array}\right) \xrightarrow[R_3-\frac{1}{2}R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & (1) & -1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right) \xrightarrow[R_3-\frac{1}{2}R_2]{} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & \frac{1}{2} \end{array}\right)$$

We get:

$$(S') : \begin{cases} 2x + 3y - z = 5 \\ y - z = 0 \\ 2z = \frac{1}{2} \end{cases}$$

, Solution is:

$$\begin{cases} x = \frac{9}{4} \\ y = \frac{1}{4} \\ z = \frac{1}{4} \end{cases}$$

Gauss-Jordan method

$$[A|B]=\left(\begin{array}{ccc|c}(2)&3&-1&5\\4&7&-3&10\\1&2&1&3\end{array}\right)\xrightarrow[R_3-\frac{1}{2}R_1]{R_2-2R_1}\left(\begin{array}{ccc|c}2&3&-1&5\\0&1&-1&0\\0&\frac{1}{2}&\frac{3}{2}&\frac{1}{2}\end{array}\right)$$

$$\left(\begin{array}{ccc|c}2&3&-1&5\\0&(1)&-1&0\\0&\frac{1}{2}&\frac{3}{2}&\frac{1}{2}\end{array}\right)\xrightarrow[R_3-\frac{1}{2}R_2]{R_1-3R_2}\left(\begin{array}{ccc|c}2&0&2&5\\0&1&-1&0\\0&0&2&\frac{1}{2}\end{array}\right)$$

$$\left(\begin{array}{ccc|c}2&0&2&5\\0&1&-1&0\\0&0&(2)&\frac{1}{2}\end{array}\right)\xrightarrow{\substack{R_1-R_3\\R_2+\frac{1}{2}R_3}}\left(\begin{array}{ccc|c}2&0&0&\frac{9}{2}\\0&1&0&\frac{1}{4}\\0&0&2&\frac{1}{2}\end{array}\right)$$

$$\left(\begin{array}{ccc|c}2&0&0&\frac{9}{2}\\0&1&0&\frac{1}{4}\\0&0&2&\frac{1}{2}\end{array}\right)\xrightarrow[\frac{R_3}{2}]{\frac{R_1}{2}}\left(\begin{array}{ccc|c}1&0&0&\frac{9}{4}\\0&1&0&\frac{1}{4}\\0&0&1&\frac{1}{4}\end{array}\right)$$

We get:

$$\begin{cases}x=\frac{9}{4}\\y=\frac{1}{4}\\z=\frac{1}{4}\end{cases}$$

The inverse method:

$$AX=B:\begin{pmatrix}2&3&-1\\4&7&-3\\1&2&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\10\\3\end{pmatrix}$$

We have:

$$A^{-1}=\begin{pmatrix}\frac{13}{4}&-\frac{5}{4}&-\frac{1}{2}\\-\frac{7}{4}&\frac{3}{4}&\frac{1}{2}\\\frac{1}{4}&-\frac{1}{4}&\frac{1}{2}\end{pmatrix}$$

Then

$$X=A^{-1}B=\begin{pmatrix}\frac{13}{4}&-\frac{5}{4}&-\frac{1}{2}\\-\frac{7}{4}&\frac{3}{4}&\frac{1}{2}\\\frac{1}{4}&-\frac{1}{4}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}5\\10\\3\end{pmatrix}$$

So

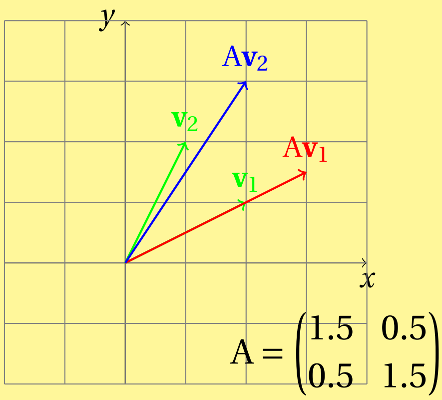
$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{9}{4}\\\frac{1}{4}\\\frac{1}{4}\end{pmatrix}$$

Chapter 4

Eigenvalues, Eigenvectors and Diagonalization

Introduction

Eigenvalues, eigenvectors, and diagonalization are key concepts in linear algebra, widely applied across various fields. In this chapter, we offer a concise overview of these topics, illustrating how matrices transform vectors and how diagonalization simplifies matrix computations.



4.1 Eigenvalues and Eigenvectors

Given a square matrix $A \in \mathbb{R}^{n\times n}$, a scalar λ is called an **eigenvalue** of A if there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that:

$$A\mathbf{v} = \lambda\mathbf{v}$$

where $\mathbf{v}$ is called the corresponding **eigenvector**.

4.1.1 Characteristic Equation

To find the eigenvalues of a matrix A, we solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

where I is the identity matrix and λ is a scalar. The solutions $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

4.1.2 Finding Eigenvectors

Once the eigenvalues are found, the eigenvectors can be determined by solving the system:

$$(A - \lambda_i I)\mathbf{v} = 0$$

for each eigenvalue λ_i.

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

Consider the matrix:

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

Step 1: Find the eigenvalues by solving the characteristic equation:

$$\det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

Solving this quadratic equation gives:

$$\lambda_1 = 5, \quad \lambda_2 = 2$$

Step 2: Find the eigenvectors for each eigenvalue.

For $\lambda_1 = 5$, solve:

$$(A - 5I)\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \mathbf{v} = 0$$

The solution is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 2$, solve:

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \mathbf{v} = 0$$

The solution is $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

4.2 Diagonalization

A matrix $A \in \mathbb{R}^{n\times n}$ is **diagonalizable** if it can be written as:

$$A = PDP^{-1}$$

where P is a matrix whose columns are the eigenvectors of A, and D is a diagonal matrix whose diagonal entries are the eigenvalues of A.

Steps for Diagonalization

1. **Find the eigenvalues** of A by solving the characteristic equation.
2. **Find the eigenvectors** corresponding to each eigenvalue.
3. **Form matrix** P, where each column is an eigenvector of A.
4. **Form matrix** D, a diagonal matrix with the eigenvalues on the diagonal.
5. **Verify** that $A = PDP^{-1}$.

Example: Diagonalizing a 2×2 Matrix

Consider the matrix:

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

From the previous example, we know the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Step 1: Construct P using the eigenvectors:

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

Step 2: Construct D using the eigenvalues:

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$$

Step 3: Verify that $A = PDP^{-1}$.

4.3 Properties of Eigenvalues and Eigenvectors

Property 4.1.

- *The sum of the eigenvalues of a matrix is equal to the trace of the matrix:* $\sum \lambda_i = tr(A)$.
- *The product of the eigenvalues of a matrix is equal to the determinant of the matrix:* $\prod \lambda_i = \det(A)$.
- *If* A *is diagonalizable, the matrix* D *will have the eigenvalues of* A *along its diagonal.*
- *Eigenvalues are the values* λ *for which the equation* $(A - \lambda I)x = 0$ *has non-trivial solutions, where* I *is the identity matrix and* x *is a non-zero vector.*
- *Eigenvectors are the corresponding vectors* x *that satisfy* $Ax = \lambda x$ *for some eigenvalue* λ.
- *If* A *is diagonalizable, its eigenvalues appear on the diagonal of the diagonal matrix* D*, and the corresponding eigenvectors form the columns of the matrix* P.
- *If* A *is a real symmetric matrix, its eigenvalues are real and its eigenvectors are orthogonal.*
- *If* A *is upper or lower triangular, its eigenvalues are the diagonal elements.*
- *If* A *is diagonal, its eigenvalues are the diagonal elements.*
- *The trace is equal to the sum of the eigenvalues of* A.
- *The determinant of a matrix* A *is equal to the product of its eigenvalues.*
- *If* A *is invertible, its eigenvalues are non-zero.*
- *The rank of a matrix* A *is equal to the number of non-zero eigenvalues.*

4.4 Exercises

Exercise 4.1. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Exercise 4.2. For the matrix

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix},$$

determine the eigenvalues and eigenvectors.

Exercise 4.3. Determine the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 4.4. Given the matrix

$$A = \begin{pmatrix} 5 & 4 \\ -2 & 1 \end{pmatrix},$$

find the eigenvalues and corresponding eigenvectors.

Exercise 4.5. Show that the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

has purely imaginary eigenvalues.

Exercise 4.6. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}.$$

Exercise 4.7. For the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

find the eigenvalues and eigenvectors.

Exercise 4.8. Determine the eigenvalues for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Exercise 4.9. For the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

find the eigenvalues and eigenvectors.

Exercise 4.10. Show that the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has a repeated eigenvalue and find the corresponding eigenvector.

Exercise 4.11. Given a matrix A, as follows:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

- Compute the characteristic polynomial of A:
- Find the eigenvalues and eigenvectors of A
- Find P, D and P^{-1} where $A = PDP^{-1}$.
- Compute A^5.

4.5 Solutions

1. For Exercise 1, the characteristic equation is:

$$\det(A-\lambda I) = \det\begin{pmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{pmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0.$$

   The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 1$.

2. For Exercise 2, the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 4$ (Because the matrix is a diagonal matrix, with eigenvectors $\mathbf{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$.

3. For Exercise 3, the characteristic polynomial is:

$$\det(A-\lambda I) = (1-\lambda)^3 = 0,$$

   giving $\lambda = 1$ with multiplicity 3.

4. For Exercise 4, the eigenvalues are complex numbers $\lambda_1 = 3-2i$, $\lambda_2 = 3+2i$, with eigenvectors $\mathbf{v}_1 = \begin{pmatrix}-1+i\\1\end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix}-1-i\\1\end{pmatrix}$.

5. For Exercise 5, solving the characteristic polynomial gives eigenvalues $\lambda = i$ and $\lambda = -i$, leading to eigenvectors $\begin{pmatrix}-i\\1\end{pmatrix}$ and $\begin{pmatrix}i\\1\end{pmatrix}$.

6. For Exercise 6, the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = -1.$, leading to eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

7. For Exercise 7, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3.$, leading to eigenvectors $\begin{pmatrix}-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$.

8. For Exercise 8, the eigenvalues are $\lambda_1 = \frac{5}{2} - \frac{1}{2}\sqrt{33}$, $\lambda_2 = \frac{5}{2} + \frac{1}{2}\sqrt{33}$.

9. For Exercise 9, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$ with eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$,$\begin{pmatrix}0\\1\end{pmatrix}$.

10. For Exercise 10, solving gives $\lambda = 1$ with eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$.

11. From the definition of the eigenvector $\mathbf{v}$ corresponding to the eigenvalue λ we have $A\mathbf{v} = \lambda\mathbf{v}$

    Then: $A\mathbf{v} - \lambda\mathbf{v} = (A - \lambda I)\mathbf{v} = 0$

    The equation has a nonzero solution if and only if $\det(A - \lambda I) = 0$

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} \\
&= -\lambda^3 - \lambda^2 + 12\lambda \\
&= -\lambda(\lambda^2 + \lambda - 12) \\
&= -\lambda(\lambda+4)(\lambda-3) = 0
\end{aligned}$$

- The characteristic polynomial is : $-\lambda^3 - \lambda^2 + 12\lambda$.
- The eigenvalues are the solutions of $-\lambda(\lambda+4)(\lambda-3) = 0$, then $\lambda_1 = 0$, $\lambda_2 = -4, \lambda_3 = 3$

Searching the corresponding igenvectors:

For every λ we find its own vectors:

- $\lambda_1 = 0$

$$A - \lambda_1 I = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$(A - \lambda I)\mathbf{v} = 0$$

So we have a homogeneous system of linear equations, we solve it by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} [1] & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right) \xrightarrow[\substack{R_2 - 6R_1 \to R_2 \\ R_3 - (-1)R_1 \to R_3}]{} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -13 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -13x_2 - 6x_3 = 0 \\ 0 = 0 \end{cases} \quad (1)$$

Find the variable x_2 from equation 2 of the system (1):

$$x_2 = -\frac{6}{13}x_3$$

Find the variable x_1 from equation 1 of the system (1):

$$x_1 = -\frac{1}{13}x_3$$

Answer:

$$x_1 = -\frac{1}{13}x_3, x_2 = -\frac{6}{13}x_3, x_3 \in \mathbb{R}$$

General Solution:

$$X = \begin{pmatrix} -\frac{1}{13}x_3 \\ -\frac{6}{13}x_3 \\ x_3 \end{pmatrix}$$

The solution set:

$$\{x_3 \begin{pmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{pmatrix}\}$$

- For $\lambda_1 = 0$, let $x_3 = 1$, $\quad v_1 = \begin{pmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{pmatrix}$

- $\lambda_2 = -4$

$$A - \lambda_2 I = \begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix}$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$(A - \lambda I)\mathbf{v} = 0$$

So we have a homogeneous system of linear equations, we solve it by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} [5] & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right) \xrightarrow[R_3 - (\frac{-1}{5})R_1 \to R_3]{R_2 - \frac{6}{5}R_1 \to R_2} \left(\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 0 & [\frac{3}{5}] & -\frac{6}{5} & 0 \\ 0 & -\frac{8}{5} & \frac{16}{5} & 0 \end{array}\right)$$

$$\xrightarrow[R_3 - (\frac{-8}{3})R_1 \to R_3]{} \left(\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 0 & \frac{3}{5} & -\frac{6}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} 5x_1 & 2x_2 & +x_3 & = & 0 \\ & -\frac{3}{5}x_2 & -\frac{6}{5}x_3 & = & 0 \\ & & 0 & = & 0 \end{cases} \quad (2)$$

Find the variable x_2 from equation 2 of the system (2):

$$x_2 = 2x_3$$

Find the variable x_1 from equation 1 of the system (2):

$$x_1 = -x_3$$

Answer:

$$x_1 = -x_3, x_2 = 2x_3, x_3 = x_3$$

General Solution:

$$X = \begin{pmatrix} -x_3 \\ 2x_3 \\ x_3 \end{pmatrix}$$

The solution set:

$$\{x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\}$$

- For $\lambda_2 = -4$, let $x_3 = 1$, $\quad v_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

- $\lambda_3 = 3$

$$A - \lambda_3 I = \begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix}$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$(A - \lambda I)\mathbf{v} = 0$$

So we have a homogeneous system of linear equations, we solve it by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} [-2] & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right) \xrightarrow[\substack{R_2 - \frac{6}{-2}R_1 \to R_2 \\ R_3 - (\frac{-1}{-2})R_1 \to R_3}]{} \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & [2] & 3 & 0 \\ 0 & -3 & -\frac{9}{2} & 0 \end{array}\right)$$

$$\xrightarrow[R_3-(\frac{-3}{2})R_2\to R_3]{} \left(\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} -2x_1 & +2x_2 & +x_3 & = & 0 \\ & +2x_3 & +3x_3 & = & 0 \\ & & 0 & = & 0 \end{cases} \quad (3)$$

Find the variable x_2 from equation 2 of the system (3):

$$x_2 = -\frac{3}{2}x_3$$

Find the variable x_1 from equation 1 of the system (3)

$$x_1 = -x_3$$

Answer:

$$x_1 = -x_3 x_2 = -\frac{3}{2}x_3 x_3 = x_3$$

General Solution:

$$X = \begin{pmatrix} -x_3 \\ -\frac{3}{2}x_3 \\ x_3 \end{pmatrix}$$

- For $\lambda_3 = 3$, let $x_3 = 1$, $\quad v_1 = \begin{pmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{pmatrix}$

P the matrix of eigenvectors of A

$$P = \begin{pmatrix} -\frac{1}{13} & -1 & -1 \\ -\frac{6}{13} & -\frac{3}{2} & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

P^{-1} the inverse of the matrix P

$$P^{-1} = \begin{pmatrix} \frac{13}{12} & 0 & \frac{13}{12} \\ -\frac{16}{21} & -\frac{2}{7} & -\frac{4}{21} \\ -\frac{9}{28} & \frac{2}{7} & \frac{3}{28} \end{pmatrix}$$

D the diagonal matrix containing the eigenvalues of A

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

We have:

$$PDP^{-1} = \begin{pmatrix} -\frac{1}{13} & -1 & -1 \\ -\frac{6}{13} & -\frac{3}{2} & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} \frac{13}{12} & 0 & \frac{13}{12} \\ -\frac{16}{21} & -\frac{2}{7} & -\frac{4}{21} \\ -\frac{9}{28} & \frac{2}{7} & \frac{3}{28} \end{pmatrix}$$

Then:

$$PDP^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} = A$$

Calculating A^5, by using the property $A^n = PD^nP^{-1}$, we get:

$$A^5 = \begin{pmatrix} -\frac{1}{13} & -1 & -1 \\ -\frac{6}{13} & -\frac{3}{2} & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0^5 & 0 & 0 \\ 0 & 3^5 & 0 \\ 0 & 0 & (-4)^5 \end{pmatrix} \begin{pmatrix} \frac{13}{12} & 0 & \frac{13}{12} \\ -\frac{16}{21} & -\frac{2}{7} & -\frac{4}{21} \\ -\frac{9}{28} & \frac{2}{7} & \frac{3}{28} \end{pmatrix}$$

Then:

$$A^5 = \begin{pmatrix} -\frac{1}{13} & -1 & -1 \\ -\frac{6}{13} & -\frac{3}{2} & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 243 & 0 \\ 0 & 0 & -1024 \end{pmatrix} \begin{pmatrix} \frac{13}{12} & 0 & \frac{13}{12} \\ -\frac{16}{21} & -\frac{2}{7} & -\frac{4}{21} \\ -\frac{9}{28} & \frac{2}{7} & \frac{3}{28} \end{pmatrix}$$

So,

$$A^5 = \begin{pmatrix} -144 & 362 & 156 \\ 936 & -481 & -150 \\ 144 & -362 & -156 \end{pmatrix}$$

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