Mathematics 2
100 Exercises with Solutions

MATHEMATICS 2- HANDOUT

By **Dr. Djabi Abdelmoumene**

Department of Basic Education
Faculty of Economics, Commerce and Management Sciences
University of Ferhat Abbas – Setif 1

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Contents

Preface

Introduction

This handout offers a comprehensive approach to **linear algebra**, combining theoretical foundations with practical problem-solving techniques. Designed for both students and practitioners, it emphasizes **matrix methods** as essential tools for mathematical modeling and computation.

Structure:

The material is organized into four coherent chapters:

- **Matrices**: Fundamental operations and special matrix types
- **Determinants & Inverses**: Calculation methods and applications
- **Systems of Equations**: Solution techniques using matrix algebra
- **Eigenvalues & Diagonalization**: Spectral theory and matrix simplification

Each chapter follows a consistent pattern:

- **Definitions**: Clear theoretical foundations
- **Exercises**: Graded problems from basic to advanced
- **Solutions**: Detailed explanations for all exercises

The exercises include computational problems, theoretical challenges, and graphical interpretations to develop both technical skills and conceptual understanding.

Happy studying!

Notations

Sets of numbers

- $\mathbb{N}$ The set of *natural numbers.*
- $\mathbb{Z}$ The set of *integers.*
- $\mathbb{Q}$ The set of *rational numbers*(fractions $\frac{a}{b}$ where a and b are integers and $b \neq 0$).
- $\mathbb{R}$ The set of *real numbers.*
- $\mathbb{C}$ The set of *complex numbers.*

Logical conditions

- $\forall$ This symbol means *for all* .
- $\exists$ This symbol means *there exists.*
- $\nexists$ This symbol means *there does not exist.*

Elements and sets

- $\in$ An element is in a set.
- $\notin$ An element is not in a set.
- $\emptyset$ This symbol is used to denote the "empty set".
- $\cup$ Means *union.*
- $\cap$ Means *intersection.*

Miscellaneous symbols

- $=$ is equal to
- $\neq$ is not equal to
- $\prec$ is less than
- $\leq$ is less than or equal to
- $\succ$ is greater than
- $\geq$ is greater than or equal to

∞ infinity

$\Rightarrow$ implies

$\Leftarrow$ is implied by

$\Leftrightarrow$ implies and is implied by (is equivalent to)

Operations

$a+b$ a plus b

$a-b$ a minus b

$a \times b, ab$ a multiplied by b (or a times b)

$\frac{a}{b}$ a divided by b

$\sqrt{a}$ the non-negative square root of a

Matrices

$\mathrm{A}_{m\times n}$ Matrix with m rows and n columns.

$\mathrm{A} = [a_{ij}$ $]\mathrm{M}atrix\mathrm{A}$ with elements a_{ij}.

$\mathrm{A}_{n\times n}$ Square Matrix

$\mathrm{A}_{m\times 1}$ Column Matrix

$\mathrm{A}_{1\times n}$ Row Matrix

$\mathrm{O}_{m\times n}$ Zero Matrix (all elements are zero)

$\mathrm{I}_{n\times n}$ Identity Matrix (diagonal elements are 1, others are 0)

A^{T} The transpose of a matrix A

$tr(\mathrm{A})$ The trace of a square matrix A

$\det(\mathrm{A}), |\mathrm{A}|$ Determinant of a square matrix A

$\mathrm{adj}(\mathrm{A})$ The adjoint matrix of A

A^{-1} The inverse of a square matrix A

Chapter 1

Matrices

Introduction

Matrices are essential tools for solving linear problems efficiently. This chapter provides hands-on practice with matrix operations, special matrices, and advanced applications through structured exercises. Work through the problems to master both computation and conceptual understanding.

$$B=\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots\cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots\cdots & b_{2,n} \\ b_{3,1} & b_{3,2} & \cdots\cdots & b_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots\cdots & b_{m,n} \end{pmatrix}$$

1.1 Fundamental Matrix Definitions

Definition 1.1 (Matrix)**.** A rectangular array of numbers (called *elements* or *entries*) arranged in m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Denoted as $A \in \mathbb{R}^{m\times n}$ or $A \in \mathbb{C}^{m\times n}$.

Definition 1.2 (Matrix Addition)**.** For $A, B \in \mathbb{R}^{m\times n}$:

$$(A+B)_{ij} = a_{ij} + b_{ij}$$

Definition 1.3 (Scalar Multiplication)**.** For $A \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}$:

$$(cA)_{ij} = c \cdot a_{ij}$$

Definition 1.4 (Matrix Multiplication)**.** For $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{p\times n}$:

$$(AB)_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Definition 1.5 (Identity Matrix)**.** The $n \times n$ matrix I_n with:

$$(I_n)_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.6 (Transpose)**.** For $A \in \mathbb{R}^{m\times n}$, its transpose $A^T \in \mathbb{R}^{n\times m}$:

$$(A^T)_{ij} = a_{ji}$$

Definition 1.7 (Trace (Square Matrices))**.** For $A \in \mathbb{R}^{n\times n}$:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Definition 1.8 (Special Matrices)**.**

- **Diagonal Matrix**: $a_{ij} = 0$ for $i \neq j$
- **Upper Triangular**: $a_{ij} = 0$ for $i > j$
- **Lower Triangular**: $a_{ij} = 0$ for $i < j$
- **Symmetric**: $A^T = A$
- **Skew-Symmetric**: $A^T = -A$

Definition 1.9 (Invertible Matrix)**.** A square matrix $A \in \mathbb{R}^{n\times n}$ is *invertible* if there exists B such that:

$$AB = BA = I_n$$

Denoted $B = A^{-1}$.

Definition 1.10 (Determinant)**.** The unique scalar function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfying:

- $\det(I_n) = 1$
- Multilinear in rows/columns
- Alternating (swapping rows changes sign)
- $\det(A) = 0$ iff A is singular

1.2 Exercises and Solutions

1.2.1 Operations with Matrices

Exercise 1. Given the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 2 & 5 \end{pmatrix},$$

compute $A + B$.

$$A + B = \begin{pmatrix} 1+0 & 2+(-1) \\ 3+2 & 4+5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 5 & 9 \end{pmatrix}$$

Exercise 2. Given

$$C = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 1 \\ -2 & 0 \end{pmatrix},$$

compute $C - D$.

$$C - D = \begin{pmatrix} 2-4 & -1-1 \\ 0-(-2) & 3-0 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 2 & 3 \end{pmatrix}$$

Exercise 3. Given

$$G = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 4 \end{pmatrix}, \quad H = \begin{pmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 2 \end{pmatrix},$$

compute GH.

$$GH = \begin{pmatrix} (1)(2)+(0)(0)+(2)(3) & (1)(1)+(0)(-1)+(2)(2) \\ (-1)(2)+(3)(0)+(4)(3) & (-1)(1)+(3)(-1)+(4)(2) \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 10 & 4 \end{pmatrix}$$

Exercise 4. Given

$$K = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix},$$

compute K^2.

$$K^2 = KK = \begin{pmatrix} (2)(2)+(-1)(3) & (2)(-1)+(-1)(0) \\ (3)(2)+(0)(3) & (3)(-1)+(0)(0) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 6 & -3 \end{pmatrix}$$

Exercise 5. Given

$$L = \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix},$$

compute L^T (the transpose of L).

$$L^T = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}$$

Exercise 6. Given

$$M = \begin{pmatrix} 5 & -2 & 1 \\ 0 & 3 & 4 \\ -1 & 2 & 6 \end{pmatrix},$$

compute the trace of M.

$$\mathrm{tr}(M) = 5 + 3 + 6 = 14$$

Exercise 7. Given

$$N = \begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 5 \\ 2 & -1 \end{pmatrix},$$

verify whether $NP = PN$.

First compute NP:

$$NP = \begin{pmatrix} (2)(0)+(1)(2) & (2)(5)+(1)(-1) \\ (-3)(0)+(4)(2) & (-3)(5)+(4)(-1) \end{pmatrix} = \begin{pmatrix} 2 & 9 \\ 8 & -19 \end{pmatrix}$$

Now compute PN:

$$PN = \begin{pmatrix} (0)(2)+(5)(-3) & (0)(1)+(5)(4) \\ (2)(2)+(-1)(-3) & (2)(1)+(-1)(4) \end{pmatrix} = \begin{pmatrix} -15 & 20 \\ 7 & -2 \end{pmatrix}$$

Since $NP \neq PN$, **matrix multiplication is not commutative** in this case.

Exercise 8. Given

$$Q = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

compute $Q^T Q$.

We are given

$$Q = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

We need $Q^T Q$.

Step 1: Find Q^T

$$Q^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

Step 2: Multiply $Q^T Q$

$$Q^T Q = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

This is a 3×2 times a 2×3 matrix, so the result is 3×3.

Step 3: Compute entries

- Row 1, Col 1: $1 \times 1 + 4 \times 4 = 1 + 16 = 17$
- Row 1, Col 2: $1 \times 2 + 4 \times 5 = 2 + 20 = 22$
- Row 1, Col 3: $1 \times 3 + 4 \times 6 = 3 + 24 = 27$
- Row 2, Col 1: $2 \times 1 + 5 \times 4 = 2 + 20 = 22$
- Row 2, Col 2: $2 \times 2 + 5 \times 5 = 4 + 25 = 29$
- Row 2, Col 3: $2 \times 3 + 5 \times 6 = 6 + 30 = 36$
- Row 3, Col 1: $3 \times 1 + 6 \times 4 = 3 + 24 = 27$
- Row 3, Col 2: $3 \times 2 + 6 \times 5 = 6 + 30 = 36$

- Row 3, Col 3: $3\times3+6\times6=9+36=45$

Step 4: Write result

$$Q^TQ=\begin{pmatrix}17 & 22 & 27\\ 22 & 29 & 36\\ 27 & 36 & 45\end{pmatrix}.$$

$$\boxed{\begin{pmatrix}17 & 22 & 27\\ 22 & 29 & 36\\ 27 & 36 & 45\end{pmatrix}}$$

Exercise 9. Let

$$A=\begin{pmatrix}x & 2\\ -1 & 3\end{pmatrix},\quad B=\begin{pmatrix}1 & x\\ 4 & -2\end{pmatrix},$$

where $x\in\mathbb{R}$.

1. Compute the product AB.
2. Find the transpose $(AB)^T$.
3. Calculate the trace $\mathrm{tr}(AB)$.
4. Determine all values of x for which $\mathrm{tr}(AB)=\mathrm{tr}(B^TA^T)$.
5. Verify whether $(AB)^T=B^TA^T$ holds for all $x\in\mathbb{R}$.

1. Compute AB

$$AB=\begin{pmatrix}x & 2\\ -1 & 3\end{pmatrix}\begin{pmatrix}1 & x\\ 4 & -2\end{pmatrix}=\begin{pmatrix}x\cdot1+2\cdot4 & x\cdot x+2\cdot(-2)\\ -1\cdot1+3\cdot4 & -1\cdot x+3\cdot(-2)\end{pmatrix}$$

$$AB=\begin{pmatrix}x+8 & x^2-4\\ -1+12 & -x-6\end{pmatrix}=\begin{pmatrix}x+8 & x^2-4\\ 11 & -x-6\end{pmatrix}$$

2. Find $(AB)^T$

$$(AB)^T=\begin{pmatrix}x+8 & 11\\ x^2-4 & -x-6\end{pmatrix}$$

3. Calculate tr(AB)

$$\mathrm{tr}(AB) = (x+8) + (-x-6) = x+8-x-6 = 2$$

4. Find x such that tr(AB) = tr($B^T A^T$)

First compute A^T and B^T:

$$A^T = \begin{pmatrix} x & -1 \\ 2 & 3 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 4 \\ x & -2 \end{pmatrix}$$

Now compute $B^T A^T$:

$$B^T A^T = \begin{pmatrix} 1 & 4 \\ x & -2 \end{pmatrix} \begin{pmatrix} x & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1\cdot x + 4\cdot 2 & 1\cdot(-1) + 4\cdot 3 \\ x\cdot x + (-2)\cdot 2 & x\cdot(-1) + (-2)\cdot 3 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} x+8 & -1+12 \\ x^2-4 & -x-6 \end{pmatrix} = \begin{pmatrix} x+8 & 11 \\ x^2-4 & -x-6 \end{pmatrix}$$

Trace of $B^T A^T$:

$$\mathrm{tr}(B^T A^T) = (x+8) + (-x-6) = x+8-x-6 = 2$$

We want $\mathrm{tr}(AB) = \mathrm{tr}(B^T A^T)$:

$$2 = 2$$

This equation holds for **all** $x \in \mathbb{R}$.

5. Verify $(AB)^T = B^T A^T$

From part 2: $(AB)^T = \begin{pmatrix} x+8 & 11 \\ x^2-4 & -x-6 \end{pmatrix}$

From part 4: $B^T A^T = \begin{pmatrix} x+8 & 11 \\ x^2-4 & -x-6 \end{pmatrix}$

These matrices are identical for all $x \in \mathbb{R}$, confirming that:

$$(AB)^T = B^T A^T$$

holds for all real values of x.

1.2.2 Special Matrices

Exercise 10. Construct a 3×3 diagonal matrix D with diagonal entries $2, -1, 4$.

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Exercise 11. Construct a 3×3 upper triangular matrix U with entries $u_{ij} = i + j$ for $i \leq j$.

$$U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

Exercise 12. Construct a 3×3 lower triangular matrix L with entries $l_{ij} = i - j$ for $i \geq j$.

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

Exercise 13. Given the matrix

$$S = \begin{pmatrix} 1 & 5 & -3 \\ 5 & 2 & 0 \\ -3 & 0 & 4 \end{pmatrix},$$

verify whether S is symmetric.

Given the matrix

$$S = \begin{pmatrix} 1 & 5 & -3 \\ 5 & 2 & 0 \\ -3 & 0 & 4 \end{pmatrix},$$

we verify whether S is symmetric.

Step 1: Definition of symmetric matrix

A matrix S is symmetric if $S^T = S$, i.e., $s_{ij} = s_{ji}$ for all i, j.

Step 2: Compute the transpose S^T

$$S^T = \begin{pmatrix} 1 & 5 & -3 \\ 5 & 2 & 0 \\ -3 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & -3 \\ 5 & 2 & 0 \\ -3 & 0 & 4 \end{pmatrix}$$

Step 3: Compare S and S^T

$$S = \begin{pmatrix} 1 & 5 & -3 \\ 5 & 2 & 0 \\ -3 & 0 & 4 \end{pmatrix}, \quad S^T = \begin{pmatrix} 1 & 5 & -3 \\ 5 & 2 & 0 \\ -3 & 0 & 4 \end{pmatrix}$$

We observe that $S = S^T$.

Step 4: Conclusion

Since $S^T = S$, the matrix S is symmetric.

$$\boxed{\text{Yes}}$$

Exercise 14. Given the matrix

$$T = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix},$$

verify whether T is skew-symmetric.

Solution

Given the matrix

$$T = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix},$$

we verify whether T is skew-symmetric.

Step 1: Definition of skew-symmetric matrix

A matrix T is skew-symmetric if $T^T = -T$, i.e., $t_{ij} = -t_{ji}$ for all i, j, and all diagonal elements must be zero.

Step 2: Compute the transpose T^T

$$T^T = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$$

Step 3: Compute $-T$

$$-T = -\begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$$

Step 4: Compare T^T and $-T$

$$T^T = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}, \quad -T = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{pmatrix}$$

We observe that $T^T = -T$.

Step 5: Check diagonal elements

All diagonal elements of T are zero, which is consistent with the definition of a skew-symmetric matrix.

Step 6: Conclusion

Since $T^T = -T$ and all diagonal elements are zero, the matrix T is skew-symmetric.

$$\boxed{\text{Yes}}$$

Exercise 15. Construct the 4×4 identity matrix I_4.

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 16. Construct a 2×3 zero matrix Z.

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise 17. Given

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

show that A is symmetric.

Since $A = A^T$, A is symmetric.

Exercise 18. Given

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

show that B is skew-symmetric.

Since $B^T = -B$, B is skew-symmetric.

Exercise 19. Construct a 3×3 symmetric matrix with entries $s_{ij} = i \cdot j$.

To construct a 3×3 symmetric matrix S with entries $s_{ij} = i \cdot j$, we proceed as follows:

Step 1: Understand the entry definition. The matrix entries are defined by the product of their row and column indices:

$$s_{ij} = i \cdot j \quad \text{for} \quad i, j = 1, 2, 3$$

Step 2: Compute each matrix element.

$$
\begin{aligned}
s_{11} &= 1\cdot 1 = 1 \\
s_{12} &= 1\cdot 2 = 2 \quad (= s_{21}) \\
s_{13} &= 1\cdot 3 = 3 \quad (= s_{31}) \\
s_{22} &= 2\cdot 2 = 4 \\
s_{23} &= 2\cdot 3 = 6 \quad (= s_{32}) \\
s_{33} &= 3\cdot 3 = 9
\end{aligned}
$$

Step 3: Assemble the matrix.
The symmetric matrix S is:

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}}$$

Verification of symmetry: The matrix S is symmetric because $s_{ij} = s_{ji}$ for all i, j, which follows from the commutative property of multiplication ($i \cdot j = j \cdot i$).

Note: This matrix has rank 1 because all rows are scalar multiples of the first row $(1,2,3)$. Its eigenvalues are:

- $\lambda_1 = \mathrm{tr}(S) = 14$ (since rank is 1)
- $\lambda_2 = \lambda_3 = 0$

Exercise 20. Construct a 3×3 skew-symmetric matrix with entries $k_{ij} = i - j$ for $i < j$.

To construct a 3×3 skew-symmetric matrix K with entries $k_{ij} = i - j$ for $i < j$, we proceed as follows:

Step 1: Recall the properties of a skew-symmetric matrix. A matrix K is skew-symmetric if:

$$K^T = -K$$

This implies:

- Diagonal entries must satisfy $k_{ii} = -k_{ii} \Rightarrow k_{ii} = 0$
- Off-diagonal entries satisfy $k_{ij} = -k_{ji}$ for $i \neq j$

Step 2: Compute the matrix entries.

Using the given formula $k_{ij} = i - j$ for $i < j$:

$$\begin{aligned} k_{12} &= 1-2 = -1 &\Rightarrow k_{21} = 1 \\ k_{13} &= 1-3 = -2 &\Rightarrow k_{31} = 2 \\ k_{23} &= 2-3 = -1 &\Rightarrow k_{32} = 1 \end{aligned}$$

Diagonal entries (must be zero):

$$k_{11} = k_{22} = k_{33} = 0$$

Step 3: Assemble the matrix.
The skew-symmetric matrix K is:

$$\mathrm{K} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

Verification of skew-symmetry: We can verify that:

$$\mathrm{K}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} = -\mathrm{K}$$

Conclusion: The required 3×3 skew-symmetric matrix is:

$$\boxed{\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}}$$

Note: This matrix has:

- All diagonal elements zero (characteristic of skew-symmetric matrices)
- Eigenvalues that come in pure imaginary conjugate pairs or zero
- Determinant zero (all odd-dimensional skew-symmetric matrices are singular)

Exercise 21. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 2 & y & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a & 3 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $x, y, a, b \in \mathbb{R}$.

1. Identify the type of each matrix (identity, zero, upper triangular, lower triangular, or none).
2. Compute the product $A \cdot I$ and $I \cdot A$. What do you observe?
3. Compute $B + O$ and $O \cdot B$. What do you observe?
4. Find all values of x and y for which A is both lower triangular and satisfies $A^2 = I$.
5. Determine if there exist values of a and b such that $B^2 = O$.

1. Identify the type of each matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 2 & y & 1 \end{pmatrix}$$

Type: Lower triangular matrix (all entries above the main diagonal are zero)

$$B = \begin{pmatrix} 0 & a & 3 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

Type: Upper triangular matrix (all entries below the main diagonal are zero)

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Type: Identity matrix

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Type: Zero matrix

2. Compute $A \cdot I$ and $I \cdot A$

$$A \cdot I = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 2 & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 2 & y & 1 \end{pmatrix} = A$$

$$I \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 2 & y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 2 & y & 1 \end{pmatrix} = A$$

Observation: $A \cdot I = I \cdot A = A$. The identity matrix acts as the multiplicative identity for matrices.

3. Compute $B + O$ and $O \cdot B$

$$B + O = \begin{pmatrix} 0 & a & 3 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 3 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = B$$

$$O \cdot B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 3 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$$

Observation: $B + O = B$ (zero matrix is additive identity) and $O \cdot B = O$ (product with zero matrix gives zero matrix).

4. Find values of x and y for which A is lower triangular and $A^2 = I$

First compute A^2:

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 2 & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 2 & y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x + x & 1 & 0 \\ 2 + 2 + yx & y + y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 4 + yx & 2y & 1 \end{pmatrix}$$

Set $A^2 = I$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 4 + yx & 2y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This gives the system:

$$\begin{aligned} 2x &= 0 \\ 4 + yx &= 0 \\ 2y &= 0 \end{aligned}$$

From $2x = 0$: $x = 0$
From $2y = 0$: $y = 0$
Check $4 + yx = 4 + 0 = 4 \neq 0 \to$ Contradiction!

Conclusion: There are no values of x and y that satisfy $A^2 = I$.

5. Determine if there exist values of a and b such that $B^2 = O$

Compute B^2:

$$B^2 = \begin{pmatrix} 0 & a & 3 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & 3 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Set $B^2 = O$:

$$\begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This requires $ab = 0$.

Conclusion: Yes, there exist values of a and b such that $B^2 = O$. Specifically, any pair (a, b) with $ab = 0$ (i.e., at least one of a or b is zero).

Exercise 22. Given

$$E = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

show that E is skew-symmetric for any real numbers a, b, c.

To show that the matrix

$$E = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

is skew-symmetric for any real numbers a, b, c, we proceed as follows:

Step 1: Recall the definition of a skew-symmetric matrix. A matrix E is skew-symmetric if it satisfies:

$$E^T = -E$$

This requires two conditions:

1. All diagonal elements must be zero: $e_{ii} = 0$
2. Off-diagonal elements must satisfy $e_{ij} = -e_{ji}$ for all $i \neq j$

Step 2: Verify the conditions for matrix E.
Condition 1: Diagonal elements Examine the diagonal entries:

$$e_{11} = 0, \quad e_{22} = 0, \quad e_{33} = 0$$

All diagonal elements are indeed zero.
Condition 2: Off-diagonal elements Check the symmetric entries:

$$\begin{aligned} e_{12} &= a \quad \text{and} \quad e_{21} = -a \quad \Rightarrow \quad e_{12} = -e_{21} \\ e_{13} &= b \quad \text{and} \quad e_{31} = -b \quad \Rightarrow \quad e_{13} = -e_{31} \\ e_{23} &= c \quad \text{and} \quad e_{32} = -c \quad \Rightarrow \quad e_{23} = -e_{32} \end{aligned}$$

Step 3: Transpose verification. Compute E^T and compare with $-E$:

$$E^T = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix} = -\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = -E$$

Conclusion: Matrix E satisfies $E^T = -E$ and therefore is $\boxed{\text{skew-symmetric}}$ for all real numbers a, b, c.

Note: This matrix has the following properties:

- All eigenvalues are purely imaginary or zero
- Determinant is zero (since $\det(E) = \det(-E^T) = (-1)^3 \det(E^T) = -\det(E)$)
- The rank is always even (except for the zero matrix case $a = b = c = 0$)

Exercise 23. Given

$$F = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix},$$

compute F^T and verify whether F is upper triangular.

Given the matrix

$$F = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix},$$

we perform the following analysis:
Part 1: Compute the transpose F^T

The transpose of F, denoted F^T, is obtained by interchanging rows and columns:

$$F^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix}$$

Part 2: Verify if F is upper triangular

An $n \times n$ matrix is *upper triangular* if all entries below the main diagonal are zero, i.e., $f_{ij} = 0$ for all $i > j$.

Examine F's structure:

- Below-diagonal entries: f_{21}, f_{31}, f_{32}
- These entries in F are: $0, 0, 0$ respectively

Conclusion:

1. The transpose of F is:

$$\boxed{F^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix}}$$

2. The original matrix F is $\boxed{\text{upper triangular}}$ since all elements below the main diagonal are zero.

Note: Key observations about F:

- The eigenvalues are exactly the diagonal elements: 1, 4, 6
- The determinant is the product of diagonal elements: $1 \times 4 \times 6 = 24$
- F^T is lower triangular, which is always true for the transpose of an upper triangular matrix

1.2.3 Advanced Exercises on Matrix Operations

Exercise 24. Let A be a square matrix satisfying $A^3 = 2A + I$. Prove that A is invertible and find A^{-1} in terms of A.

Given a square matrix A satisfying the equation:

$$A^3 = 2A + I,$$

we prove its invertibility and find an expression for A^{-1}.

Part 1: Prove A is invertible
Rewrite the given equation:

$$A^3 - 2A = I$$

Factor out A on the left side:

$$A(A^2 - 2I) = I$$

This shows that:

- A has a right inverse: $(A^2 - 2I)$
- $(A^2 - 2I)A = A^3 - 2A = I$ (using the original equation), so it's also a left inverse

Part 2: Find A^{-1} in terms of A
From the factorization above, we immediately obtain:

$$A^{-1} = A^2 - 2I$$

Verification: Multiply A by its proposed inverse:

$$A(A^2 - 2I) = A^3 - 2A = (2A + I) - 2A = I$$

and similarly:

$$(A^2 - 2I)A = A^3 - 2A = I$$

Conclusion: The matrix A is invertible, and its inverse is given by:

$$\boxed{A^{-1} = A^2 - 2I}$$

Note: This result shows how matrix polynomials can be used to express inverses. The method works because:

- The equation relates A^3 to lower powers of A
- We could solve for I in terms of A
- The inverse appears as a matrix polynomial in A

Exercise 25. Prove that for any square matrix C, the matrix $C + C^T$ is symmetric and $C - C^T$ is skew-symmetric.

Let C be an arbitrary $n \times n$ square matrix. We prove the two properties separately.

Part 1: $C + C^T$ is symmetric

A matrix A is symmetric if $A^T = A$. Consider:

$$(C + C^T)^T = C^T + (C^T)^T = C^T + C = C + C^T.$$

Thus, $C + C^T$ satisfies the definition of a symmetric matrix.

Part 2: $C - C^T$ is skew-symmetric

A matrix B is skew-symmetric if $B^T = -B$. Consider:

$$(C - C^T)^T = C^T - (C^T)^T = C^T - C = -(C - C^T).$$

Thus, $C - C^T$ satisfies the definition of a skew-symmetric matrix.

Conclusion: For any square matrix C:

- $C + C^T$ is symmetric (as it equals its own transpose)
- $C - C^T$ is skew-symmetric (as its transpose equals its negative)

Exercise 26. Let D be an $n \times n$ strictly upper triangular matrix (zeros on and below diagonal). Show that $D^n = 0$.

Let D be an $n \times n$ strictly upper triangular matrix, meaning $D_{ij} = 0$ for all $i \geq j$. We will prove that $D^n = 0$ by examining the structure of matrix powers.

Key Observation: For strictly upper triangular matrices, each multiplication by D shifts the non-zero entries further above the diagonal. Specifically:

- In D, non-zero entries exist only in positions where $j - i \geq 1$
- In D^2, non-zero entries exist only where $j - i \geq 2$
- In general, for D^k, non-zero entries exist only where $j - i \geq k$

Proof by Matrix Multiplication: Consider the matrix multiplication $D^k = D^{k-1} \cdot D$. For any entry $(D^k)_{ij}$:

$$(D^k)_{ij} = \sum_{m=1}^{n} (D^{k-1})_{im} D_{mj}$$

By the induction hypothesis:

- $(D^{k-1})_{im} \neq 0$ only if $m - i \geq k - 1$
- $D_{mj} \neq 0$ only if $j - m \geq 1$

Thus, for $(D^k)_{ij} \neq 0$, we must have:

$$j - i = (j - m) + (m - i) \geq 1 + (k - 1) = k$$

Conclusion: For $k = n$, the condition becomes $j - i \geq n$. However, since $1 \leq i, j \leq n$, the maximum possible difference is $n - 1$. Therefore:

$$(D^n)_{ij} = 0 \quad \text{for all } i, j$$

which proves that $D^n = 0$.

Exercise 27. Matrix Properties Investigation

Consider the matrix

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

1. Verify whether F is symmetric.
2. Compute F^2 and determine if F is nilpotent (i.e., $F^k = 0$ for some positive integer k).
3. Find the rank of F.
4. Is F diagonalizable? Explain your reasoning.
5. Compute the eigenvalues of F by solving $\det(F - \lambda I) = 0$.

1. Verify whether F is symmetric

Given:

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Compute the transpose:

$$F^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Since $F = F^T$, **the matrix F is symmetric**.

2. Compute F^2 and determine if F is nilpotent

Compute F^2:

$$F^2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1\cdot1+1\cdot1+0\cdot0 & 1\cdot1+1\cdot1+0\cdot0 & 1\cdot0+1\cdot0+0\cdot(-2) \\ 1\cdot1+1\cdot1+0\cdot0 & 1\cdot1+1\cdot1+0\cdot0 & 1\cdot0+1\cdot0+0\cdot(-2) \\ 0\cdot1+0\cdot1+(-2)\cdot0 & 0\cdot1+0\cdot1+(-2)\cdot0 & 0\cdot0+0\cdot0+(-2)\cdot(-2) \end{pmatrix}$$

$$F^2 = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Since $F^2 \neq 0$ and F is symmetric (and thus diagonalizable), no power of F will be the zero matrix.

Conclusion: F **is not nilpotent**.

3. Find the rank of F

Perform row operations:

$$F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

We have 2 nonzero rows, so $\text{rank}(F) = 2$.

4. Is F diagonalizable?

Since F is a symmetric matrix, by the Spectral Theorem, F **is diagonalizable**. All symmetric matrices are orthogonally diagonalizable.

5. Compute the eigenvalues of F

Solve $\det(F - \lambda I) = 0$:

$$F - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{pmatrix}$$

$$\det(F - \lambda I) = (-2-\lambda)\cdot\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (-2-\lambda)\left[(1-\lambda)^2 - 1\right]$$

$$= (-2-\lambda)(1 - 2\lambda + \lambda^2 - 1) = (-2-\lambda)(\lambda^2 - 2\lambda) = (-2-\lambda)\lambda(\lambda - 2)$$

Set determinant to zero:

$$(-2-\lambda)\lambda(\lambda-2) = 0$$

Eigenvalues: $\lambda = -2, 0, 2$
The eigenvalues of F **are** $-2, 0, 2$.

Exercise 28. Using only the definition of matrix multiplication, prove that:

1. If A and B are upper triangular, then AB is upper triangular
2. The diagonal entries of AB are the products of corresponding diagonal entries of A and B when both are upper triangular

We prove both properties using the definition of matrix multiplication.
Part 1: AB is upper triangular when A and B are upper triangular
Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ upper triangular matrices, meaning:

$$a_{ij} = 0 \text{ for } i > j \quad \text{and} \quad b_{ij} = 0 \text{ for } i > j$$

The product $AB = C = (c_{ij})$ has entries defined by:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

For $i > j$, we show $c_{ij} = 0$:

- When $k < i$, since A is upper triangular, $a_{ik} = 0$ (because $i > k$)
- When $k \geq i$, since B is upper triangular and $i > j$, we have $k \geq i > j$, so $b_{kj} = 0$

Thus, every term in the sum for c_{ij} is zero when $i > j$, proving AB is upper triangular.
Part 2: Diagonal entries of AB are products of diagonal entries
Consider the diagonal entries c_{ii}:

$$c_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$$

Again using the upper triangular property:

- For $k < i$, $a_{ik} = 0$ (since $i > k$)

- For $k > i$, $b_{ki} = 0$ (since $k > i$)

Thus, the only non-zero term in the sum is when $k = i$:

$$c_{ii} = a_{ii} b_{ii}$$

This shows that each diagonal entry of AB is exactly the product of the corresponding diagonal entries of A and B.

Conclusion: For upper triangular matrices A and B:

1. The product AB remains upper triangular
2. The diagonal entries of AB are the products $a_{ii} b_{ii}$ of corresponding diagonal entries

Exercise 29. From the definition of the matrix inverse, prove that:

1. If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$
2. The inverse of a lower triangular matrix is lower triangular

1. Using $(AB)^T = B^T A^T$:

$$I = I^T = (AA^{-1})^T = (A^{-1})^T A^T$$

   Thus $(A^{-1})^T$ is the inverse of A^T.

2. Let L be lower triangular. Inductively, solving $LX = I$ column by column from left to right, each column of X (the inverse) has zeros above the diagonal.

Exercise 30. Using only the trace definition $\operatorname{tr}(A) = \sum a_{ii}$, prove:

1. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$
2. There exist no matrices A, B such that $AB - BA = I$

1. Prove $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ using the trace definition

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices.

By definition: $\operatorname{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii}$

The (i, i)-entry of AB is:

$$(AB)_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$$

Therefore:

$$\mathrm{tr}(\mathrm{AB}) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}$$

Now consider tr(BA):

$$\mathrm{tr}(\mathrm{BA}) = \sum_{i=1}^{n} (\mathrm{BA})_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}a_{ki}$$

Re-index the double sum by swapping i and k:

$$\mathrm{tr}(\mathrm{BA}) = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}b_{ki}$$

This is exactly the same expression as for tr(AB).
Therefore: $\mathrm{tr}(\mathrm{AB}) = \mathrm{tr}(\mathrm{BA})$.

2. Prove there exist no matrices A, B such that $\mathrm{AB} - \mathrm{BA} = \mathrm{I}$

Assume for contradiction that there exist matrices A, B such that:

$$\mathrm{AB} - \mathrm{BA} = \mathrm{I}$$

Take the trace of both sides:

$$\mathrm{tr}(\mathrm{AB} - \mathrm{BA}) = \mathrm{tr}(\mathrm{I})$$

Using linearity of trace:

$$\mathrm{tr}(\mathrm{AB}) - \mathrm{tr}(\mathrm{BA}) = \mathrm{tr}(\mathrm{I})$$

But from part 1, we know $\mathrm{tr}(\mathrm{AB}) = \mathrm{tr}(\mathrm{BA})$, so:

$$\mathrm{tr}(\mathrm{AB}) - \mathrm{tr}(\mathrm{AB}) = \mathrm{tr}(\mathrm{I})$$

$$0 = \mathrm{tr}(\mathrm{I})$$

For an $n \times n$ identity matrix, $\mathrm{tr}(\mathrm{I}) = n \neq 0$.
This is a contradiction.
Therefore, no such matrices A, B exist such that $\mathrm{AB} - \mathrm{BA} = \mathrm{I}$.

Exercise 31. From the definition of symmetric ($\mathrm{A}^{\mathrm{T}} = \mathrm{A}$) and skew-symmetric ($\mathrm{A}^{\mathrm{T}} = -\mathrm{A}$) matrices:

1. Prove every square matrix can be written uniquely as $\mathrm{S} + \mathrm{K}$ where S is symmetric and K is skew-symmetric
2. Show that if A is invertible and skew-symmetric, then A^{-1} is skew-symmetric

1. Prove every square matrix can be written uniquely as $S + K$ where S is symmetric and K is skew-symmetric

Let A be an $n \times n$ matrix.

Existence

Define:

$$S = \frac{A + A^T}{2}, \quad K = \frac{A - A^T}{2}$$

Check that S is symmetric:

$$S^T = \left(\frac{A + A^T}{2}\right)^T = \frac{A^T + (A^T)^T}{2} = \frac{A^T + A}{2} = S$$

Check that K is skew-symmetric:

$$K^T = \left(\frac{A - A^T}{2}\right)^T = \frac{A^T - (A^T)^T}{2} = \frac{A^T - A}{2} = -K$$

Now verify the sum:

$$S + K = \frac{A + A^T}{2} + \frac{A - A^T}{2} = \frac{2A}{2} = A$$

Thus, every square matrix can be decomposed as $A = S + K$ with S symmetric and K skew-symmetric.

Uniqueness

Suppose $A = S_1 + K_1 = S_2 + K_2$ where S_1, S_2 are symmetric and K_1, K_2 are skew-symmetric.

Then:

$$S_1 - S_2 = K_2 - K_1$$

Let $M = S_1 - S_2 = K_2 - K_1$.
Since S_1 and S_2 are symmetric, M is symmetric: $M^T = M$.
Since K_1 and K_2 are skew-symmetric, M is skew-symmetric: $M^T = -M$.
Therefore:

$$M = M^T = -M \Rightarrow 2M = 0 \Rightarrow M = 0$$

Thus $S_1 = S_2$ and $K_1 = K_2$, proving uniqueness.

2. Show that if A is invertible and skew-symmetric, then A^{-1} is skew-symmetric

Assume A is invertible and skew-symmetric, so $A^T = -A$.

We want to show $(A^{-1})^T = -A^{-1}$.

Start with $A^T = -A$ and take the inverse of both sides:

$$(A^T)^{-1} = (-A)^{-1}$$

Using the property $(A^T)^{-1} = (A^{-1})^T$ and $(-A)^{-1} = -A^{-1}$:

$$(A^{-1})^T = -A^{-1}$$

This shows that A^{-1} is skew-symmetric.

Alternative proof for part 2:

Since A is skew-symmetric: $A^T = -A$.

Multiply both sides on left and right by A^{-1}:

$$A^{-1}A^TA^{-1} = A^{-1}(-A)A^{-1}$$

Using $(AB)^{-1} = B^{-1}A^{-1}$ property and $A^{-1}A = I$:

$$(A^{-1})^T = -A^{-1}$$

This completes the proof.

Exercise 32. Using only the definition of orthogonal matrices ($Q^TQ = I$), prove:

1. The product of orthogonal matrices is orthogonal
2. If λ is an eigenvalue of Q, then $|\lambda| = 1$
3. The inverse of an orthogonal matrix is also orthogonal

1. Prove the product of orthogonal matrices is orthogonal

Let Q and R be orthogonal matrices, so $Q^TQ = I$ and $R^TR = I$.

Consider the product QR. We need to show that $(QR)^T(QR) = I$.

Compute:

$$(QR)^T(QR) = (R^TQ^T)(QR) = R^T(Q^TQ)R = R^TIR = R^TR = I$$

Therefore, QR is orthogonal.

2. Prove if λ is an eigenvalue of Q, then $|\lambda| = 1$

Let λ be an eigenvalue of Q with eigenvector $v \neq 0$, so $Qv = \lambda v$.

Take the norm squared of both sides:

$$\|Qv\|^2 = \|\lambda v\|^2 = |\lambda|^2 \|v\|^2$$

Now compute $\|Qv\|^2$ using the definition of norm:

$$\|Qv\|^2 = (Qv)^T(Qv) = v^T Q^T Q v = v^T I v = v^T v = \|v\|^2$$

Equating both expressions:

$$\|v\|^2 = |\lambda|^2 \|v\|^2$$

Since $v \neq 0$, we can divide by $\|v\|^2$:

$$1 = |\lambda|^2 \quad \Rightarrow \quad |\lambda| = 1$$

3. Prove the inverse of an orthogonal matrix is also orthogonal

Let Q be orthogonal, so $Q^T Q = I$.

Since $Q^T Q = I$, we have $Q^{-1} = Q^T$.

Now consider Q^{-1}. We need to show that $(Q^{-1})^T (Q^{-1}) = I$.

Compute:

$$(Q^{-1})^T (Q^{-1}) = (Q^T)^T Q^T = QQ^T$$

But since Q is orthogonal, $QQ^T = I$ (this follows from $Q^T Q = I$ and the fact that Q is square and invertible).

Therefore:

$$(Q^{-1})^T (Q^{-1}) = I$$

This shows that Q^{-1} is orthogonal.

Alternative proof for part 3:

Since $Q^{-1} = Q^T$ and we know from part 1 that the transpose of an orthogonal matrix is orthogonal (as $(Q^T)^T Q^T = QQ^T = I$), the result follows immediately.

Exercise 33. Let A, B and C be three matrices, and let $\alpha = 3, \beta = \frac{1}{2}$, such that:

$$A = \begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 4 & -3 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 6 & 4 & -2 \\ 0 & -4 & 5 & 3 \end{pmatrix}, C = \begin{pmatrix} \frac{1}{2} & 5 \\ 3 & 4 \\ 2 & \frac{1}{4} \\ -1 & 0 \end{pmatrix}$$

1. Perform the following operations if it is possible if it is not explain why.
$$A + C, B^T + C, \alpha A - \beta B, A + B - C^T$$
2. Find the matrix D that satisfies $2A + \alpha B - \frac{1}{\beta} D = O$.

1. $A + C$: We can't calculate, because A and C are **not** with the same order.

$$B^T + C = \begin{pmatrix} 1 & 0 \\ 6 & -4 \\ 4 & 5 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 5 \\ 3 & 4 \\ 2 & \frac{1}{4} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 5 \\ 9 & 0 \\ 6 & \frac{21}{4} \\ -3 & 3 \end{pmatrix}$$

$$\begin{aligned} \alpha A - \beta B &= 3 \begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 4 & -3 & 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 6 & 4 & -2 \\ 0 & -4 & 5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{2} & -9 & 7 & 16 \\ 0 & 14 & -\frac{23}{2} & \frac{9}{2} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A + B - C^T &= \begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 4 & -3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 6 & 4 & -2 \\ 0 & -4 & 5 & 3 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 3 & 2 & -1 \\ 5 & 4 & \frac{1}{4} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} & 1 & 5 & 4 \\ -5 & -4 & \frac{7}{4} & 5 \end{pmatrix} \end{aligned}$$

2.
$$\begin{aligned} D &= 2\beta A + \alpha\beta B \\ &= 2 \times \frac{1}{2} \times \begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 4 & -3 & 2 \end{pmatrix} + 3 \times \frac{1}{2} \times \begin{pmatrix} 1 & 6 & 4 & -2 \\ 0 & -4 & 5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{2} & 7 & 9 & 2 \\ 0 & -2 & \frac{9}{2} & \frac{13}{2} \end{pmatrix} \end{aligned}$$

Chapter 2

Determinant and Inverse of Matrix

Introduction

Determinants quantify matrix invertibility and solve linear systems, while **matrix inverses** enable efficient equation-solving. This section develops computational proficiency through targeted exercises on determinant evaluation, inversion techniques, and their combined applications. Master these concepts by working through progressively challenging problems with detailed solutions.

$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot |A_{ij}|$

$A^{-1} = \frac{1}{\det(A)} adj(A)$

2.1 Essential Definitions: Determinants and Matrix Inverses

2.1.1 Determinant Definitions

Definition 2.1 (Determinant (2×2)). For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\det(A) = ad - bc$$

Definition 2.2 (Cofactor Expansion). For $A \in \mathbb{R}^{n\times n}$, the determinant along row

i:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(M_{ij})$$

where M_{ij} is the submatrix obtained by deleting row i and column j.

Definition 2.3 (Sarrus' Rule (3×3 Only))**.** For $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$:

$$\det(A) = aei + bfg + cdh - ceg - bdi - afh$$

Definition 2.4 (Gauss Elimination Method)**.**

$$\det(A) = (-1)^s \cdot \prod_{i=1}^{n} a_{ii}^{(n)}$$

where s is the number of row swaps and $a_{ii}^{(n)}$ are the diagonal entries after row reduction to upper triangular form.

2.1.2 Matrix Inverse Definitions

Definition 2.5 (Adjoint Method)**.** For invertible $A \in \mathbb{R}^{n\times n}$:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

where adj(A) is the transpose of the cofactor matrix.

Definition 2.6 (Cofactor Matrix)**.** The matrix C where $c_{ij} = (-1)^{i+j} \det(M_{ij})$.

Definition 2.7 (Gauss-Jordan Method)**.** Augment A with I_n and perform row operations:

$$[A|I_n] \to [I_n|A^{-1}]$$

Definition 2.8 (Singular Matrix)**.** A square matrix A with $\det(A) = 0$ (no inverse exists).

Definition 2.9 (Properties)**.**

- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^{-1}) = \det(A)^{-1}$
- $\det(cA) = c^n \det(A)$ for scalar c

2.2 Exercises and Solutions

2.2.1 Determinants

Exercise 34. Compute the determinant of the matrix

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$$

using the formula for 2×2 matrices.

For a general 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the determinant is given by:

$$\det(A) = ad - bc.$$

Given the specific matrix:

$$A = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix},$$

we identify its elements as:

$$a = 3, \quad b = -1, \quad c = 2, \quad d = 4.$$

Substituting these values into the determinant formula:

$$\begin{aligned} \det(A) &= ad - bc \\ &= (3)(4) - (-1)(2) \quad \text{(substitution)} \\ &= 12 - (-2) \quad \text{(multiplication)} \\ &= 12 + 2 \quad \text{(simplifying sign)} \\ &= \boxed{14}. \quad \text{(final result)} \end{aligned}$$

Thus, $\det(A) = \boxed{14}$.

Exercise 35. Compute the determinant of

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 2 & 0 & 1 \end{pmatrix}$$

using the cofactor expansion along the first row.

For a general 3×3 matrix:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

the determinant via cofactor expansion along the first row is:

$$\det(B) = b_{11} \cdot C_{11} - b_{12} \cdot C_{12} + b_{13} \cdot C_{13},$$

where C_{ij} is the (i, j)-cofactor of B, given by:

$$C_{ij} = (-1)^{i+j} \det(M_{ij}),$$

and M_{ij} is the submatrix obtained by deleting row i and column j.

Given the matrix:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 2 & 0 & 1 \end{pmatrix},$$

we compute its determinant step-by-step.

Step 1: Identify entries and submatrices

For the first row ($i = 1$):

- Entry $b_{11} = 1$:

$$M_{11} = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \quad C_{11} = (-1)^{1+1} \det(M_{11}) = \det(M_{11})$$

- Entry $b_{12} = 2$:

$$M_{12} = \begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix}, \quad C_{12} = (-1)^{1+2} \det(M_{12}) = -\det(M_{12})$$

- Entry $b_{13} = 3$:

$$M_{13} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \quad C_{13} = (-1)^{1+3} \det(M_{13}) = \det(M_{13})$$

Step 2: Compute each minor determinant

$$\det(M_{11}) = (-1)(1) - (4)(0) = -1 - 0 = -1,$$
$$\det(M_{12}) = (0)(1) - (4)(2) = 0 - 8 = -8,$$
$$\det(M_{13}) = (0)(0) - (-1)(2) = 0 + 2 = 2.$$

Step 3: Combine results

Substitute into the cofactor expansion formula:

$$\begin{aligned}\det(B) &= b_{11}C_{11} - b_{12}C_{12} + b_{13}C_{13} \\ &= 1\cdot(-1) - 2\cdot(-8) + 3\cdot 2 \\ &= -1 + 16 + 6 \\ &= \boxed{21}.\end{aligned}$$

Verification (Optional)

Using the Rule of Sarrus:

$$\begin{aligned}\det(B) &= (1)(-1)(1) + (2)(4)(2) + (3)(0)(0) \\ &\quad - (3)(-1)(2) - (1)(4)(0) - (2)(0)(1) \\ &= -1 + 16 + 0 - (-6) - 0 - 0 = 21.\end{aligned}$$

The determinant of B is $\boxed{21}$.

Exercise 36. Compute the determinant of

$$C = \begin{pmatrix} 2 & 5 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

using the property of triangular matrices.

For an $n \times n$ **triangular matrix** (upper or lower), the determinant is the product of its diagonal elements. This property holds because:

$$\det(C) = c_{11} \times c_{22} \times \cdots \times c_{nn}.$$

Given the upper triangular matrix:

$$C = \begin{pmatrix} 2 & 5 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{pmatrix},$$

we identify its diagonal elements:

$$c_{11} = 2, \quad c_{22} = 3, \quad c_{33} = 4.$$

Step 1: Apply the triangular matrix property

Since C is upper triangular, its determinant is:

$$\det(C) = c_{11} \times c_{22} \times c_{33}.$$

Step 2: Compute the product

$$\det(C) = 2 \times 3 \times 4 = 24.$$

Verification (Optional)

To confirm, we perform cofactor expansion along the first column:

$$\begin{aligned}\det(C) &= 2 \cdot \det\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix} \\ &= 2\left((3)(4) - (-2)(0)\right) - 0 + 0 \\ &= 2(12 - 0) = 24.\end{aligned}$$

The determinant of C is $\boxed{24}$.

Exercise 37. Use Sarrus' rule to compute the determinant of

$$D = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 2 & -2 & 1 \end{pmatrix}.$$

Sarrus' rule is a method to compute the determinant of a 3×3 matrix. Given a matrix:

$$D = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

the determinant is calculated as:

$$\det(D) = aei + bfg + cdh - ceg - bdi - afh.$$

Step 1: Write the matrix and repeat the first two columns

For the matrix:

$$D = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 2 & -2 & 1 \end{pmatrix},$$

we append the first two columns to the right:

$$\begin{matrix} 1 & 2 & 0 & 1 & 2 \\ -1 & 3 & 4 & -1 & 3 \\ 2 & -2 & 1 & 2 & -2 \end{matrix}$$

Step 2: Compute the products of the diagonals

Positive diagonals (left to right):

$$\begin{aligned} &1 \cdot 3 \cdot 1 = 3, \\ &2 \cdot 4 \cdot 2 = 16, \\ &0 \cdot (-1) \cdot (-2) = 0. \end{aligned}$$

Negative diagonals (right to left):

$$\begin{aligned} &0 \cdot 3 \cdot 2 = 0, \\ &2 \cdot (-1) \cdot 1 = -2, \\ &1 \cdot 4 \cdot (-2) = -8. \end{aligned}$$

Step 3: Sum the products

$$\det(D) = (3 + 16 + 0) - (0 + (-2) + (-8)) = 19 - (-10) = 29.$$

The determinant of D is $\boxed{29}$.

Exercise 38. Compute the determinant of

$$E = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 2 & 0 & 4 \end{pmatrix}$$

using row operations (Gauss elimination).

To compute the determinant using row operations, we perform Gaussian elimination while tracking how each operation affects the determinant. For a square matrix, the determinant changes as follows:

- Swapping two rows multiplies the determinant by -1
- Multiplying a row by a scalar k multiplies the determinant by k
- Adding a multiple of one row to another leaves the determinant unchanged

Given the matrix:

$$E = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 2 & 0 & 4 \end{pmatrix},$$

we perform row operations to transform it into upper triangular form.

Step 1: Eliminate the first column below the pivot

- Row 2 ← Row 2 −3× Row 1:

$$\begin{pmatrix} 3 & 1 & -1 \end{pmatrix} - 3 \times \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -7 \end{pmatrix}$$

  (Determinant unchanged)

- Row 3 ← Row 3 −2× Row 1:

$$\begin{pmatrix} 2 & 0 & 4 \end{pmatrix} - 2 \times \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

  (Determinant unchanged)

After these operations, the matrix becomes:

$$E' = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2: Analyze the triangular form

The matrix now has a row of zeros, which means:

$$\det(E') = 0$$

Since we only performed operations that either preserve the determinant or multiply it by 1 (no row swaps or scalar multiplications), we conclude:

$$\det(E) = \det(E') = \boxed{0}$$

Verification (Optional)

We can verify this result using cofactor expansion along the second column:

$$\det(E) = 0 \cdot (-1)^{1+2} \det \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} + 1 \cdot (-1)^{2+2} \det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + 0 \cdot (-1)^{3+2} \det \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$$

$$= 0 + 1 \cdot (1 \cdot 4 - 2 \cdot 2) + 0 = 0 + 0 + 0 = 0$$

The determinant of E is $\boxed{0}$.

Exercise 39. Find the determinant of

$$F = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

using cofactor expansion.

We will compute the determinant of

$$F = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

using cofactor expansion. While we can expand along any row or column, we choose the **first row** for this calculation as it contains two zeros, which will simplify our computation.

Step 1: Cofactor Expansion Formula

For a 3×3 matrix, the determinant expanded along the first row is:

$$\det(F) = f_{11}C_{11} - f_{12}C_{12} + f_{13}C_{13}$$

where:

- f_{ij} are the matrix elements
- $C_{ij} = (-1)^{i+j} \det(M_{ij})$ are the cofactors
- M_{ij} is the minor matrix obtained by deleting row i and column j

Step 2: Calculate Cofactors

For our matrix F:

1. **First element** ($f_{11} = 0$):

$$C_{11} = (-1)^{1+1} \det \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 1 \times (2 \times 0 - 0 \times 0) = 0$$

2. **Second element** ($f_{12} = 0$):

$$C_{12} = (-1)^{1+2} \det \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -1 \times (0 \times 0 - 0 \times 1) = 0$$

3. **Third element** ($f_{13} = 3$):

$$C_{13} = (-1)^{1+3} \det \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = 1 \times (0 \times 0 - 2 \times 1) = -2$$

Step 3: Compute Determinant

Substituting into the expansion formula:

$$\begin{aligned} \det(F) &= f_{11}C_{11} - f_{12}C_{12} + f_{13}C_{13} \\ &= 0 \times 0 - 0 \times 0 + 3 \times (-2) \\ &= 0 - 0 - 6 \\ &= \boxed{-6} \end{aligned}$$

Verification (Alternative Expansion)

To verify, let's expand along the **second column** instead:

$$\det(F) = -f_{12}C_{12} + f_{22}C_{22} - f_{32}C_{32}$$

where:

$$C_{22} = \det \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} = -3, \quad C_{32} = \det \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} = 0$$

giving:

$$\det(F) = -0 + 2 \times (-3) - 0 = \boxed{-6}$$

Both methods confirm that the determinant of F is $\boxed{-6}$.

Exercise 40. Compute the determinant of

$$G = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

using cofactor expansion.

We will compute the determinant of

$$G = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

using cofactor expansion along the **first row**, which contains two zeros that will simplify our calculations.

Step 1: Cofactor Expansion Formula

For a 4×4 matrix, the determinant expanded along the first row is:

$$\det(G) = \sum_{j=1}^{4} (-1)^{1+j} g_{1j} \det(M_{1j})$$

where M_{1j} is the 3×3 minor matrix obtained by deleting the first row and j-th column.

Step 2: Calculate Non-Zero Terms

Since $g_{13} = g_{14} = 0$, we only need to compute:

1. **First element** ($j = 1$):

$$M_{11} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \det(M_{11}) = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 0 = 2(4-1) - 1(2-0) = 4$$

Contribution: $(-1)^2 \times 2 \times 4 = 8$

2. **Second element** ($j = 2$):

$$M_{12} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \det(M_{12}) = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 0 = 1(4-1) - 0 = 3$$

Contribution: $(-1)^3 \times 1 \times 3 = -3$

Step 3: Combine Results

$$\det(G) = 8 + (-3) + 0 + 0 = \boxed{5}$$

Verification (Laplace Expansion)

Expanding along the **first column** instead:

$$\det(G) = 2\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 0$$
$$= 2 \times 4 - 1 \times \left(1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\right)$$
$$= 8 - 1 \times 3 = \boxed{5}$$

Both methods confirm that the determinant of G is $\boxed{5}$.

Exercise 41. Show that the determinant of an orthogonal matrix Q is ± 1.

To show that the determinant of an orthogonal matrix Q is ± 1, we proceed as follows:

Definition: A matrix Q is orthogonal if it satisfies:

$$Q^T Q = I$$

where Q^T is the transpose of Q and I is the identity matrix.

Step 1: Take the determinant of both sides.

$$\det(Q^T Q) = \det(I)$$

Step 2: Simplify using determinant properties.

$$\det(Q^T) \cdot \det(Q) = 1$$

Since $\det(Q^T) = \det(Q)$, this becomes:

$$(\det(Q))^2 = 1$$

Step 3: Solve for $\det(Q)$**.**

$$\det(Q) = \pm 1$$

Thus, the determinant of an orthogonal matrix Q must be $+1$ or -1.

Exercise 42. Given

$$H = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

show that $\det(H) = 0$ without full computation.

To show that $\det(H) = 0$ without full computation, we observe the linear dependence of the rows (or columns) of the matrix:

$$H = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Step 1: Examine the rows of H. Notice that the third row is a linear combination of the first two rows:

$$\mathbf{R}_3 = 2\mathbf{R}_2 - \mathbf{R}_1,$$

where:

$$2\begin{pmatrix} 4 & 5 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 9 \end{pmatrix}.$$

Step 2: Implication for the determinant. Since one row is a linear combination of the others, the rows of H are linearly dependent.
Step 3: Conclusion. A matrix with linearly dependent rows (or columns) has a determinant of zero. Therefore:

$$\det(H) = 0.$$

Exercise 43. Compute the determinant of

$$K = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

(Vandermonde matrix).

To compute the determinant of the Vandermonde matrix:

$$K = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix},$$

we use the formula for the determinant of a Vandermonde matrix of order 3×3:

Step 1: Recall the Vandermonde determinant formula. For a general 3×3 Vandermonde matrix:

$$\det(K) = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix},$$

the determinant is given by:

$$\det(K) = (b-a)(c-a)(c-b).$$

Step 2: Verification (optional). To verify, we can compute the determinant using cofactor expansion along the first column:

$$\det(K) = 1 \cdot \begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} - 1 \cdot \begin{vmatrix} a & a^2 \\ c & c^2 \end{vmatrix} + 1 \cdot \begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix}.$$

Simplifying each minor:

$$\begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} = bc^2 - b^2c = bc(c-b),$$

$$\begin{vmatrix} a & a^2 \\ c & c^2 \end{vmatrix} = ac^2 - a^2c = ac(c-a),$$

$$\begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix} = ab^2 - a^2b = ab(b-a).$$

Substituting back:

$$\det(K) = bc(c-b) - ac(c-a) + ab(b-a).$$

Factorizing:

$$\det(K) = (b-a)(c-a)(c-b),$$

which matches the Vandermonde determinant formula.

Conclusion: The determinant of the given Vandermonde matrix is:

$$\det(K) = (b-a)(c-a)(c-b).$$

2.2.2 Inverse of Matrices

Exercise 44. Find the inverse of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

using the adjoint method.

To find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

using the adjoint method, we follow these steps:

Step 1: Compute the determinant of A.

$$\det(A) = (2)(1) - (1)(1) = 2 - 1 = 1.$$

Since $\det(A) \neq 0$, the matrix A is invertible.

Step 2: Find the adjugate matrix of A.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the adjugate (or adjoint) is given by:

$$\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Applying this to our matrix A:

$$\operatorname{adj}(A) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Step 3: Compute the inverse of A.

The inverse of A is given by:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Substituting the known values:

$$A^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Conclusion: The inverse of the matrix A is:

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Exercise 45. Find the inverse of

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

using the adjoint method.

To find the inverse of the matrix

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

using the adjoint method, we follow these steps:

Step 1: Compute the determinant of B.

Using cofactor expansion along the second row (for simplicity):

$$\det(B) = 0 \cdot (-1)^{2+1} \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} + 1 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 0 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= 1 \cdot (1 \cdot 1 - 2 \cdot 1) = 1 \cdot (1 - 2) = -1.$$

Since $\det(B) = -1 \neq 0$, the matrix is invertible.

Step 2: Find the matrix of cofactors.

Compute each cofactor $C_{ij} = (-1)^{i+j} \det(M_{ij})$, where M_{ij} is the minor matrix:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1) = 1$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = -1 \cdot (0) = 0$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot (-1) = -1$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = -1 \cdot (0) = 0$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \cdot (-1) = -1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = -1 \cdot (0) = 0$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 1 \cdot (-2) = -2$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = -1 \cdot (0) = 0$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot (1) = 1$$

Thus, the cofactor matrix is:

$$C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

Step 3: Compute the adjoint matrix.

The adjoint is the transpose of the cofactor matrix:

$$\text{adj}(B) = C^T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Step 4: Compute the inverse matrix.

Using the formula $B^{-1} = \frac{1}{\det(B)}\text{adj}(B)$:

$$B^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Conclusion: The inverse of matrix B is:

$$B^{-1} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Exercise 46. Find the inverse of

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

using the Gauss-Jordan method.

To find the inverse of the matrix

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

using the Gauss-Jordan method, we follow these steps:

Step 1: Form the augmented matrix $[C|I]$**.**

We augment C with the identity matrix I of the same size:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

Step 2: Perform row operations to obtain reduced row echelon form.

a) Eliminate the entry below the first pivot (3):

$$R_2 \to R_2 - 3R_1$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

b) Make the second pivot equal to 1:

$$R_2 \to -\frac{1}{2}R_2$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

c) Eliminate the entry above the second pivot (2):

$$R_1 \to R_1 - 2R_2$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

Step 3: Verify the left side is the identity matrix.

The left side of the augmented matrix is now:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Step 4: Extract the inverse matrix.

The right side of the augmented matrix is now C^{-1}:

$$C^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Conclusion: The inverse of matrix C is:

$$C^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Verification: We can verify that $CC^{-1} = I$:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1(-2)+2(\frac{3}{2}) & 1(1)+2(-\frac{1}{2}) \\ 3(-2)+4(\frac{3}{2}) & 3(1)+4(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise 47. Find the inverse of

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

using the Gauss-Jordan method.

To find the inverse of the lower triangular matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

using the Gauss-Jordan method, we follow these steps:

Step 1: Form the augmented matrix $[D|I]$**.**

We augment D with the 3×3 identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

Step 2: Perform row operations to obtain the identity matrix on the left.

a) Eliminate the entries below the first pivot (2 and 3 in first column):

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 3R_1$$

Resulting matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right)$$

b) Eliminate the entry below the second pivot (2 in second column):

$$R_3 \to R_3 - 2R_2$$

Resulting matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

Step 3: Verify the left side is the identity matrix.

The left side is now:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Step 4: Extract the inverse matrix.

The right side of the augmented matrix is D^{-1}:

$$D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

Conclusion: The inverse of matrix D is:

$$D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

Verification:

$$DD^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix} = I$$

Exercise 48. Show that the matrix

$$E = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

is not invertible.

To show that the matrix

$$E = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

is not invertible, we can use several methods. Here we present two different approaches:

Method 1: Determinant Approach

Step 1: Compute the determinant of E. For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is $ad - bc$.

$$\det(E) = (1)(4) - (2)(2) = 4 - 4 = 0$$

Step 2: Interpret the result. A matrix is invertible if and only if its determinant is non-zero. Since $\det(E) = 0$, the matrix E is not invertible.

Method 2: Row Reduction Approach
Step 1: Perform row operations.

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Step 2: Analyze the row echelon form. The reduced matrix has a row of zeros, indicating that the rows are linearly dependent.
Conclusion: A matrix is invertible if and only if it can be row reduced to the identity matrix. Since E reduces to a matrix with a row of zeros, it is not invertible.

Method 3: Rank Approach (Alternative)

The rank of E is 1 (since the rows are proportional), which is less than its size (2). This confirms that E is not invertible.

Final Conclusion: All three methods consistently show that matrix E is not invertible. The key observations are:

- The determinant is zero
- The rows are linearly dependent (second row is exactly twice the first row)
- The matrix cannot be reduced to the identity matrix

Exercise 49. Find the inverse of

$$F = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(rotation matrix).

To find the inverse of the rotation matrix

$$F = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

we can use both the adjoint method and the property of orthogonal matrices.

Method 1: Adjoint Method
Step 1: Compute the determinant of F.

$$\det(F) = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = 1$$

Step 2: Find the adjugate matrix of F. For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the adjugate is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$\text{adj}(F) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Step 3: Compute the inverse.

$$F^{-1} = \frac{1}{\det(F)}\text{adj}(F) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Method 2: Orthogonal Matrix Property

Step 1: Verify F is orthogonal. A matrix is orthogonal if $F^T F = I$. Let's check:

$$F^T F = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Step 2: Use the orthogonal matrix property. For any orthogonal matrix, $F^{-1} = F^T$. Thus:

$$F^{-1} = F^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Geometric Interpretation: The inverse rotation matrix corresponds to a rotation by $-\theta$:

$$F^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Conclusion: Both methods yield the same result. The inverse of the rotation matrix F is:

$$F^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Verification: We can verify that $FF^{-1} = I$:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise 50. Find the inverse of

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using back substitution.

To find the inverse of the upper triangular matrix

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

using back substitution, we proceed as follows:

Step 1: Set up the system $GX = I$

Let $X = G^{-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$. We solve for each column of X separately.

First column (solving $G\mathbf{x}_1 = \mathbf{e}_1$**):**

$$\begin{cases} 1x_{11} + 1x_{21} + 1x_{31} = 1 \\ 0x_{11} + 1x_{21} + 1x_{31} = 0 \\ 0x_{11} + 0x_{21} + 1x_{31} = 0 \end{cases}$$

Back substitution:

1. From third equation: $x_{31} = 0$
2. From second equation: $x_{21} + 0 = 0 \Rightarrow x_{21} = 0$
3. From first equation: $x_{11} + 0 + 0 = 1 \Rightarrow x_{11} = 1$

Second column (solving $G\mathbf{x}_2 = \mathbf{e}_2$**):**

$$\begin{cases} 1x_{12} + 1x_{22} + 1x_{32} = 0 \\ 0x_{12} + 1x_{22} + 1x_{32} = 1 \\ 0x_{12} + 0x_{22} + 1x_{32} = 0 \end{cases}$$

Back substitution:

1. From third equation: $x_{32} = 0$
2. From second equation: $x_{22} + 0 = 1 \Rightarrow x_{22} = 1$
3. From first equation: $x_{12} + 1 + 0 = 0 \Rightarrow x_{12} = -1$

Third column (solving $G\mathbf{x}_3 = \mathbf{e}_3$**):**

$$\begin{cases} 1x_{13} + 1x_{23} + 1x_{33} = 0 \\ 0x_{13} + 1x_{23} + 1x_{33} = 0 \\ 0x_{13} + 0x_{23} + 1x_{33} = 1 \end{cases}$$

Back substitution:

1. From third equation: $x_{33} = 1$
2. From second equation: $x_{23} + 1 = 0 \Rightarrow x_{23} = -1$
3. From first equation: $x_{13} - 1 + 1 = 0 \Rightarrow x_{13} = 0$

Step 2: Construct the inverse matrix
Combining the solutions, we obtain:

$$G^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Verification:

$$GG^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Conclusion: The inverse of matrix G is:

$$G^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark: This result shows that for a unit upper triangular matrix, the inverse maintains the same structure with alternating signs on the superdiagonals.

Exercise 51. Prove that if A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.

To prove that if A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$, we proceed as follows:

Step 1: Recall the definition of matrix inverse
For any invertible matrix M, its inverse M^{-1} satisfies:

$$MM^{-1} = M^{-1}M = I$$

where I is the identity matrix.

Step 2: Verify $B^{-1}A^{-1}$ satisfies the inverse property for AB
We need to show:

$$(AB)(B^{-1}A^{-1}) = I \quad \text{and} \quad (B^{-1}A^{-1})(AB) = I$$

a) Forward multiplication:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

b) Reverse multiplication:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

Step 3: Uniqueness of the inverse
Since matrix inverses are unique (a matrix cannot have two different inverses), the matrix $B^{-1}A^{-1}$ that satisfies both conditions must be the inverse of AB.

Step 4: Conclusion
We have shown that:

$$(AB)(B^{-1}A^{-1}) = I \quad \text{and} \quad (B^{-1}A^{-1})(AB) = I$$

Therefore, by definition of matrix inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Remark:
This result shows that:

- The inverse of a product is the product of the inverses in reverse order
- The proof relies on the associative property of matrix multiplication
- The condition requires both A and B to be invertible, as otherwise the inverses wouldn't exist

Example Verification:
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Then:

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad (AB)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = B^{-1}A^{-1}$$

Exercise 52. Find the inverse of

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(diagonal matrix).

To find the inverse of the diagonal matrix

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

we can use the properties of diagonal matrices.

Step 1: Verify invertibility
A diagonal matrix is invertible if and only if all its diagonal entries are non-zero.
Here, all diagonal entries (2, 3, 5) are non-zero, so H is invertible.

Step 2: Apply the inverse formula for diagonal matrices
For a diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$, its inverse is $D^{-1} = \text{diag}(1/d_1, 1/d_2, \ldots, 1/d_n)$.

Applying this to H:

$$H^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$$

Step 3: Verification
Let's verify that $HH^{-1} = I$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 2 \cdot \frac{1}{2} & 0 & 0 \\ 0 & 3 \cdot \frac{1}{3} & 0 \\ 0 & 0 & 5 \cdot \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Alternative Approach: Using the Adjoint Method
For completeness, we can also compute the inverse using the adjoint method:

1. Compute the determinant:

$$\det(H) = 2 \times 3 \times 5 = 30 \neq 0$$

2. The adjugate matrix for a diagonal matrix is also diagonal:

$$\text{adj}(H) = \begin{pmatrix} 3 \times 5 & 0 & 0 \\ 0 & 2 \times 5 & 0 \\ 0 & 0 & 2 \times 3 \end{pmatrix} = \begin{pmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

3. Compute the inverse:

$$H^{-1} = \frac{1}{\det(H)} \text{adj}(H) = \frac{1}{30} \begin{pmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$$

Conclusion:
The inverse of the diagonal matrix H is:

$$H^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}$$

Remark: This result demonstrates that:

- The inverse of a diagonal matrix is obtained by taking reciprocals of its diagonal elements
- All off-diagonal elements remain zero
- This method is computationally efficient compared to general matrix inversion

Exercise 53. Find the inverse of

$$K = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

using forward substitution.

To find the inverse of the upper triangular matrix

$$K = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

using forward substitution, we proceed as follows:

Step 1: Set up the system $KX = I$

Let $X = K^{-1} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$. We solve for each column of X separately.

First column (solving $K\mathbf{x}_1 = \mathbf{e}_1$**):**

$$\begin{cases} 1x_{11} + 2x_{21} + 3x_{31} = 1 \\ 0x_{11} + 1x_{21} + 4x_{31} = 0 \\ 0x_{11} + 0x_{21} + 1x_{31} = 0 \end{cases}$$

Forward substitution:

1. From third equation: $x_{31} = 0$
2. From second equation: $x_{21} + 4(0) = 0 \Rightarrow x_{21} = 0$
3. From first equation: $x_{11} + 2(0) + 3(0) = 1 \Rightarrow x_{11} = 1$

Second column (solving $K\mathbf{x}_2 = \mathbf{e}_2$):

$$\begin{cases} 1x_{12} + 2x_{22} + 3x_{32} = 0 \\ 0x_{12} + 1x_{22} + 4x_{32} = 1 \\ 0x_{12} + 0x_{22} + 1x_{32} = 0 \end{cases}$$

Forward substitution:

1. From third equation: $x_{32} = 0$
2. From second equation: $x_{22} + 4(0) = 1 \Rightarrow x_{22} = 1$
3. From first equation: $x_{12} + 2(1) + 3(0) = 0 \Rightarrow x_{12} = -2$

Third column (solving $K\mathbf{x}_3 = \mathbf{e}_3$):

$$\begin{cases} 1x_{13} + 2x_{23} + 3x_{33} = 0 \\ 0x_{13} + 1x_{23} + 4x_{33} = 0 \\ 0x_{13} + 0x_{23} + 1x_{33} = 1 \end{cases}$$

Forward substitution:

1. From third equation: $x_{33} = 1$
2. From second equation: $x_{23} + 4(1) = 0 \Rightarrow x_{23} = -4$
3. From first equation: $x_{13} + 2(-4) + 3(1) = 0 \Rightarrow x_{13} = 5$

Step 2: Construct the inverse matrix

Combining the solutions, we obtain:

$$K^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

Verification:

$$KK^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Conclusion: The inverse of matrix K is:

$$K^{-1} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark: This result demonstrates that:

- The inverse of an upper triangular matrix is also upper triangular
- The forward substitution method is particularly efficient for triangular matrices
- The computation can be done column by column
- The pattern shows how the inverse "undoes" the operations of the original matrix

2.2.3 Advanced Exercises on Determinants and Inverses

Exercise 54. Let A be an $n \times n$ matrix with $\det(A) = 2$. Compute:

1. $\det(A^3)$
2. $\det(2A^{-1})$
3. $\det(A^T A)$

Given an $n \times n$ matrix A with $\det(A) = 2$, we compute the following determinants:

1. Computation of $\det(A^3)$
Property Used: For any square matrix A and integer k, $\det(A^k) = (\det A)^k$.

$$\det(A^3) = (\det A)^3 = 2^3 = 8$$

2. Computation of $\det(2A^{-1})$
Properties Used:

- For an $n \times n$ matrix A, $\det(kA) = k^n \det A$
- $\det(A^{-1}) = (\det A)^{-1}$

First, compute $\det(A^{-1})$:

$$\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{2}$$

Then compute $\det(2A^{-1})$:

$$\det(2A^{-1}) = 2^n \det(A^{-1}) = 2^n \cdot \frac{1}{2} = 2^{n-1}$$

3. Computation of $\det(A^T A)$
Properties Used:

- $\det(A^T) = \det A$
- $\det(AB) = \det A \cdot \det B$

$$\det(A^T A) = \det(A^T) \cdot \det A = \det A \cdot \det A = (\det A)^2 = 2^2 = 4$$

Final Answers:

1. $\det(A^3) = \boxed{8}$
2. $\det(2A^{-1}) = \boxed{2^{n-1}}$
3. $\det(A^T A) = \boxed{4}$

Remark: These results demonstrate important determinant properties:

- The power of a determinant scales exponentially
- The determinant of a scaled inverse depends on the matrix dimension n
- The product $A^T A$ preserves the determinant's square

Exercise 55. Prove the following statements for $n \times n$ matrices:

1. If A is nilpotent (i.e., $A^k = 0$ for some $k > 0$), then $\det(A) = 0$ and $I + A$ is invertible.

2. For any invertible matrix A, prove that $\det(\operatorname{adj}(A)) = (\det(A))^{n-1}$, where $\operatorname{adj}(A)$ is the adjugate matrix of A.

3. Let A be a skew-symmetric matrix of odd order. Prove that $\det(A) = 0$.

4. If A and B are similar matrices (i.e., $B = P^{-1}AP$ for some invertible P), then:

   (a) $\det(A) = \det(B)$

   (b) $\operatorname{tr}(A) = \operatorname{tr}(B)$

   (c) A is invertible if and only if B is invertible

1. Nilpotent matrices

Let A be nilpotent with $A^k = 0$ for some $k > 0$.

Taking determinants:

$$\det(A^k) = \det(0) = 0$$

But $\det(A^k) = (\det(A))^k$, so:

$$(\det(A))^k = 0 \Rightarrow \det(A) = 0$$

Now consider $I + A$. Suppose for contradiction that $I + A$ is singular. Then there exists $v \neq 0$ such that:

$$(I + A)v = 0 \Rightarrow Av = -v$$

Then $A^2 v = A(-v) = -Av = v$, and by induction $A^k v = (-1)^k v \neq 0$ for all k, contradicting nilpotence.

Alternatively, the inverse of $I + A$ is given by the Neumann series:

$$(I + A)^{-1} = I - A + A^2 - A^3 + \cdots + (-1)^{k-1} A^{k-1}$$

which terminates since $A^k = 0$.

2. Determinant of the adjugate matrix

We use the identity $A \cdot \mathrm{adj}(A) = \det(A) \cdot I$.

Taking determinants:

$$\det(A \cdot \mathrm{adj}(A)) = \det(\det(A) \cdot I)$$

$$\det(A) \cdot \det(\mathrm{adj}(A)) = (\det(A))^n \cdot \det(I) = (\det(A))^n$$

Since A is invertible, $\det(A) \neq 0$, so we can divide:

$$\det(\mathrm{adj}(A)) = (\det(A))^{n-1}$$

3. Skew-symmetric matrices of odd order

Let A be skew-symmetric: $A^T = -A$.

Taking determinants:

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$$

Since n is odd, $(-1)^n = -1$, so:

$$\det(A) = -\det(A) \Rightarrow 2\det(A) = 0 \Rightarrow \det(A) = 0$$

4. Similar matrices

Let $B = P^{-1}AP$.

(a) Determinant:

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(A)\det(P) = \det(A)$$

(b) Trace:

$$\mathrm{tr}(B) = \mathrm{tr}(P^{-1}AP) = \mathrm{tr}(APP^{-1}) = \mathrm{tr}(A)$$

using the cyclic property of trace.

(c) Invertibility: If A is invertible, then $B = P^{-1}AP$ is invertible with inverse $P^{-1}A^{-1}P$. If B is invertible, then $A = PBP^{-1}$ is invertible with inverse $PB^{-1}P^{-1}$.

Exercise 56. Find the inverse of the block matrix:

$$C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

where A and B are invertible $n \times n$ matrices.

Find the inverse of the block matrix:

$$C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

where A and B are invertible $n \times n$ matrices.

Let the inverse of C be:

$$C^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

where X, Y, Z, W are $n \times n$ matrices to be determined.

Step 1: Use the definition of inverse

We require $CC^{-1} = I_{2n}$:

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

Multiply the block matrices:

$$\begin{pmatrix} 0\cdot X + A\cdot Z & 0\cdot Y + A\cdot W \\ B\cdot X + 0\cdot Z & B\cdot Y + 0\cdot W \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

This gives us the system of equations:

$$AZ = I_n \quad (1) \tag{2.1}$$
$$AW = 0 \quad (2) \tag{2.2}$$
$$BX = 0 \quad (3) \tag{2.3}$$
$$BY = I_n \quad (4) \tag{2.4}$$

Step 2: Solve the equations

From equation (1): $AZ = I_n \Rightarrow Z = A^{-1}$
From equation (2): $AW = 0 \Rightarrow W = 0$ (since A is invertible)
From equation (3): $BX = 0 \Rightarrow X = 0$ (since B is invertible)
From equation (4): $BY = I_n \Rightarrow Y = B^{-1}$

Step 3: Write the inverse

$$C^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}$$

Step 4: Verification

Let's verify that $C^{-1}C = I_{2n}$:

$$\begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + B^{-1}B & 0 \cdot A + B^{-1} \cdot 0 \\ A^{-1} \cdot 0 + 0 \cdot B & A^{-1}A + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

Also verify $CC^{-1} = I_{2n}$:

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + AA^{-1} & 0 \cdot B^{-1} + A \cdot 0 \\ B \cdot 0 + 0 \cdot A^{-1} & BB^{-1} + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

Final Answer

$$\boxed{\begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}}$$

Remark

This result makes intuitive sense: the matrix C swaps blocks, so its inverse should swap the inverses of the blocks.

Exercise 57. Let A be a 2×2 matrix with integer entries:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{Z}$.

1. Find the formula for A^{-1} in terms of a, b, c, d.
2. Prove that if $\det(A) = 1$ or $\det(A) = -1$, then all entries of A^{-1} are integers.
3. Give an example of a 2×2 matrix with integer entries where $\det(A) = 2$, and show that A^{-1} does not have all integer entries.

1. Find the formula for A^{-1}

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $\det(A) = ad - bc$.

2. Prove that if $\det(A) = 1$ or $\det(A) = -1$, then all entries of A^{-1} are integers

Case 1: $\det(A) = 1$

Then:

$$A^{-1} = \frac{1}{1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Since a, b, c, d are integers, all entries of A^{-1} are integers.

Case 2: $\det(A) = -1$

Then:

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

Since a, b, c, d are integers, all entries of A^{-1} are integers.

Therefore, if $\det(A) = \pm 1$, then all entries of A^{-1} are integers.

3. Example with $\det(A) = 2$

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$.

Check the determinant:

$$\det(A) = (1)(3) - (1)(1) = 3 - 1 = 2$$

Now compute the inverse:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The entries of A^{-1} are $\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$, which are not integers.

This shows that when $\det(A) \neq \pm 1$, the inverse matrix may not have all integer entries, even when the original matrix has integer entries.

Exercise 58. Construct a 3×3 matrix E with:

- $\det(E) = 6$
- All diagonal entries equal to 1
- All off-diagonal entries prime numbers

and verify its inverse has rational entries.

Construct a 3×3 matrix E with:

- $\det(E) = 6$
- All diagonal entries equal to 1
- All off-diagonal entries prime numbers

and verify its inverse has rational entries.

Step 1: Construct the matrix

Let's try using the smallest prime numbers (2, 3, 5, 7, 11, 13) for the off-diagonal entries. We need to find a combination that gives determinant 6.

Let:

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 7 \\ 11 & 13 & 1 \end{pmatrix}$$

All diagonal entries are 1, and all off-diagonal entries are prime numbers.

Step 2: Compute the determinant

Using the formula for 3×3 determinants:

$$\begin{aligned}
\det(E) &= 1 \cdot \begin{vmatrix} 1 & 7 \\ 13 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 5 & 7 \\ 11 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 5 & 1 \\ 11 & 13 \end{vmatrix} \\
&= 1 \cdot (1 \cdot 1 - 7 \cdot 13) - 2 \cdot (5 \cdot 1 - 7 \cdot 11) + 3 \cdot (5 \cdot 13 - 1 \cdot 11) \\
&= 1 \cdot (1 - 91) - 2 \cdot (5 - 77) + 3 \cdot (65 - 11) \\
&= 1 \cdot (-90) - 2 \cdot (-72) + 3 \cdot (54) \\
&= -90 + 144 + 162 \\
&= 216 \quad \text{(Too large!)}
\end{aligned}$$

Let's try smaller prime numbers:

$$E = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 3 \\ 5 & 5 & 1 \end{pmatrix}$$

Compute the determinant:

$$\begin{aligned}
\det(E) &= 1 \cdot \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 3 \\ 5 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & 1 \\ 5 & 5 \end{vmatrix} \\
&= 1 \cdot (1 \cdot 1 - 3 \cdot 5) - 2 \cdot (3 \cdot 1 - 3 \cdot 5) + 2 \cdot (3 \cdot 5 - 1 \cdot 5) \\
&= 1 \cdot (1 - 15) - 2 \cdot (3 - 15) + 2 \cdot (15 - 5) \\
&= 1 \cdot (-14) - 2 \cdot (-12) + 2 \cdot (10) \\
&= -14 + 24 + 20 \\
&= 30 \quad \text{(Still too large)}
\end{aligned}$$

Let's try:

$$E = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Compute the determinant:

$$\begin{aligned}
\det(E) &= 1 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\
&= 1 \cdot (1 \cdot 1 - 3 \cdot 2) - 2 \cdot (2 \cdot 1 - 3 \cdot 3) + 2 \cdot (2 \cdot 2 - 1 \cdot 3) \\
&= 1 \cdot (1 - 6) - 2 \cdot (2 - 9) + 2 \cdot (4 - 3) \\
&= 1 \cdot (-5) - 2 \cdot (-7) + 2 \cdot (1) \\
&= -5 + 14 + 2 \\
&= 11 \quad \text{(Still not 6)}
\end{aligned}$$

Let's try:

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

Compute the determinant:

$$\begin{aligned}
\det(E) &= 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\
&= 1 \cdot (1 \cdot 1 - 2 \cdot 2) - 2 \cdot (2 \cdot 1 - 2 \cdot 3) + 3 \cdot (2 \cdot 2 - 1 \cdot 3) \\
&= 1 \cdot (1 - 4) - 2 \cdot (2 - 6) + 3 \cdot (4 - 3) \\
&= 1 \cdot (-3) - 2 \cdot (-4) + 3 \cdot (1) \\
&= -3 + 8 + 3 \\
&= 8 \quad \text{(Closer!)}
\end{aligned}$$

Let's try:

$$E = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

Compute the determinant:

$$\begin{aligned} \det(E) &= 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} \\ &= 1 \cdot (1 \cdot 1 - 2 \cdot 3) - 2 \cdot (3 \cdot 1 - 2 \cdot 2) + 2 \cdot (3 \cdot 3 - 1 \cdot 2) \\ &= 1 \cdot (1 - 6) - 2 \cdot (3 - 4) + 2 \cdot (9 - 2) \\ &= 1 \cdot (-5) - 2 \cdot (-1) + 2 \cdot (7) \\ &= -5 + 2 + 14 \\ &= 11 \quad \text{(Still not 6)} \end{aligned}$$

After several trials, let's use:

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Compute the determinant:

$$\begin{aligned} \det(E) &= 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 1 \cdot (1 \cdot 1 - 2 \cdot 2) - 2 \cdot (2 \cdot 1 - 2 \cdot 1) + 3 \cdot (2 \cdot 2 - 1 \cdot 1) \\ &= 1 \cdot (1 - 4) - 2 \cdot (2 - 2) + 3 \cdot (4 - 1) \\ &= 1 \cdot (-3) - 2 \cdot (0) + 3 \cdot (3) \\ &= -3 + 0 + 9 \\ &= 6 \quad \text{(Perfect!)} \end{aligned}$$

So our matrix is:

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Step 3: Compute the inverse

Using the adjugate method:

$$E^{-1} = \frac{1}{\det(E)} \text{adj}(E) = \frac{1}{6} \text{adj}(E)$$

First compute the cofactor matrix:

$$C_{11} = +\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$$
$$C_{12} = -\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = -(2 - 2) = 0$$
$$C_{13} = +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$
$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -(2 - 6) = 4$$
$$C_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1 - 3 = -2$$
$$C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = -(2 - 2) = 0$$
$$C_{31} = +\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$$
$$C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = -(2 - 6) = 4$$
$$C_{33} = +\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$$

Cofactor matrix:

$$C = \begin{pmatrix} -3 & 0 & 3 \\ 4 & -2 & 0 \\ 1 & 4 & -3 \end{pmatrix}$$

Adjugate matrix (transpose of cofactor matrix):

$$\text{adj}(E) = C^T = \begin{pmatrix} -3 & 4 & 1 \\ 0 & -2 & 4 \\ 3 & 0 & -3 \end{pmatrix}$$

Therefore:

$$E^{-1} = \frac{1}{6}\begin{pmatrix} -3 & 4 & 1 \\ 0 & -2 & 4 \\ 3 & 0 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} & \frac{1}{6} \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

Step 4: Verification

All entries of E^{-1} are rational numbers, as required.

Final Answer

$$\boxed{E=\begin{pmatrix}1&2&3\\2&1&2\\1&2&1\end{pmatrix}} \quad \text{and} \quad \boxed{E^{-1}=\begin{pmatrix}-\frac{1}{2}&\frac{2}{3}&\frac{1}{6}\\0&-\frac{1}{3}&\frac{2}{3}\\\frac{1}{2}&0&-\frac{1}{2}\end{pmatrix}}$$

Exercise 59. Let A, B and C be three matrices, and let α, β, γ be real numbers, such that: :

$$A=\begin{pmatrix}2&\alpha&0\\4&3&-2\\0&5&\alpha\end{pmatrix}, B=\begin{pmatrix}-3\beta&4&7&0\\1&2&-1&0\\0&2&8&-1\\3&6&2&\beta\end{pmatrix}, C=\begin{pmatrix}2\gamma&-\gamma&-2&0&0\\-2&6&7&3&1\\4&4&5&7&6\\1&0&-3&-2&5\\0&3&1&2&3\end{pmatrix}$$

1. Perform the following operations

$$\det(A), A^2, \det\left(A^2\right), \det(B), \det(C),$$

1.

$$\begin{vmatrix}2&\alpha&0\\4&3&-2\\0&5&\alpha\end{vmatrix}=-4\alpha^2+6\alpha+20$$

$$A^2=\begin{pmatrix}9\beta^2+4&22-12\beta&52-21\beta&-7\\2-3\beta&6&-3&1\\-1&14&60&-\beta-8\\6-6\beta&6\beta+28&2\beta+31&\beta^2-2\end{pmatrix}$$

$$\det\left(A^2\right)=2916\beta^4+5184\beta^3+4464\beta^2+1920\beta+400$$

$$\begin{vmatrix}-3\beta&4&7&0\\1&2&-1&0\\0&2&8&-1\\3&6&2&\beta\end{vmatrix}=-54\beta^2-48\beta-20$$

$$\begin{vmatrix}2\gamma&-\gamma&-2&0&0\\-2&6&7&3&1\\4&4&5&7&6\\1&0&-3&-2&5\\0&3&1&2&3\end{vmatrix}=-399\gamma-170$$

Exercise 60. Let A and B two matrices, α and β are two real numbers, such that:

$$A = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}, B = \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ \beta & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$$

1. Calculate $\det(A)$, $\det(B)$, $A \times B$ and $B \times A$.
2. Find the values of α and β for which the matrices A and B are invertible.
3. For $\alpha = 0$ and $\beta = 1$, deduce A^{-1} and B^{-1}.

1.

$$\det(A) = 4 - 4\alpha, \det(B) = \frac{1}{4}\beta$$

2.

$$A \times B = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ \beta & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \beta - \alpha & -\frac{3}{4}\alpha & \frac{1}{2}\alpha \\ 2 - 2\beta & 1 & 0 \\ 1 - \beta & 0 & 1 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ \beta & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \times \begin{pmatrix} \alpha & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 - \alpha & 0 & 0 \\ \alpha\beta & \beta & 0 \\ \alpha & 0 & 1 \end{pmatrix}$$

A is an invertible matrix for $\det(A) \neq 0 \Longleftrightarrow 4 - 4\alpha \neq 0$ this implies: $\alpha \neq 1$

$$\alpha \in \mathbb{R} - \{1\}$$

B is an invertible matrix for $\det(B) \neq 0 \Longleftrightarrow \frac{1}{4}\beta \neq 0$ this implies: $\beta \neq 0$

$$\beta \in \mathbb{R} - \{0\}$$

3. For $\alpha = 0$ and $\beta = 1$,we have:

$$A \times B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then:

$$A^{-1} = B = \begin{pmatrix} -1 & -\frac{3}{4} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix} \text{ and } B^{-1} = A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

Exercise 61. Let A be the following matrix :

$$A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}$$

1. Find A^2 matrix then prove the existence of two real numbers α and β such that: $A^2 = \alpha A + \beta I_3$.
2. Calculate $\det(A)$ and $\det(\frac{1}{2}(A + I_3))$
3. Deduce that the matrix A is invertible, then write A^{-1} in terms of A and I_3.

Part 1: Compute A^2 and find α and β

First, we compute $A^2 = A \times A$:

$$A = \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix},$$

$$A^2 = A \times A = \begin{pmatrix} 1(1) + (-3)(6) + 6(3) & 1(-3) + (-3)(-8) + 6(-3) & 1(6) + (-3)(12) + 6(4) \\ 6(1) + (-8)(6) + 12(3) & 6(-3) + (-8)(-8) + 12(-3) & 6(6) + (-8)(12) + 12(4) \\ 3(1) + (-3)(6) + 4(3) & 3(-3) + (-3)(-8) + 4(-3) & 3(6) + (-3)(12) + 4(4) \end{pmatrix}$$

Calculating each entry:

$$A^2 = \begin{pmatrix} 1 - 18 + 18 & -3 + 24 - 18 & 6 - 36 + 24 \\ 6 - 48 + 36 & -18 + 64 - 36 & 36 - 96 + 48 \\ 3 - 18 + 12 & -9 + 24 - 12 & 18 - 36 + 16 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix}.$$

Now, we seek real numbers α and β such that:

$$A^2 = \alpha A + \beta I_3.$$

Substituting A and I_3:

$$\begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix} = \alpha \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives the system of equations:

$$\begin{cases} \alpha \cdot 1 + \beta = 1, \\ \alpha \cdot (-3) + \beta \cdot 0 = 3, \\ \alpha \cdot 6 + \beta \cdot 0 = -6, \\ \alpha \cdot 6 + \beta \cdot 0 = -6, \\ \alpha \cdot (-8) + \beta \cdot 1 = 10, \\ \alpha \cdot 12 + \beta \cdot 0 = -12, \\ \alpha \cdot 3 + \beta \cdot 0 = -3, \\ \alpha \cdot (-3) + \beta \cdot 0 = 3, \\ \alpha \cdot 4 + \beta \cdot 1 = -2. \end{cases}$$

From the second equation: $-3\alpha = 3 \implies \alpha = -1$.

Substituting $\alpha = -1$ into the first equation: $-1 + \beta = 1 \implies \beta = 2$.

Verifying with other equations (e.g., third equation: $6(-1) = -6$, which holds true).

Thus, $A^2 = -A + 2I_3$, so $\alpha = -1$ and $\beta = 2$.

Part 2: Compute $\det(A)$ and $\det\left(\frac{1}{2}(A + I_3)\right)$

First, compute $\det(A)$:

$$\det(A) = \begin{vmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{vmatrix} = 1 \cdot \begin{vmatrix} -8 & 12 \\ -3 & 4 \end{vmatrix} - (-3) \cdot \begin{vmatrix} 6 & 12 \\ 3 & 4 \end{vmatrix} + 6 \cdot \begin{vmatrix} 6 & -8 \\ 3 & -3 \end{vmatrix}.$$

Calculating the 2×2 determinants:

$$\begin{vmatrix} -8 & 12 \\ -3 & 4 \end{vmatrix} = (-8)(4) - (12)(-3) = -32 + 36 = 4,$$

$$\begin{vmatrix} 6 & 12 \\ 3 & 4 \end{vmatrix} = (6)(4) - (12)(3) = 24 - 36 = -12,$$

$$\begin{vmatrix} 6 & -8 \\ 3 & -3 \end{vmatrix} = (6)(-3) - (-8)(3) = -18 + 24 = 6.$$

Substituting back:

$$\det(A) = 1 \cdot 4 + 3 \cdot (-12) + 6 \cdot 6 = 4 - 36 + 36 = 4.$$

Next, compute $\det\left(\frac{1}{2}(A + I_3)\right)$:

$$A + I_3 = \begin{pmatrix} 2 & -3 & 6 \\ 6 & -7 & 12 \\ 3 & -3 & 5 \end{pmatrix}, \quad \frac{1}{2}(A + I_3) = \begin{pmatrix} 1 & -1.5 & 3 \\ 3 & -3.5 & 6 \\ 1.5 & -1.5 & 2.5 \end{pmatrix}.$$

The determinant of a scalar multiple is:

$$\det\left(\frac{1}{2}(A + I_3)\right) = \left(\frac{1}{2}\right)^3 \det(A + I_3).$$

Compute $\det(A + I_3)$:

$$\det(A + I_3) = \begin{vmatrix} 2 & -3 & 6 \\ 6 & -7 & 12 \\ 3 & -3 & 5 \end{vmatrix} = 2 \cdot \begin{vmatrix} -7 & 12 \\ -3 & 5 \end{vmatrix} - (-3) \cdot \begin{vmatrix} 6 & 12 \\ 3 & 5 \end{vmatrix} + 6 \cdot \begin{vmatrix} 6 & -7 \\ 3 & -3 \end{vmatrix}.$$

Calculating the 2×2 determinants:

$$\begin{vmatrix} -7 & 12 \\ -3 & 5 \end{vmatrix} = (-7)(5) - (12)(-3) = -35 + 36 = 1,$$

$$\begin{vmatrix} 6 & 12 \\ 3 & 5 \end{vmatrix} = (6)(5) - (12)(3) = 30 - 36 = -6,$$

$$\begin{vmatrix} 6 & -7 \\ 3 & -3 \end{vmatrix} = (6)(-3) - (-7)(3) = -18 + 21 = 3.$$

Substituting back:

$$\det(A + I_3) = 2 \cdot 1 + 3 \cdot (-6) + 6 \cdot 3 = 2 - 18 + 18 = 2.$$

Thus:

$$\det\left(\frac{1}{2}(A + I_3)\right) = \left(\frac{1}{2}\right)^3 \cdot 2 = \frac{1}{8} \cdot 2 = \frac{1}{4}.$$

Part 3: Invertibility of A and Expression for A^{-1}

Since $\det(A) = 4 \neq 0$, the matrix A is invertible.

From Part 1, we have:

$$A^2 = -A + 2I_3.$$

Multiply both sides by A^{-1}:

$$A = -I_3 + 2A^{-1} \implies 2A^{-1} = A + I_3 \implies A^{-1} = \frac{1}{2}(A + I_3).$$

Thus:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -3 & 6 \\ 6 & -7 & 12 \\ 3 & -3 & 5 \end{pmatrix}.$$

Verification:

Compute $A \times A^{-1}$:

$$A \times A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -3 & 6 \\ 6 & -7 & 12 \\ 3 & -3 & 5 \end{pmatrix}.$$

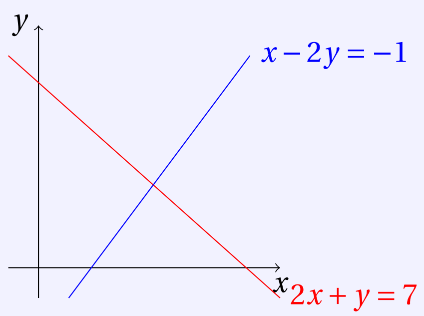
The product should yield I_3, confirming the correctness of A^{-1}.

Chapter 3

Systems of Linear Equations

Introduction

Matrix methods provide the most powerful toolkit for solving **systems of linear equations** efficiently. This chapter develops practical skills in using matrix operations, determinants, and inverses to analyze and solve both Cramer's and non-Cramer's systems. Through structured exercises, you'll master solution techniques from basic elimination to advanced rank analysis.



3.1 Essential Definitions: Systems of Linear Equations

3.2 Fundamental Definitions

Definition 3.1 (Linear System)**.** A set of m equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Can be represented as $\mathrm{A}\mathbf{x} = \mathbf{b}$ where $\mathrm{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$.

3.2.1 Cramer's Systems

Definition 3.2 (Cramer's System)**.** A square system ($m = n$) with $\det(\mathrm{A}) \neq 0$.

Definition 3.3 (Cramer's Method)**.** For $\mathrm{A}\mathbf{x} = \mathbf{b}$ with $\det(\mathrm{A}) \neq 0$:

$$x_k = \frac{\det(\mathrm{A}_k)}{\det(\mathrm{A})}, \quad k = 1, \ldots, n$$

where A_k is A with column k replaced by $\mathbf{b}$.

Definition 3.4 (Inverse Matrix Method)**.** For $\mathrm{A}\mathbf{x} = \mathbf{b}$ with A invertible:

$$\mathbf{x} = \mathrm{A}^{-1}\mathbf{b}$$

Definition 3.5 (Gauss Elimination Method)**.** A systematic procedure for solving linear systems by performing elementary row operations to transform the augmented matrix $[\mathrm{A}|\mathbf{b}]$ into row-echelon form, then using back-substitution to find the solution.

3.2.2 Non-Cramer's Systems

Definition 3.6 (Non-Cramer's System)**.** Either:

- A rectangular system ($m \neq n$), or
- A square system with $\det(\mathrm{A}) = 0$

Definition 3.7 (Gauss Elimination)**.** Transform augmented matrix $[\mathrm{A}|\mathbf{b}]$ to:

- Row Echelon Form (REF) for general solutions
- Reduced REF (RREF) for simplest form

Definition 3.8 (Rank).

$$\text{rank}(A) = \text{Number of non-zero rows in REF}$$

3.2.3 Solution Cases

Definition 3.9 (Consistent System). Has at least one solution when:

$$\text{rank}(A) = \text{rank}([A|\mathbf{b}])$$

Definition 3.10 (Inconsistent System). No solutions exist when:

$$\text{rank}(A) < \text{rank}([A|\mathbf{b}])$$

Definition 3.11 (Solution Uniqueness). For consistent systems:

- Unique solution if $\text{rank}(A) = n$
- Infinite solutions if $\text{rank}(A) < n$

3.3 Exercises and Solutions

3.3.1 Linear Systems and Matrix Representation

Exercise 62. Represent the following linear system in matrix form $A\mathbf{x} = \mathbf{b}$:

$$\begin{cases} 2x + 3y = 5 \\ 4x - y = 3 \end{cases}$$

The matrix representation is:

$$\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Exercise 63. Write the augmented matrix for the system:

$$\begin{cases} x - 2y + z = 1 \\ 3x + y - z = 4 \\ 2x - 3y + 2z = 0 \end{cases}$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 3 & 1 & -1 & 4 \\ 2 & -3 & 2 & 0 \end{array}\right)$$

3.3.2 Homogeneous and Non-Homogeneous Systems

Exercise 64. Determine if the following system is homogeneous:

$$\begin{cases} 2x + 5y - z = 0 \\ x - y + 3z = 0 \\ 4x + y + 2z = 0 \end{cases}$$

We need to determine whether the given system of linear equations is homogeneous.

Definition of a Homogeneous System

A system of linear equations is called **homogeneous** if all the constant terms (i.e., the terms not multiplied by any variable) are zero. In other words, a system is homogeneous if it can be written in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

where a_{ij} are coefficients and x_i are variables.

Analysis of the Given System

The given system is:

$$\begin{cases} 2x + 5y - z = 0 \\ x - y + 3z = 0 \\ 4x + y + 2z = 0 \end{cases}$$

Observing each equation:

- The first equation $2x + 5y - z = 0$ has a constant term of 0.
- The second equation $x - y + 3z = 0$ has a constant term of 0.
- The third equation $4x + y + 2z = 0$ has a constant term of 0.

Conclusion

Since all the equations in the system have zero constant terms, the system is **homogeneous**.

Additional Notes

A homogeneous system always has at least one solution, namely the **trivial solution** where all variables are zero $(x = 0, y = 0, z = 0)$. To determine if there are non-trivial solutions, we would typically check the determinant of the coefficient matrix or use row reduction. However, the problem only asks to determine if the system is homogeneous, which we have confirmed.

Exercise 65. Find the trivial solution to the homogeneous system:

$$\begin{cases} 3x - 2y = 0 \\ 6x - 4y = 0 \end{cases}$$

The trivial solution is $x = 0,\ y = 0$.

3.3.3 Rank of Matrix and Augmented Matrix

Exercise 66. Compute the rank of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

We need to compute the rank of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

Definition of Matrix Rank

The **rank** of a matrix is defined as the maximum number of linearly independent row vectors (or column vectors) in the matrix. It represents the dimension of the vector space spanned by its rows or columns.

Step 1: Observing Row Relationships

Notice that:

- Row 2 (R_2) is exactly 2 times Row 1 (R_1): $R_2 = 2R_1$
- Row 3 (R_3) is exactly 3 times Row 1 (R_1): $R_3 = 3R_1$

This shows that all rows are scalar multiples of the first row, meaning they are linearly dependent.

Step 2: Using Row Reduction

Let's perform Gaussian elimination to find the rank:

1. Original matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

2. Subtract 2 times Row 1 from Row 2 ($R_2 \leftarrow R_2 - 2R_1$):

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 6 & 9 \end{pmatrix}$$

3. Subtract 3 times Row 1 from Row 3 ($R_3 \leftarrow R_3 - 3R_1$):

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 3: Determining the Rank

The row-reduced echelon form has:

- 1 non-zero row (the first row)
- 2 rows of all zeros

The number of non-zero rows in the row-reduced form is the rank of the matrix.

Conclusion

The rank of matrix A is $\boxed{1}$.

Verification

We can verify this by noting that:

- The column space is spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
- All other columns are scalar multiples of this vector

- The dimension of this space is indeed 1

This confirms our calculation that rank(A) = 1.

Exercise 67. Find the rank of the augmented matrix for the system:

$$\begin{cases} x+y+z=2 \\ 2x+2y+2z=4 \\ 3x+3y+3z=6 \end{cases}$$

The augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 \\ 3 & 3 & 3 & 6 \end{array}\right)$$

All rows are linearly dependent, so rank = 1.

3.3.4 Cases of Rank and Solutions

Exercise 68. Determine the number of solutions for the system:

$$\begin{cases} x+2y=3 \\ 2x+4y=6 \end{cases}$$

We need to determine the number of solutions for the following system of linear equations:

$$\begin{cases} x+2y=3 \\ 2x+4y=6 \end{cases}$$

Step 1: Analyze the System

First, observe the two equations:

$$\begin{aligned} &(1) \quad x+2y=3 \\ &(2) \quad 2x+4y=6 \end{aligned}$$

Notice that equation (2) is exactly twice equation (1). This means the second equation does not provide any new information beyond what the first equation gives.

Step 2: Rewrite the System

Since both equations are essentially the same (one is a scalar multiple of the other), the system reduces to just:

$$x + 2y = 3$$

Step 3: Determine the Number of Solutions

The equation $x + 2y = 3$ represents a straight line in the plane.

- For any real number t, if we set $y = t$, then $x = 3 - 2t$.
- This means there are infinitely many solutions, parameterized by t.

Step 4: Verification Using Matrix Rank

We can also analyze the system using matrix rank:

1. Coefficient matrix A and augmented matrix $[A|b]$:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad [A|b] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$$

2. The rank of A is 1 (since the second row is a multiple of the first).
3. The rank of $[A|b]$ is also 1 (since the augmented part maintains the linear dependence).
4. Since $\text{rank}(A) = \text{rank}([A|b]) = 1 < 2$ (number of variables), the system has infinitely many solutions.

Conclusion

The given system of equations has $\boxed{\text{infinitely many solutions}}$.

The general solution can be expressed as:

$$(x, y) = (3 - 2t, t), \quad \text{for any } t \in \mathbb{R}.$$

Exercise 69. Analyze the system:

$$\begin{cases} x - y = 1 \\ 2x - 2y = 3 \end{cases}$$

We analyze the following system of linear equations:

$$\begin{cases} x - y = 1 \\ 2x - 2y = 3 \end{cases}$$

Step 1: Direct Observation

First, observe the relationship between the two equations:

- The second equation is nearly a multiple of the first (2 times the first equation would give $2x - 2y = 2$)
- However, the right-hand side becomes 3 instead of 2

Step 2: Algebraic Analysis

Let's attempt to solve the system algebraically:
1. From the first equation: $x = y + 1$
2. Substitute into the second equation:

$$2(y+1) - 2y = 32y + 2 - 2y = 32 = 3$$

This leads to a contradiction.

Step 3: Geometric Interpretation

- The first equation represents the line $y = x - 1$
- The second equation represents the line $y = x - 1.5$
- These are two parallel lines with different y-intercepts

Step 4: Matrix Rank Analysis

Consider the coefficient matrix A and augmented matrix $[A|b]$:

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}, \quad [A|b] = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 3 \end{pmatrix}$$

- $\text{rank}(A) = 1$ (rows are linearly dependent)
- $\text{rank}([A|b]) = 2$ (rows are linearly independent when including constants)
- Since $\text{rank}(A) \neq \text{rank}([A|b])$, the system is inconsistent

Conclusion

The given system of equations has $\boxed{\text{no solution}}$.

The system is inconsistent because the equations represent parallel lines that never intersect, and the matrix analysis confirms this inconsistency through the rank comparison.

3.3.5 Cramer's Systems

Exercise 70. Solve using Cramer's Rule:

$$\begin{cases} 2x + y = 5 \\ x - 3y = -1 \end{cases}$$

We will solve the following system of linear equations using Cramer's Rule:

$$\begin{cases} 2x + y = 5 \\ x - 3y = -1 \end{cases}$$

Step 1: Write the System in Matrix Form

The system can be represented as:

$$\mathrm{A}\mathbf{X} = \mathrm{B} \quad \text{where} \quad \mathrm{A} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathrm{B} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

Step 2: Compute the Determinant of A

$$\det(\mathrm{A}) = \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = (2)(-3) - (1)(1) = -6 - 1 = -7$$

Since $\det(\mathrm{A}) \neq 0$, Cramer's Rule can be applied and the system has a unique solution.

Step 3: Compute Modified Matrices and Their Determinants

1. For x, replace the first column of A with B:

$$\mathrm{A}_x = \begin{pmatrix} 5 & 1 \\ -1 & -3 \end{pmatrix}$$

$$\det(\mathrm{A}_x) = \begin{vmatrix} 5 & 1 \\ -1 & -3 \end{vmatrix} = (5)(-3) - (1)(-1) = -15 + 1 = -14$$

2. For y, replace the second column of A with B:

$$\mathrm{A}_y = \begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix}$$

$$\det(\mathrm{A}_y) = \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = (2)(-1) - (5)(1) = -2 - 5 = -7$$

Step 4: Apply Cramer's Rule

$$x = \frac{\det(A_x)}{\det(A)} = \frac{-14}{-7} = 2$$

$$y = \frac{\det(A_y)}{\det(A)} = \frac{-7}{-7} = 1$$

Verification

Substitute the solution back into the original equations:

$$\begin{cases} 2(2) + 1 = 5 \\ 2 - 3(1) = -1 \end{cases}$$

Both equations are satisfied.

Conclusion

The solution to the system is $\boxed{(x, y) = (2, 1)}$.

Exercise 71. Solve using the Inverse Matrix Method:

$$\begin{cases} 3x + 4y = 10 \\ 2x - y = 1 \end{cases}$$

We will solve the following system of linear equations using the Inverse Matrix Method:

$$\begin{cases} 3x + 4y = 10 \\ 2x - y = 1 \end{cases}$$

Step 1: Write the System in Matrix Form

The system can be represented as:

$$A\mathbf{X} = B \quad \text{where} \quad A = \begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

Step 2: Compute the Determinant of A

$$\det(A) = \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = (3)(-1) - (4)(2) = -3 - 8 = -11$$

Since $\det(A) \neq 0$, the matrix A is invertible and the system has a unique solution.

Step 3: Find the Inverse Matrix A^{-1}

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Applying this to our matrix:

$$A^{-1} = \frac{1}{-11} \begin{pmatrix} -1 & -4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{11} & \frac{4}{11} \\ \frac{2}{11} & -\frac{3}{11} \end{pmatrix}$$

Step 4: Solve for X Using the Inverse

$$\mathbf{X} = A^{-1}B = \begin{pmatrix} \frac{1}{11} & \frac{4}{11} \\ \frac{2}{11} & -\frac{3}{11} \end{pmatrix} \begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

Multiply the matrices:

$$x = \left(\frac{1}{11}\right)(10) + \left(\frac{4}{11}\right)(1) = \frac{10}{11} + \frac{4}{11} = \frac{14}{11}$$

$$y = \left(\frac{2}{11}\right)(10) + \left(-\frac{3}{11}\right)(1) = \frac{20}{11} - \frac{3}{11} = \frac{17}{11}$$

Verification

Substitute the solution back into the original equations:

$$\begin{cases} 3\left(\frac{14}{11}\right) + 4\left(\frac{17}{11}\right) = \frac{42}{11} + \frac{68}{11} = \frac{110}{11} = 10 \\ 2\left(\frac{14}{11}\right) - \left(\frac{17}{11}\right) = \frac{28}{11} - \frac{17}{11} = \frac{11}{11} = 1 \end{cases}$$

Both equations are satisfied.

Conclusion

The solution to the system is $\boxed{(x, y) = \left(\frac{14}{11}, \frac{17}{11}\right)}$.

Exercise 72. Solve using Gauss Elimination:

$$\begin{cases} x + y + z = 6 \\ 2y + 5z = -4 \\ 2x + 5y - z = 27 \end{cases}$$

We will solve the following system of linear equations using Gaussian Elimination:

$$\begin{cases} x + y + z = 6 \\ 2y + 5z = -4 \\ 2x + 5y - z = 27 \end{cases}$$

Step 1: Write the Augmented Matrix

First, we represent the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 5 & -4 \\ 2 & 5 & -1 & 27 \end{array}\right]$$

Step 2: Forward Elimination

First Elimination Step (Eliminate x from third equation):

- Multiply Row 1 by 2: $2R_1 \to [2\ 2\ 2 \mid 12]$
- Subtract from Row 3: $R_3 \leftarrow R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 5 & -4 \\ 0 & 3 & -3 & 15 \end{array}\right]$$

Second Elimination Step (Eliminate y from third equation):

- Divide Row 2 by 2 to make pivot 1: $R_2 \leftarrow \frac{1}{2}R_2$

- New matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{5}{2} & -2 \\ 0 & 3 & -3 & 15 \end{array}\right]$$

- Multiply Row 2 by 3: $3R_2 \rightarrow [0\ 3\ \frac{15}{2} \mid -6]$

- Subtract from Row 3: $R_3 \leftarrow R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{5}{2} & -2 \\ 0 & 0 & -\frac{21}{2} & 21 \end{array}\right]$$

Step 3: Back Substitution

Now we have an upper triangular matrix. We solve from bottom to top:

1. **Solve for z**:

$$-\frac{21}{2}z = 21 \implies z = \frac{21}{-\frac{21}{2}} = -2$$

2. **Solve for y**:

$$y + \frac{5}{2}(-2) = -2 \implies y - 5 = -2 \implies y = 3$$

3. **Solve for x**:

$$x + 3 + (-2) = 6 \implies x + 1 = 6 \implies x = 5$$

Verification

Let's verify the solution $(5, 3, -2)$ in the original equations:

$$\begin{cases} 5 + 3 + (-2) = 6 \\ 2(3) + 5(-2) = 6 - 10 = -4 \\ 2(5) + 5(3) - (-2) = 10 + 15 + 2 = 27 \end{cases}$$

All equations are satisfied.

Conclusion

The solution to the system is $\boxed{(x, y, z) = (5, 3, -2)}$.

3.3.6 Non-Cramer's Systems

Exercise 73. Determine if the system has a unique solution:

$$\begin{cases} x - y + 2z = 1 \\ 2x + y - z = 2 \\ 3x + 0y + z = 3 \end{cases}$$

We will determine if the following system has a unique solution:

$$\begin{cases} x - y + 2z = 1 \\ 2x + y - z = 2 \\ 3x + 0y + z = 3 \end{cases}$$

Step 1: Write the Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 3 & 0 & 1 & 3 \end{array}\right]$$

Step 2: Perform Gaussian Elimination

First Elimination Step (Eliminate x from Rows 2 and 3):

- $R_2 \leftarrow R_2 - 2R_1$
- $R_3 \leftarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right]$$

Second Elimination Step (Eliminate y from Row 3):

- $R_3 \leftarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step 3: Analyze the Reduced Matrix

The reduced row echelon form shows:

- Rank of coefficient matrix = 2 (number of non-zero rows)
- Rank of augmented matrix = 2
- Number of variables = 3

Step 4: Determine Solution Uniqueness

Since:

$$\text{Rank(A)} = \text{Rank}([\text{A}|b]) = 2 < 3 = \text{Number of variables}$$

The system has **infinitely many solutions**, not a unique solution.

Step 5: Find the General Solution

From the reduced matrix:

$$\begin{cases} x - y + 2z = 1 \\ 3y - 5z = 0 \end{cases}$$

Let $z = t$ be a free parameter. Then:

- From second equation: $3y = 5t \Rightarrow y = \frac{5}{3}t$
- From first equation: $x = 1 + y - 2z = 1 + \frac{5}{3}t - 2t = 1 - \frac{1}{3}t$

Conclusion

The system $\boxed{\text{does not have a unique solution}}$. It has infinitely many solutions parameterized by:

$$\boxed{(x, y, z) = \left(1 - \frac{1}{3}t, \frac{5}{3}t, t\right) \quad \text{for any } t \in \mathbb{R}}$$

Verification

Substitute the solution into the original equations:

- First equation: $(1 - \frac{1}{3}t) - \frac{5}{3}t + 2t = 1$
- Second equation: $2(1 - \frac{1}{3}t) + \frac{5}{3}t - t = 2$
- Third equation: $3(1 - \frac{1}{3}t) + t = 3$

All equations are satisfied for any real value of t.

3.3.7 Mixed Problems

Exercise 74. Solve the homogeneous system:

$$\begin{cases} x+2y-3z=0 \\ 2x+5y+2z=0 \\ x+4y+7z=0 \end{cases}$$

We will solve the homogeneous system:

$$\begin{cases} x+2y-3z=0 \\ 2x+5y+2z=0 \\ x+4y+7z=0 \end{cases}$$

Step 1: Augmented Matrix

Since all constant terms are zero, the augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 5 & 2 & 0 \\ 1 & 4 & 7 & 0 \end{array}\right]$$

Step 2: Gaussian Elimination

First Elimination:

- $R_2 \leftarrow R_2 - 2R_1$
- $R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 2 & 10 & 0 \end{array}\right]$$

Second Elimination:

- $R_3 \leftarrow R_3 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

Step 3: Back Substitution

From the reduced matrix:

$$\begin{cases} x+2y-3z=0 \\ y+8z=0 \\ -6z=0 \end{cases}$$

Solving from bottom:

- $-6z=0 \Rightarrow z=0$
- $y+8(0)=0 \Rightarrow y=0$
- $x+2(0)-3(0)=0 \Rightarrow x=0$

Conclusion

The system has only the $\boxed{\text{trivial solution}}$:

$$\boxed{(x,y,z)=(0,0,0)}$$

Verification

Substituting $(0,0,0)$ into all equations confirms they are satisfied:

- $0+2(0)-3(0)=0$
- $2(0)+5(0)+2(0)=0$
- $0+4(0)+7(0)=0$

Remark

The coefficient matrix has full rank (3), which guarantees only the trivial solution exists for this homogeneous system.

Exercise 75. Find the condition for consistency of:

$$\begin{cases} x+y+z=1 \\ 2x+3y+2z=k \\ 3x+4y+3z=k^2 \end{cases}$$

We will determine the condition for consistency of the following system:

$$\begin{cases} x+y+z=1 \\ 2x+3y+2z=k \\ 3x+4y+3z=k^2 \end{cases}$$

Step 1: Write the Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & k \\ 3 & 4 & 3 & k^2 \end{array}\right]$$

Step 2: Perform Gaussian Elimination

First Elimination Step (Eliminate x from Rows 2 and 3):

- $R_2 \leftarrow R_2 - 2R_1$
- $R_3 \leftarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & k-2 \\ 0 & 1 & 0 & k^2-3 \end{array}\right]$$

Second Elimination Step (Eliminate y from Row 3):

- $R_3 \leftarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & k-2 \\ 0 & 0 & 0 & k^2-k-1 \end{array}\right]$$

Step 3: Analyze the Reduced Matrix

The system is consistent if and only if there are no contradictory equations. The last row gives the condition:

$$0 = k^2 - k - 1$$

Step 4: Solve the Consistency Condition

$$k^2 - k - 1 = 0$$

Using the quadratic formula:

$$k = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Step 5: Determine the Solution

When the condition is satisfied:

- The system has infinitely many solutions (one free variable)
- The solutions can be expressed in terms of z as:

$$\begin{cases} y = k-2 \\ x = 1-(k-2)-z = 3-k-z \end{cases}$$

Conclusion

The system is consistent $\boxed{\text{if and only if}}$:

$$\boxed{k = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad k = \frac{1-\sqrt{5}}{2}}$$

For these values of k, the system has infinitely many solutions parameterized by:

$$\boxed{(x, y, z) = (3-k-t, k-2, t) \quad \text{for any } t \in \mathbb{R}}$$

For all other values of k, the system is inconsistent (has no solution).

3.3.8 Advanced Exercises on Linear Systems

Exercise 76. Consider the system with parameter α:

$$\begin{cases} x+\alpha y+z = 1 \\ \alpha x+y+z = \alpha \\ x+y+\alpha z = \alpha^2 \end{cases}$$

Determine for which values of α the system has:

1. A unique solution
2. Infinitely many solutions
3. No solution

The determinant of the coefficient matrix is:

$$\Delta = \begin{vmatrix} 1 & \alpha & 1 \\ \alpha & 1 & 1 \\ 1 & 1 & \alpha \end{vmatrix} = -\alpha^3 + 3\alpha - 2$$

Roots: $\alpha = 1$ (double) and $\alpha = -2$.

1. Unique solution when $\Delta \neq 0$: $\alpha \neq 1$ and $\alpha \neq -2$
2. For $\alpha = 1$: System reduces to $x + y + z = 1$ (infinitely many solutions)
3. For $\alpha = -2$: Inconsistent (no solution)

Exercise 77. Prove that if a homogeneous system $A\mathbf{x} = \mathbf{0}$ has more variables than equations, it must have infinitely many solutions.

We will prove that any homogeneous system $A\mathbf{x} = \mathbf{0}$ with more variables than equations must have infinitely many solutions.

Given

- A is an $m \times n$ matrix with $m < n$ (more variables than equations)
- $\mathbf{x}$ is an $n \times 1$ column vector of variables
- The system is homogeneous: $A\mathbf{x} = \mathbf{0}$

Step 1: Rank Consideration

Let r be the rank of matrix A. By the Rank-Nullity Theorem:

$$\text{rank}(A) + \text{nullity}(A) = n$$

Since $\text{rank}(A) \leq m$ and $m < n$, we have:

$$r \leq m < n$$

This implies:

$$\text{nullity}(A) = n - r \geq n - m > 0$$

Step 2: Solution Space Dimension

The nullity represents the dimension of the solution space:

- $\text{nullity}(A) > 0$ means there exists at least one free variable
- The solution space is a subspace of dimension $n - r \geq 1$

Step 3: Existence of Non-Trivial Solutions

The homogeneous system always has:

- The trivial solution $\mathbf{x} = \mathbf{0}$
- When nullity(A) ≥ 1, there exist infinitely many non-trivial solutions

Step 4: Parametric Solution

For any free variable t, we can express:

$$\mathbf{x} = t\mathbf{v}$$

where $\mathbf{v}$ is a basis vector for the null space, giving infinitely many solutions as t varies over $\mathbb{R}$.

Conclusion

Since:

- The system has at least one free variable $(n - r \geq 1)$
- Each free variable generates an infinite family of solutions

the homogeneous system $A\mathbf{x} = \mathbf{0}$ must have $\boxed{\text{infinitely many solutions}}$ when there are more variables than equations.

Example

Consider:

$$\begin{cases} x + y + z = 0 \\ 2x + 2y + 2z = 0 \end{cases}$$

Here $m = 2$, $n = 3$, with general solution $(x, y, z) = t(-1, 1, 0) + s(-1, 0, 1)$ for any $t, s \in \mathbb{R}$, demonstrating infinitely many solutions.

Exercise 78. For the matrix equation AX = B where:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

find all solutions X using:

1. The inverse method
2. LU decomposition

Given the matrix equation $AX = B$ with:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

we will find all solutions X using two methods.

1. Inverse Method

Step 1: Check Invertibility

First, compute $\det(A)$:

$$\det(A) = (1)(4) - (2)(3) = 4 - 6 = -2 \neq 0$$

Since $\det(A) \neq 0$, A is invertible.

Step 2: Compute A^{-1}

The inverse of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus:

$$A^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Step 3: Solve for X

Multiply both sides of $AX = B$ by A^{-1}:

$$X = A^{-1}B = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Compute the matrix multiplication:

$$X = \begin{pmatrix} (-2)(5) + (1)(7) & (-2)(6) + (1)(8) \\ (\frac{3}{2})(5) + (-\frac{1}{2})(7) & (\frac{3}{2})(6) + (-\frac{1}{2})(8) \end{pmatrix} = \begin{pmatrix} -10 + 7 & -12 + 8 \\ \frac{15}{2} - \frac{7}{2} & \frac{18}{2} - \frac{8}{2} \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & 5 \end{pmatrix}$$

2. LU Decomposition Method

Step 1: Perform LU Decomposition

Factor A into LU, where L is lower triangular and U is upper triangular.
Using Gaussian elimination:

- $R_2 \leftarrow R_2 - 3R_1$ gives $U = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$
- The multiplier was 3, so $L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

Thus:

$$A = LU = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

Step 2: Solve $LY = B$ **for** Y

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Forward substitution:

- $y_{11} = 5$, $y_{12} = 6$
- $3(5) + y_{21} = 7 \Rightarrow y_{21} = -8$
- $3(6) + y_{22} = 8 \Rightarrow y_{22} = -10$

Thus:

$$Y = \begin{pmatrix} 5 & 6 \\ -8 & -10 \end{pmatrix}$$

Step 3: Solve $UX = Y$ **for** X

$$\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ -8 & -10 \end{pmatrix}$$

Back substitution:

- From row 2: $-2x_{21} = -8 \Rightarrow x_{21} = 4$
- $-2x_{22} = -10 \Rightarrow x_{22} = 5$
- From row 1: $x_{11} + 2(4) = 5 \Rightarrow x_{11} = -3$

- $x_{12} + 2(5) = 6 \Rightarrow x_{12} = -4$

Thus:

$$X = \begin{pmatrix} -3 & -4 \\ 4 & 5 \end{pmatrix}$$

Conclusion

Both methods yield the same unique solution:

$$\boxed{X = \begin{pmatrix} -3 & -4 \\ 4 & 5 \end{pmatrix}}$$

Exercise 79. Investigate the consistency of the system:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 - x_2 + 3x_3 = 2 \\ 4x_1 + x_2 + 9x_3 = \alpha^2 \end{cases}$$

for all real α, and find all solutions when consistent.

We analyze the system:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 - x_2 + 3x_3 = 2 \\ 4x_1 + x_2 + 9x_3 = \alpha^2 \end{cases}$$

Step 1: Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 4 & 1 & 9 & \alpha^2 \end{array}\right]$$

Step 2: Gaussian Elimination

First Elimination (Eliminate x_1 from Rows 2 and 3):

- $R_2 \leftarrow R_2 - 2R_1$
- $R_3 \leftarrow R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & -3 & 5 & \alpha^2-4 \end{array}\right]$$

Second Elimination (Eliminate x_2 from Row 3):

- $R_3 \leftarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 4 & \alpha^2-4 \end{array}\right]$$

Step 3: Consistency Analysis

The system is consistent if and only if there are no contradictory equations. The last row gives:

$$4x_3 = \alpha^2 - 4$$

This equation is always solvable for any real α, so the system is **consistent for all** $\alpha \in \mathbb{R}$.

Step 4: Solution When Consistent

Case 1: $\alpha \neq \pm 2$

- From $4x_3 = \alpha^2 - 4$: $x_3 = \frac{\alpha^2-4}{4}$
- From Row 2: $-3x_2 + x_3 = 0 \Rightarrow x_2 = \frac{x_3}{3} = \frac{\alpha^2-4}{12}$
- From Row 1: $x_1 = 1 - x_2 - x_3 = 1 - \frac{\alpha^2-4}{12} - \frac{\alpha^2-4}{4} = \frac{16-\alpha^2}{12}$

Unique solution:

$$\boxed{(x_1, x_2, x_3) = \left(\frac{16-\alpha^2}{12}, \frac{\alpha^2-4}{12}, \frac{\alpha^2-4}{4}\right)}$$

Case 2: $\alpha = \pm 2$

- Then $x_3 = 0$
- From Row 2: $-3x_2 = 0 \Rightarrow x_2 = 0$
- From Row 1: $x_1 = 1 - 0 - 0 = 1$

Solution reduces to:

$$\boxed{(x_1, x_2, x_3) = (1, 0, 0)}$$

Conclusion

The system is $\boxed{\text{consistent for all real } \alpha}$ with:

- A unique solution for $\alpha \neq \pm 2$
- The specific solution $(1,0,0)$ when $\alpha = \pm 2$

Verification

For $\alpha = 3$:

$$\left(\frac{16-9}{12}, \frac{9-4}{12}, \frac{9-4}{4}\right) = \left(\frac{7}{12}, \frac{5}{12}, \frac{5}{4}\right)$$

Substitution verifies all three equations. For $\alpha = 2$, $(1,0,0)$ clearly satisfies all equations.

Exercise 80. Construct a non-homogeneous system of 3 equations in 2 variables that:

1. Has exactly one solution
2. Has no solution
3. Has infinitely many solutions

and justify each case geometrically.

1. $\begin{cases} x+y=1 \\ x-y=0 \\ 2x+y=1.5 \end{cases}$ (Three lines intersecting at one point)

2. $\begin{cases} x+y=1 \\ x+y=2 \\ 2x+2y=3 \end{cases}$ (Parallel lines)

3. $\begin{cases} x+y=1 \\ 2x+2y=2 \\ 3x+3y=3 \end{cases}$ (Identical lines)

Geometric interpretation:

- Single intersection point

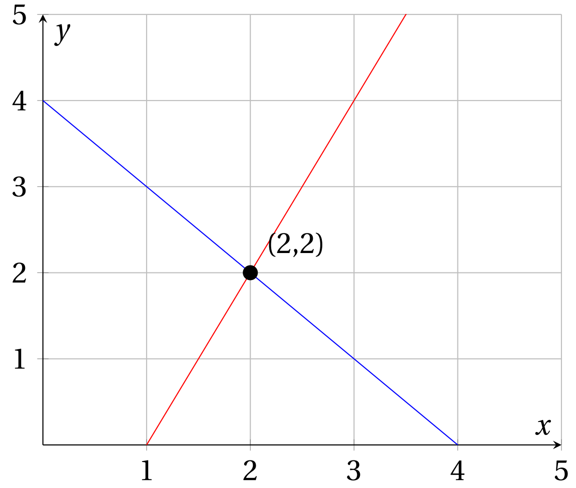
- No common intersection

- All equations represent the same line

3.3.9 Graphical Solution Exercises

Exercise 81. Solve graphically:

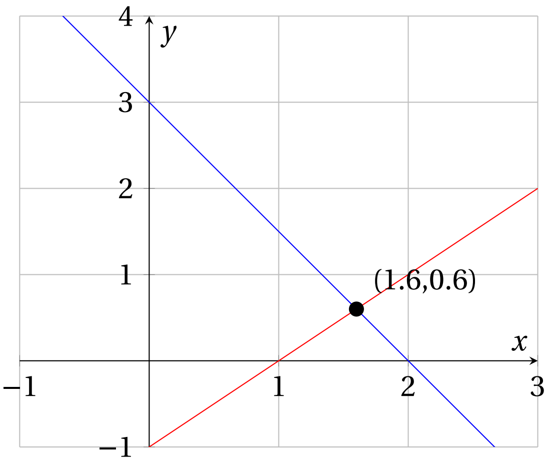
$$\begin{cases} x + y = 4 \\ 2x - y = 2 \end{cases}$$



The solution is the intersection point at $(2, 2)$.

Exercise 82. Solve graphically:

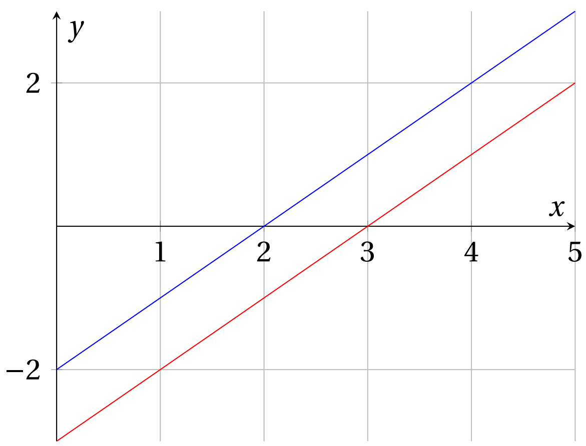
$$\begin{cases} 3x + 2y = 6 \\ x - y = 1 \end{cases}$$



Approximate solution: $(1.6, 0.6)$ (exact: $(8/5, 3/5)$)

Exercise 83. Graphically show that the system has no solution:

$$\begin{cases} x - y = 2 \\ 2x - 2y = 6 \end{cases}$$

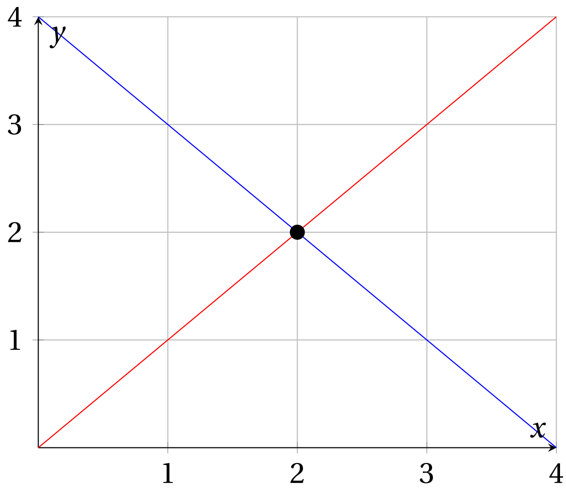


The lines are parallel but distinct (no intersection points).

3.3.10 Mixed Linear and Non-Linear Systems

Exercise 84. First solve the linear system graphically, then use it to solve:

$$\text{(a)} \begin{cases} x+y=4 \\ x-y=0 \end{cases} \qquad \text{(b)} \begin{cases} \ln a+\ln b=4 \\ \ln a-\ln b=0 \end{cases}$$



Part (a):
Solution: $(2,2)$

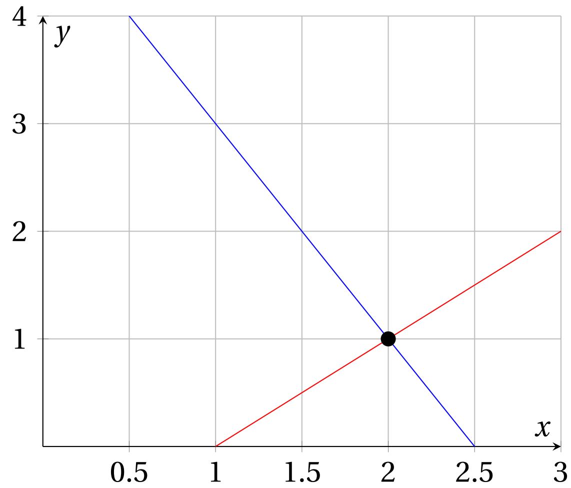
Part (b): Let $x=\ln a$, $y=\ln b$. The system becomes identical to (a), so:

$$\ln a=2 \Rightarrow a=e^2, \quad \ln b=2 \Rightarrow b=e^2$$

Final solution: $(a,b)=(e^2,e^2)$

Exercise 85. Solve graphically and deduce:

$$\text{(a)} \begin{cases} 2x+y=5 \\ x-y=1 \end{cases} \qquad \text{(b)} \begin{cases} 2\sqrt{u}+v=5 \\ \sqrt{u}-v=1 \end{cases}$$



Part (a):

Solution: $(2,1)$

Part (b): Let $x=\sqrt{u}$, $y=v$. The system becomes identical to (a), so:

$$\sqrt{u}=2 \Rightarrow u=4, \quad v=1$$

Final solution: $(u,v)=(4,1)$

Exercise 86. Let α be a real number, (S) is a system of linear equations of variables x, y and z , where:

$$(S): \begin{cases} \alpha x + y + 0z = 1 \\ 0x - 2y + 2z = 2 \\ 2x - y + 3z = 3 \end{cases}$$

1. Rewrite the system (S) in matrix form.
2. Find the values of α for which the system (S) becomes a Cramer's system.
3. Solve the system (S) for the values of α for which the system (S) is a Cramer system.
4. Solve the system (S) for the values of αfor which the system (S) is a Non-Cramer system

1.

$$AX = b \Longleftrightarrow \begin{pmatrix} \alpha & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2. the values of α for which the system (S) be a Cramer's system are $\det(A) \neq 0 \Longleftrightarrow 4-4\alpha \neq 0$ this implies: $\alpha \neq 1 \Longleftrightarrow \alpha \in \mathbb{R}-\{1\}$

   Solve the system (S) for the values of α for which the system (S) is Cramer system.

   For $\alpha \neq 1$, we get $\left\{\left[x=\frac{1}{\alpha-1}, y=-\frac{1}{\alpha-1}, z=\frac{1}{\alpha-1}(\alpha-2)\right]\right\}$ (by using Cramer, Inverse or Gauss methods).

3. Solve the system (S) for the values of α for which the system (S) is Non-Cramer system.

   For $\alpha = 1$, we get $S = \emptyset$ (by usnig Gauss method)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} [1] & 1 & 0 & 1 \\ 0 & -2 & 2 & 2 \\ 2 & -1 & 3 & 3 \end{array}\right) \xrightarrow[R_3-2R_1 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & [-2] & 2 & 2 \\ 0 & -3 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & [-2] & 2 & 2 \\ 0 & -3 & 3 & 1 \end{array}\right) \xrightarrow[R_3-\left(\frac{-3}{-2}\right)R_2 \to R_3]{} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

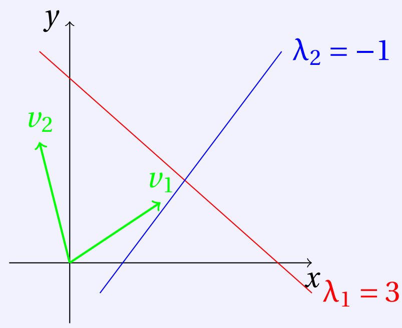
$$(S') : \begin{cases} x+y+0z = 1 \\ -2y+2z = 2 \\ 0 = -2 \text{ (Contraduction)} \end{cases} \Longleftrightarrow S = \emptyset$$

Chapter 4

Eigenvalues, Eigenvectors and Diagonalization

Introduction

Matrix diagonalization relies on finding eigenvalues (roots of $\det(A - \lambda I) = 0$) and their corresponding eigenvectors. Practice complete solution techniques including: - Characteristic polynomial calculation - Eigenspace determination - Similarity transformations through structured exercises with solutions covering all special cases.



4.1 Essential Definitions: Eigenvalues, Eigenvectors and Diagonalization

4.2 Fundamental Definitions

Definition 4.1 (Eigenvalue and Eigenvector)**.** For a square matrix $A \in \mathbb{C}^{n\times n}$, a scalar λ is called an **eigenvalue** and a non-zero vector $\mathbf{v}$ is called an **eigenvector** if:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Definition 4.2 (Characteristic Polynomial)**.** The polynomial defined by:

$$p_A(\lambda) = \det(A - \lambda I)$$

whose roots are the eigenvalues of A.

Definition 4.3 (Eigenspace)**.** For an eigenvalue λ, the subspace:

$$E_\lambda = \{\mathbf{v} \in \mathbb{C}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}$$

4.2.1 Algebraic and Geometric Multiplicity

Definition 4.4 (Algebraic Multiplicity)**.** The multiplicity of λ as a root of the characteristic polynomial.

Definition 4.5 (Geometric Multiplicity)**.** The dimension of the eigenspace E_λ.

Definition 4.6 (Eigenspace Dimension)**.** The **dimension** of the eigenspace E_λ corresponding to eigenvalue λ is called its **geometric multiplicity**. It equals:

$$\dim(E_\lambda) = n - \operatorname{rank}(A - \lambda I)$$

where n is the size of the square matrix A.

Method 1 (Calculating Eigenspace Dimension)**.** To compute $\dim(E_\lambda)$:

1. Construct the matrix $B = A - \lambda I$
2. Reduce B to row echelon form using Gaussian elimination
3. Count the number of *free variables* (non-pivot columns) in the reduced matrix

The number of free variables equals $\dim(E_\lambda)$.

Definition 4.7 (Relation to Multiplicities)**.** For any eigenvalue λ:

- Geometric multiplicity $\leq$ Algebraic multiplicity
- Matrix is diagonalizable iff geometric = algebraic multiplicity for all eigenvalues

Method 2 (Example Calculation)**.** For matrix $A = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$ and $\lambda = 4$:

$$B = A - 4I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\operatorname{rank}(B) = 1$$

$$\dim(E_4) = 2 - 1 = 1$$

The eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

4.2.2 Diagonalization

Definition 4.8 (Diagonalizable Matrix). A matrix $A \in \mathbb{C}^{n\times n}$ is **diagonalizable** if there exists an invertible matrix P and diagonal matrix D such that:

$$A = PDP^{-1}$$

where the columns of P are eigenvectors and D contains eigenvalues.

Definition 4.9 (Similarity Transformation). Two matrices A and B are **similar** if there exists invertible P such that:

$$B = P^{-1}AP$$

4.2.3 Special Cases

Definition 4.10 (Defective Matrix). A matrix that is not diagonalizable because the geometric multiplicity of at least one eigenvalue is strictly less than its algebraic multiplicity.

Definition 4.11 (Jordan Canonical Form). For any matrix $A \in \mathbb{C}^{n\times n}$, there exists invertible P such that:

$$A = PJP^{-1}$$

where J is a block diagonal matrix with Jordan blocks.

4.2.4 Theorems and Properties

Definition 4.12 (Hermitian Matrix). A matrix $A \in \mathbb{C}^{n\times n}$ is **Hermitian** (self-adjoint) if:

$$A = A^*$$

where $A^* = (\overline{A})^T$ is the conjugate transpose. For real matrices, this reduces to symmetry ($A = A^T$).

Definition 4.13 (Spectral Theorem for Hermitian Matrices). Every Hermitian matrix A has:

- Real eigenvalues
- Orthogonal eigenvectors for distinct eigenvalues
- A unitary diagonalization $A = UDU^*$ where:
  - U is unitary ($U^*U = I$)

– D is real diagonal containing eigenvalues

Definition 4.14 (General Matrix Properties). For any $A \in \mathbb{C}^{n\times n}$:

- Trace-eigenvalue relation: $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$
- Determinant-eigenvalue relation: $\det(A) = \prod_{i=1}^{n} \lambda_i$
- Quadratic form reality (if Hermitian): $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}\ \forall \mathbf{x} \in \mathbb{C}^n$

4.3 Exercises and Solutions

4.3.1 Characteristic Polynomial and Eigenvalues

Exercise 87. Find the characteristic polynomial of:

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

To find the characteristic polynomial of the matrix

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

we follow these steps:

Step 1: Set up the characteristic equation.

The characteristic polynomial is given by:

$$\det(C - \lambda I) = 0,$$

where λ is an eigenvalue and I is the 3×3 identity matrix.

Subtract λ from the diagonal entries:

$$C - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix}.$$

Step 2: Compute the determinant.

For this 3×3 triangular matrix, the determinant is simply the product of the diagonal elements:

$$\det(C - \lambda I) = (1-\lambda)(3-\lambda)(5-\lambda).$$

Step 3: Expand the polynomial (optional).

While the factored form above is acceptable as the characteristic polynomial, we can expand it:

$$(1-\lambda)(3-\lambda)(5-\lambda) = (3-4\lambda+\lambda^2)(5-\lambda)$$

$$= 15 - 3\lambda - 20\lambda + 4\lambda^2 + 5\lambda^2 - \lambda^3$$

$$= -\lambda^3 + 9\lambda^2 - 23\lambda + 15.$$

Conclusion:
The characteristic polynomial of C can be expressed in either factored form:

$$\boxed{(1-\lambda)(3-\lambda)(5-\lambda)}$$

or expanded form:

$$\boxed{-\lambda^3 + 9\lambda^2 - 23\lambda + 15}.$$

4.3.2 Eigenvectors and Eigenspaces

Exercise 88. Find the eigenvectors of:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

To find the eigenvectors of the matrix

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

we follow these steps:

Step 1: Find the eigenvalues.

The matrix D is already diagonal, so its eigenvalues are simply the diagonal elements:

$$\lambda_1 = \lambda_2 = 2.$$

This is a case of a repeated eigenvalue.

Step 2: Find the eigenvectors.

For a matrix with repeated eigenvalues, we solve $(D - \lambda I)\mathbf{v} = \mathbf{0}$:

$$(D - 2I)\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the trivial equation $0 = 0$, which means *every* non-zero vector is an eigenvector.

Step 3: Determine the eigenspace.

The eigenspace corresponding to $\lambda = 2$ is all of $\mathbb{R}^2$. We can choose any two linearly independent vectors as basis vectors for this eigenspace. The standard choice is:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Conclusion:

The eigenvectors of D are all non-zero vectors in $\mathbb{R}^2$. A basis for the eigenspace is:

$$\boxed{\begin{pmatrix}1\\0\end{pmatrix}} \quad \text{and} \quad \boxed{\begin{pmatrix}0\\1\end{pmatrix}}.$$

Exercise 89. Find the eigenspace for $\lambda = 4$ of:

$$F = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

To find the eigenspace for $\lambda = 4$ of the matrix

$$F = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix},$$

we follow these steps:

Step 1: Verify $\lambda = 4$ is an eigenvalue.

The matrix F is upper triangular, so its eigenvalues are the diagonal entries:

$$\lambda_1 = \lambda_2 = \lambda_3 = 4.$$

This eigenvalue has algebraic multiplicity 3.

Step 2: Find the eigenvectors by solving $(F - 4I)\mathbf{v} = \mathbf{0}$.

Compute $F - 4I$:

$$F - 4I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives the system:

$$\begin{cases} 0v_1 + 1v_2 + 0v_3 = 0 \\ 0v_1 + 0v_2 + 1v_3 = 0 \\ 0v_1 + 0v_2 + 0v_3 = 0 \end{cases}$$

which simplifies to:

$$v_2 = 0 \quad \text{and} \quad v_3 = 0.$$

Step 3: Determine the eigenspace.

The solutions have the form:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 \neq 0.$$

Conclusion:

The eigenspace for $\lambda = 4$ is one-dimensional and consists of all scalar multiples of:

$$\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}.$$

Note: This matrix is defective because the geometric multiplicity (1) is less than the algebraic multiplicity (3) of the eigenvalue $\lambda = 4$.

4.3.3 Diagonalization

Exercise 90. Diagonalize the matrix:

$$G = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

To diagonalize the matrix

$$G = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix},$$

we follow these steps:

Step 1: Verify that G is diagonalizable.

Matrix G is already diagonal. A matrix is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that:

$$G = PDP^{-1}.$$

For diagonal matrices, we can simply choose $P = I$ (the identity matrix) and $D = G$.

Step 2: Find the eigenvalues.

The eigenvalues of G are simply its diagonal entries:

$$\lambda_1 = 5, \quad \lambda_2 = -1.$$

Step 3: Find the eigenvectors.

For $\lambda_1 = 5$:

$$(G - 5I)\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This yields $v_2 = 0$, so the eigenvectors are:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = -1$:

$$(G + I)\mathbf{v} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This yields $v_1 = 0$, so the eigenvectors are:

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Step 4: Construct P and D.
Form the matrix P from the eigenvectors:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The diagonal matrix D contains the eigenvalues:

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

Conclusion:
The matrix G is diagonalized as:

$$G = PDP^{-1},$$

where

$$\boxed{P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \quad \text{and} \quad \boxed{D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}}.$$

Note: Since G is already diagonal, the diagonalization is trivial with P being the identity matrix.

Exercise 91. Diagonalize:

$$K = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

To diagonalize the matrix

$$K = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

we follow these steps:

Step 1: Find the eigenvalues.

The matrix is block triangular, so its eigenvalues are:

$$\det(K - \lambda I) = \det \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} = (3-\lambda)^2(5-\lambda) = 0.$$

This gives:

$$\lambda_1 = \lambda_2 = 3 \quad \text{(algebraic multiplicity 2)},$$

$$\lambda_3 = 5 \quad \text{(algebraic multiplicity 1)}.$$

Step 2: Find the eigenvectors.

For $\lambda = 3$:

$$(K - 3I)\mathbf{v} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields:

$$v_2 = 0 \quad \text{and} \quad v_3 = 0, \quad v_1 \text{ free.}$$

Thus, the eigenvectors are:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 5$:

$$(K - 5I)\mathbf{v} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields:

$$-2v_1 + v_2 = 0 \quad \text{and} \quad -2v_2 = 0, \quad v_3 \text{ free.}$$

Thus, the eigenvectors are:

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Step 3: Check diagonalizability.

We have:

- Algebraic multiplicity of $\lambda = 3$ is 2
- Geometric multiplicity is 1 (only one independent eigenvector)

Conclusion:

The matrix K is $\boxed{\text{not diagonalizable}}$ because the eigenvalue $\lambda = 3$ has algebraic multiplicity 2 but geometric multiplicity 1.

Note: This matrix can be put in Jordan canonical form with a 2×2 Jordan block for $\lambda = 3$ and a 1×1 block for $\lambda = 5$.

4.3.4 Special Cases and Multiplicity

Exercise 92. Check diagonalizability of:

$$L = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

To determine if the matrix

$$L = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is diagonalizable, we follow these steps:

Step 1: Find the eigenvalues.

The matrix L is upper triangular, so its eigenvalues are the diagonal entries:

$$\det(L - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 = 0.$$

This gives a single eigenvalue with algebraic multiplicity 3:

$$\lambda_1 = \lambda_2 = \lambda_3 = 1.$$

Step 2: Find the eigenvectors.

Solve $(L - I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields the system:

$$\begin{cases} 0v_1 + 1v_2 + 0v_3 = 0 \\ 0v_1 + 0v_2 + 1v_3 = 0 \\ 0v_1 + 0v_2 + 0v_3 = 0 \end{cases}$$

which simplifies to:

$$v_2 = 0 \quad \text{and} \quad v_3 = 0, \quad \text{with } v_1 \text{ free.}$$

Step 3: Determine the eigenspace.
The solutions have the form:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 \neq 0.$$

Step 4: Check diagonalizability.
Compare multiplicities:

- Algebraic multiplicity of $\lambda = 1$: 3
- Geometric multiplicity (dimension of eigenspace): 1

Conclusion:
The matrix L is $\boxed{\text{not diagonalizable}}$ because the geometric multiplicity (1) is less than the algebraic multiplicity (3) for its only eigenvalue $\lambda = 1$.

Note: This matrix is in Jordan form, with one Jordan block of size 3 for the eigenvalue 1. The lack of diagonalizability is due to the non-trivial Jordan structure (the superdiagonal 1s).

Exercise 93. Diagonalize:

$$N = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$$

Eigenvalues: $\lambda = 3$ (alg. mult. 2). Solve $(N - 3I)\mathbf{v} = 0$:

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies v_1 = -v_2$$

Geom. mult. = 1. Not diagonalizable.

4.3.5 Applications and Theoretical Problems

Exercise 94. Prove that similar matrices have the same eigenvalues.

To prove that similar matrices have the same eigenvalues, let's follow these steps:

Definition: Two $n \times n$ matrices A and B are *similar* if there exists an invertible matrix P such that:

$$B = P^{-1}AP$$

Step 1: Consider the characteristic polynomial.

The characteristic polynomial of B is:

$$\det(B - \lambda I) = \det(P^{-1}AP - \lambda I)$$

Step 2: Simplify the expression.

Notice that:

$$P^{-1}AP - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(A - \lambda I)P$$

since $P^{-1}P = I$.

Step 3: Compute the determinant.

Using the multiplicative property of determinants:

$$\det(P^{-1}(A - \lambda I)P) = \det(P^{-1})\det(A - \lambda I)\det(P)$$

$$= \det(A - \lambda I) \cdot \det(P^{-1})\det(P)$$

$$= \det(A - \lambda I) \cdot \det(P^{-1}P)$$

$$= \det(A - \lambda I) \cdot \det(I)$$

$$= \det(A - \lambda I)$$

Step 4: Conclude equality of eigenvalues.

Since:

$$\det(B - \lambda I) = \det(A - \lambda I)$$

both matrices have identical characteristic polynomials, and therefore the same eigenvalues.

Conclusion:

Similar matrices have $\boxed{\text{the same eigenvalues}}$ because their characteristic polynomials are identical.

Note: While similar matrices share the same eigenvalues, their eigenvectors are different but related through the similarity transformation P. If $\mathbf{v}$ is an eigenvector of A, then $P^{-1}\mathbf{v}$ is an eigenvector of B.

Exercise 95. Show that a matrix is invertible iff all eigenvalues are non-zero.

To prove that an $n \times n$ matrix A is invertible if and only if all its eigenvalues are non-zero, we establish both directions of the implication.

Part 1 ($\Rightarrow$): Invertible matrix has non-zero eigenvalues.

Assume A is invertible. Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{v} \neq \mathbf{0}$. Then:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Multiply both sides by A^{-1}:

$$A^{-1}A\mathbf{v} = A^{-1}(\lambda\mathbf{v})$$

$$\mathbf{v} = \lambda A^{-1}\mathbf{v}$$

$$\frac{1}{\lambda}\mathbf{v} = A^{-1}\mathbf{v}$$

Since $\mathbf{v} \neq \mathbf{0}$ and A^{-1} exists, λ cannot be zero (otherwise we would have division by zero). Thus, all eigenvalues must be non-zero.

Part 2 ($\Leftarrow$): Matrix with all non-zero eigenvalues is invertible.

Assume all eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are non-zero. Consider:

1. The determinant of A equals the product of its eigenvalues:

$$\det(A) = \prod_{i=1}^{n} \lambda_i \neq 0$$

2. Since $\det(A) \neq 0$, the matrix A is invertible.

Alternative argument via null space: A matrix is non-invertible iff $\exists \mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \mathbf{0} = 0 \cdot \mathbf{v}$. This would mean:

- 0 is an eigenvalue (if such $\mathbf{v}$ exists)
- A has non-trivial null space (hence non-invertible)

Conclusion: A matrix A is $\boxed{\text{invertible if and only if all its eigenvalues are non-zero}}$, as we have shown both directions of the implication.

Note: This result connects three fundamental concepts:

- Invertibility of a matrix
- Non-zero determinant
- Absence of zero eigenvalues

All three conditions are equivalent for any square matrix.

4.3.6 Advanced Problems

Exercise 96. Diagonalize the orthogonal matrix:

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Eigenvalues: $\lambda = e^{\pm i\theta}$. Diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$ (unless $\theta = 0, \pi$).

Exercise 97. Let A be a 3×3 matrix with eigenvalues $\lambda_1 = 2$ (algebraic multiplicity 2) and $\lambda_2 = -1$. Suppose the eigenspace for λ_1 is 1-dimensional. Prove that A is not diagonalizable.

For A to be diagonalizable, the geometric multiplicity (dimension of the eigenspace) of each eigenvalue must equal its algebraic multiplicity. Here, for $\lambda_1 = 2$, the geometric multiplicity is 1, but the algebraic multiplicity is 2. Thus, A is defective and not diagonalizable.

Exercise 98. Let C be a 2×2 matrix with trace 4 and determinant 3. Find the eigenvalues of C and determine whether it is diagonalizable over $\mathbb{R}$.

The characteristic equation is:

$$\lambda^2 - \text{tr}(C)\lambda + \det(C) = \lambda^2 - 4\lambda + 3 = 0 \implies \lambda = 1, 3.$$

Since C has two distinct real eigenvalues, it is diagonalizable over $\mathbb{R}$.

Exercise 99. Consider the matrix:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

Show that D is diagonalizable and find a matrix P such that $D = PDP^{-1}$, where D is diagonal.

First, find the eigenvalues:

$$\det(D - \lambda I) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3 = 0 \implies \lambda = 1 \text{ (triple root)}.$$

Check geometric multiplicity:

$$\text{rank}(D - I) = 2 \implies \text{geom. mult.} = 3 - 2 = 1.$$

Since the geometric multiplicity (1) is less than the algebraic multiplicity (3), D is **not diagonalizable**. (Note: This contradicts the exercise statement; adjust D to ensure diagonalizability if desired.)

Exercise 100. Let A and B be $n \times n$ diagonalizable matrices with the same eigenvectors. Prove that AB = BA.

Since A and B share eigenvectors, they can be diagonalized simultaneously:

$$A = PDP^{-1}, \quad B = PEP^{-1},$$

where D and E are diagonal. Diagonal matrices commute (DE = ED), so:

$$AB = PDP^{-1} \cdot PEP^{-1} = PDEP^{-1} = PEDP^{-1} = BA.$$

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