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# Doctoral Thesis

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for the Degree of Doctor in Mathematics**

**Option : Nonlinear analysis and PDE**

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## Global existence and asymptotic behavior for some evolution systems

Presented by

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# Dedication

Gratefully and thankfully, I dedicate this thesis to my loving parents, whose words of encouragement. My dear sisters 'Sonia', have never left my side and are very special. My friends who have encouraged and supported me. To all the people in my life who touch my heart, I dedicate this research.

# Contents

# General introduction and preliminaries

A laminated beam system is a structural arrangement composed of multiple layers of materials bonded together to form a single, strong, and durable beam. It is designed to provide enhanced strength, stiffness, and load-bearing capacity compared to traditional solid beams. Laminated beams are widely used in various industries and construction applications where high structural performance is required.
The construction of a laminated beam involves stacking multiple thin layers of compatible materials, such as wood, metal, or composite materials, and bonding them together using adhesives or other bonding agents. The layers are typically oriented such that the grain or fibers run in different directions, which helps distribute loads evenly and increase overall structural integrity.
The use of different materials in the laminated beam system allows for the optimization of specific properties. For example, wood layers provide excellent compressive strength, while fiberglass or carbon fiber layers offer high tensile strength and stiffness. By combining these materials, laminated beams can be tailored to meet specific design requirements, such as increased strength, flexibility, or resistance to environmental factors like moisture or temperature variations.
In summary, laminated beam systems provide a reliable and versatile solution for structural requirements. By combining different materials in layered configurations, laminated beams offer enhanced strength, durability, and performance characteristics, making them a preferred choice in modern construction and engineering projects.
The best contribution in this field is the work of Hansen et al.[22], the laminated beam model describes the vibrations in a structure consisting of two layered identical beams of uniform thickness stuck together by an adhesive (of negligible thickness), in such a way that a slip is permitted while they are continuously in contact with each other. In the absence of interfering forces, the system of the model takes the

following form

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, \end{cases} \tag{0.1}$$

with $x \in (0,1)$ and $t \geq 0$. Hence $\rho, I_\rho, G, D, \beta,$ and $\gamma$ are density, mass moment of inertia, shear stiffness, flexural rigidity, adhesive damping parameter and, adhesive stiffness respectively. Similarly, $w = w(x,t)$ denotes the transverse displacement of the beam from its equilibrium position, $\psi = \psi(x,t)$ is the rotation angle, $3s - \psi$ denotes the effective rotation angle and, $s = s(x,t)$ is proportional to the amount of slip along the interface. The first two equations of (0.1) are derived on the assumption of Timoshenko beam theory and, the third equation describes the dynamics of the slip. Moreover, if $s(x,t)$ is identically zero, then the standard Timoshenko system is recovered. Furthermore, if $\beta \neq 0,$ then the adhesion at the interface produces a restoration comparable force to counteract the interfacial slip. Otherwise, in absence of adhesive damping (i.e. $\beta = 0)$ ), the third equation describes the dynamics slip coupled laminated beam without structural damping.

Laminated beams have wider applications in engineering as structures are often made out of more than one beam or plate stuck together using the appropriate substance depending on their intended purposes. Among other applications, the closest examples one can think of in recent times are the layered glass gorilla screen protection for smart gadgets, windscreens, among others. Being a controlled system, stability is very important. Thus, many researchers among mathematicians and engineers have focused a lot of attention on the study of well-posedness and more importantly, the stability behavior of this differential model, majorly by exploiting different damping mechanisms introduced to the system. We discuss some of the results below. The asymptotic behavior of system (0.1) with boundary feedback controls of the form

$$\begin{cases} w(0,t) = \psi(0,t) = s(0,t), \quad (\psi - w_x)(1,t) = k_1 w_t(1,t), \\ (3s_x - \psi_x)(1,t) = -k_2(3s_t - \psi_t)(1,t), \end{cases}$$

was studied by Wang et al. [47]. The authors established an exponential stability of the system provided that $r_1 = \sqrt{\dfrac{\rho}{G}} \neq \sqrt{\dfrac{I_\rho}{D}} = r_2$, $k_i \neq r_i (i = 1, 2)$. Interestingly, Tatar [45] and Mustafa [35] extended the result of [47] under weaker conditions on the parameters $\rho, G, I_\rho$, and $D$. Some related results were also obtained by Cao et al. [13] with different boundary controls. Apart from stabilization through boundary damping mechanisms, researchers have considered other interesting damping techniques. For example, Raposo [41] introduced extra linear frictional damping terms in the first two equations of (0.1) in addition to structural damping, and proved exponential stability without further restrictions. Apalara et al. [9] established that a single linear frictional damping in the effective rotation angle is sufficient for exponential decay in case of equal wave speeds. Similarly, in [2], the authors consider system (0.1) with structural damping, and prove that if it is coupled with boundary feedback controls acting through complementary displacements, then no further dissipation or restrictions on parameters are required for exponential decay, otherwise the assumption of equal wave speeds is necessary. Regarding dissipation through material damping, for laminated beam with infinite memory, we mention the work in [27], in which with only structural damping and suitable assumptions on the relaxation function, the authors established general exponential decay results in case of equal wave speeds and polynomial stability otherwise. For earlier results concerning stabilization of laminated beam through viscoelastic damping, we refer the reader to [14],[31],[36]. Furthermore, regarding stabilization through thermal effects, we refer in [30]. The authors investigated a thermoelastic laminated beam with past history, and proved that in presence of structural damping, the solution decays exponentially and polynomially without any restrictions on the parameters. For a system without structural damping, exponential and polynomial decay of the solution are possible in case of equal wave speeds, otherwise, the system lacks exponential stability. Other interesting results about damping through thermal effects can be found in [4], [5], [20] for thermoelasticity, and [28] for thermoelasticity of type III. In all these works, authors mainly established that the system decays exponentially in the case of equal wave speeds and polynomially otherwise, with and without structural damping. In control systems, time delays are inherent since propagation and transport of material and/or information are involved. Time delay may manifest in form of lags between the input and processing the output, or lags in attaining or restoring the desired system stability after perturbations due to internal or external factors, among others. To explicitly analyse the delay effect on physical properties especially stability, it is preferred that control systems are modeled and represented by delay differential equations. Although there are isolated cases where that voluntary inclusion of delay may benefit control (see [7]) or may not significantly disturb the general system stability, for instance, in [46], time delays have been established as one of the underlying causes of instability and deterioration of the system performance. For example, consider the following system of wave equation

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0, & \text{in } \Omega \times (0,\infty), \\ \varphi = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial\varphi}{\partial\nu} = -\mu_1\varphi_t - \mu_2\varphi_t(x,t-\tau), & \text{on } \Gamma_1 \times (0,\infty), \end{cases} \tag{0.2}$$

where $\varphi = \varphi(x,t), \Omega \subset \mathbb{R}^2$ is open and bounded having a smooth boundary $\partial\Omega \equiv \Gamma_0 \cup \Gamma_1$ and, $\nu = \nu(x)$ is the unit normal to $\partial\Omega$. It is long established that in the absence of delay ($\mu_2 = 0, \mu_1 > 0$), the system (0.2) is exponentially stable, see [25],[26],[50]. Whereas, on including delay ($\mu_2 > 0$). Nicaise et al. [37] established that the solution decays exponentially provided that $\mu_2 < \mu_1$, and in case of a reversed scenario ($\mu_2 \geq \mu_1$), the authors proved that the system solutions become chaotic by introducing a correlating sequence of delays to the solution. Similar conclusions were reached by [17],[49]. For more works regarding constant time delay effect on stability, we refer the reader to [16],[32],[40],[42] and references therein.

In the dynamic Timoshenko beam model, the amplitude of vibrations of the complementary displacements vanishes due to damping. A constant time delay translates into a forward phase shift increasing early time response, which is seen to cause frequency dispersion in displacements [34]. This may require stronger damping to counteract the longer time needed for decay. This delay effect is inherent in the laminated beam model as it is derived on assumption of Timoshenko beam theory. The presence of structural damping in a laminated beam provides some dissipation, which is sufficient for exponential stability in absence of delay on assumption of equal wave speeds [6],[8]. It is yet to be established if the internal structural damping can still solely stabilize the system in presence of delay, rather authors have chosen other damping mechanisms. For instance, Feng [19], considered a laminated beam with three internal constant delay feedbacks, withe help of three external boundary controls and some conditions on the system parameters, he established exponential decay result. Seghour et al. [43] on the other hand, investigated a thermoelastic laminated beam with neutral delay in dynamics of slip equation, and established uniform stability provided $\rho = GI_\rho$. . The required dissipation was obtained through thermal effects in addition to linear frictional damping in the transverse displacement.

Recently, numerous investigations have focused on the stabilization of laminated beams, crucial components within multiple engineering disciplines. Hansen and Spies [22] introduced a model for two-layered beams featuring structural damping attributed to interfacial slip. Their work led to the formulation of the subsequent system

$$\begin{cases} \rho\nu_{tt} + G(r - \nu_x)_x = 0, \\ I_\rho(3\varpi_{tt} - r_{tt}) - G(r - \nu_x) - D(3\varpi_{xx} - r_{xx}) = 0, \\ I_\rho\varpi_{tt} - G(r - \nu_x) + \dfrac{4}{3}\gamma\varpi + \dfrac{4}{3}\beta\varpi_t - D\varpi_{xx} = 0. \end{cases} \tag{0.3}$$

The equations governing the motion in the representation of system (0.3) are obtained through the utilization of Timoshenko theory. Within this framework, a third equation is interlinked with the initial two equations, delineating the dynamics of slip and incorporating internal frictional (Kelvin-Voight) damping. It is noteworthy that when assuming the slip, denoted as $\varpi$, to be identically zero, the first two equations of (0.3) can be precisely reduced to the well-established Timoshenko beam see [15, 20, 32, 44, 46]. When creating models in the fields of biology, physics, and the social sciences, there are instances where it becomes essential to consider time delays that are inherent to the observed phenomena. Explicitly incorporating these delays into equations often serves as a simplification or an idealization. This simplification is necessary when providing a detailed mathematical representation of the underlying processes proves overly complex or when some specifics remain unknown. In a broader sense, it prompts questions about how the qualitative characteristics of the system are influenced by the nature and extent of these time delays (see[7, 16, 18, 33]).

In [38], the following problem involving a wave equation and a time-varying delay was studied

$$\begin{cases} u_{tt} - \Delta u = 0, \\ u(x,t) = 0, \\ \dfrac{du}{d\nu}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t - \varsigma(t)), \end{cases}$$

the authors managed prove that the problem is exponentially stable given that condition

$$|\mu_2| \leq \sqrt{1-d}\mu_1,$$

holds, where the constant $d$ satisfies

$$\varsigma'(t) \leq d < 1, \qquad \forall t > 0.$$

Recently, a model of laminated Timoshenko beam with microtemperature effects and nonlinear time varying delay was the subject study of Djeradi et al. [18], in which the authors proved a general decay result assuming that specific assumptions concerning nonlinear terms hold.

Now, we recall some basic knowledge in functional analysis, most of which will be used in the subsequent chapters. The reader can easily find the details in the related literature, see, e.g. [1], [11], [12], [39], and [48].

## Functional Spaces

We denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subset \mathbb{R}^n$ is bonded smoth domain,$C^k(\Omega)$ is the $k^{th}$ differentiable continuous function space in $\Omega$, $C^\infty(\Omega)$ is the $\infty^{th}$ differentiable continuous function space in $\Omega$, $C_c^\infty(\Omega)$ is the $\infty^{th}$ differentiable continuous function space with compact support in $\Omega$

**Definition 0.1.** Let $X$ be a vector space over the filed $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on $X$ is a function $\|.\| : X \longrightarrow \mathbb{R}$, such that :
a) $\|x\| \geqslant 0$ for all $x \in X$,
b) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$,
c) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$.
A norm on $X$ is a semi-norm wich also satisfies :
d) $\|x\| = 0 \Rightarrow x = 0$. A vector space $X$ toghether with a norm $\|.\|$ is called a normed linear space, a normed vector space or simply, a normed space.

**Definition 0.2.** (Convergent and Cauchy sequences ). Let $X$ be a normed space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of $X$.
$a)$ $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

i.e. if

$$\forall\varepsilon > 0; \exists N > 0, \forall n \geq N, \|x_n - x\| < \varepsilon.$$

$b)$ $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if

$$\forall\varepsilon > 0; \exists N > 0, \forall m, n \geq N, \|x_m - x_n\| < \varepsilon.$$

Normed spaces in which every Cauchy sequence is convergent are called complet normed spaces. In general a normed space is not complete.

**Definition 0.3.** (Banach Spaces). A normed spaces is called a Banach space if it is complet i.e. if any Cauchy sequence inside the space converges to a point of the space. Its dual space $X'$ is the linear space of all contnuous linear functional $f : X \to \mathbb{R}$.

**Proposition 0.4.** *$X'$ equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\},$$

*is also a Banach space.*

**Remark 0.5.** From $X'$ we construct the bidual or secend dual $X'' = (X')'$. Futhermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u)$, $f \in X'$, this satisfies clearly $\|\varphi(x)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$, so it follows that $\|\varphi(x)\| = \|u\|$.

**Definition 0.6.** Since $\varphi$ is linear we see that

$$\varphi : X \to X'',$$

is a linear isometry of $X$ onto a closed subspace of $X''$, we denote this by

$$X \hookrightarrow X''.$$

**Definition 0.7.** if $\varphi$ ( in the above definition ) is into $X''$ we say $X$ is reflexive, $X \approxeq X''$.

## The weak and weak star topologies:

Let $X$ be a Banach space and $f \in X'$ . Denoted by

$$\begin{aligned} \varphi_f : & X \to \mathbb{R} \\ & x \mapsto \varphi_f. \end{aligned}$$

When $f$ cover $X'$, we obtain a family $(\varphi_f)_{f \in X'}$ of applications to $X$ in $\mathbb{R}$.

**Definition 0.8.** The weak topology on $X$, denoted by $\sigma(X, X')$, is the weakest topology on $X$ for which every $(\varphi_f)_{f \in X'}$ is continuous.

We will define the topology on $X'$, the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_x : & X' \to \mathbb{R} \\ & f \mapsto \varphi_x(f) = \langle f, x \rangle_{X', X}. \end{aligned}$$

**Definition 0.9.** The weak star topology on $X'$ is the weakest topology on $X'$ for wich every $(\varphi_x)_{x \in X'}$ is continuous.

**Remark 0.10.** Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.

**Definition 0.11.** A sequence $(x_n)$ in $X$ is weakly convergent to $x$ if and only if

$$\lim_{n \to \infty} f(x_n) = f(x),$$

for every $f \in X'$, and this is denoted by $x_n \rightharpoonup x$.

**Remark 0.12.** 1. If the weak limit exist, it is unique.
2. If $x_n \to x \in X$(strongly), then $x_n \rightharpoonup x$ (weakly).
3. If $\dim X < \infty$, then the weak convergent implies the strong convergent.

## Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analysis, known as Hilbert spaces. Then, we most give some important result on these spaces here.

**Definition 0.13.** A Hilbert space $H$is a vectorial space supplied with inner product $\langle u, v \rangle$ such that $\|u\| = \sqrt{\langle u, u \rangle}$ is the norm which let $H$ complete.

**Theorem 0.14.** *Let $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence in the Hilbert space $H$, then it possess a subsequence which converges in the weak topology of $H$.*

**Theorem 0.15.** *In the Hilbert space, all sequence which converges in the weak topology is bounded.*

**Theorem 0.16.** *Let $(x_n)_{n \in \mathbb{N}}$ be sequence which converges to $x$, in the weak topology and $(y_n)_{n \in \mathbb{N}}$ is an other sequence which converge weakly to $y$, then*

$$\lim_{n \to \infty} \langle x_n, y_n \rangle = \langle x, y \rangle.$$

**Proposition 0.17.** *Let $X$ and $Y$ be two Hilbert space, let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence which converges weakly to $x \in X$, let $A \in \mathcal{L}(X,Y)$. Then, the sequence $(A(x_n))_{n\in\mathbb{N}}$ converges to $A(x)$ in the weak topology of $Y$.*

**Theorem 0.18.** *( The Lax-Milgram Theorem)*
*Let $X$ be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear functional. Assume that there existe two constants $C < \infty, \alpha > 0$ such that:*
*(i)* $|a(u,v)| \leq C\|u\|.\|v\|$ *for all* $(u,v) \in X \times X$ *(continuity);*
*(ii)* $a(u,u) \geq \alpha\|u\|^2$ *for all* $u \in X$ *(coerciveness).*
*Then, for every $f \in X^*$ ( the dual space of $X$), there exist a unique $u \in X$ such that* $a(u,v) = \langle f, v\rangle$ *for all $v \in X$.*

## The $L^p(\Omega)$ spaces

**Definition 0.19.** Let $1 \leq p \leq \infty$, and let $\Omega$ be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \ \ is \ \ measurable \ \ and \int_\Omega |f(x)|^p dx < \infty \right\}.$$

Notation 1 : for $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by

$$\|f\|_p = \left( \int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^p(\Omega) = \{ f : \Omega \to \mathbb{R} : f \ \ is \ \ measurable \ \ and \ \ there \ \ exists \ \ C \ \ such \ \ that, |f(x)| \leq C \ \ in \ \ \Omega \}.$$

Notation 2 : Let $1 \leq p \leq \infty$, we denote by $q$ the conjugate of $p$ i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

**Theorem 0.20.** *It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all $1 \leq p \leq \infty$.*

**Remark 0.21.** In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

is a Hilbert space .

**Theorem 0.22.** *For $1 < p < \infty$, $L^p(\Omega)$ is reflexive space.*

## The Sobolev space $W^{m,p}(\Omega)$

**Definition 0.23.**

i) Let $m \in \mathbb{N}$ and $p \in [1, \infty]$ . The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega), \text{ such that } \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^m\},$$

such that $|\alpha| = \sum_{j=1}^{n} \alpha_j \leq m$ *where*, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} .... \partial_n^{\alpha_n}$.

ii) If $f \in W^{m,p}(\Omega)$, we define its norm to be

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} (\sum_{|\alpha|<k} \int_\Omega |D^\alpha f|^p dx)^{\frac{1}{p}} & ,(1 \leq p < \infty), \\ \sum_{|\alpha|<k} ess \sup |D^\alpha f| & ,(p = \infty). \end{cases}$$

**Definition 0.24.** We denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

**Remark 0.25.** If $p = 2$ we usully write

$$W^{m,2}(\Omega) = H^m(\Omega), \quad W_0^{m,2}(\Omega) = H_0^m(\Omega).$$

Supplied with the norm

$$\|f\|_{H^m} = \left( \sum_{|\alpha| \leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}},$$

we will see $H^m(\Omega)$ is a Hilbert space. With usual scalar product

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Note that $H^0(\Omega) = L^2(\Omega)$.

**Theorem 0.26.** *1. $H^m(\Omega)$ supplied with inner product $\langle ., . \rangle_{H^m(\Omega)}$ is Hilbert space.*

*2. If $m \geq m'$ , $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$.*

**Theorem 0.27.** *Assume that $\Omega$ is an open domain in $\mathbb{R}^n, n \geq 1$, with smooth boundary $\Gamma$. Then,*

*i) If $1 \leq p \leq n$, we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, p^*]$, where $p^* = \dfrac{np}{n-p}$.*

*ii) If $p = n$ we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, \infty)$ .*

*iii) If $p > n$ we have $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = \dfrac{p-n}{p}$.*

### The $L^p(0,T,X)$ space

**Definition 0.28.** Let $X$ be a Banach space, denote by $L^p(0,T,X)$ the space of measurable functions

$$f : ]0,T[ \to X$$
$$t \mapsto f(t),$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \qquad 1 \leq p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X.$$

**Theorem 0.29.** *$L^p(0,T,X)$ equipped with the norm $\|.\|_{L^p(0,T,X)}$ is a Banach space .*

**Proposition 0.30.** *Let $X$ be a reflexive Banach space, $X'$ it's dual, and $1 \leq p < \infty$, $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0,T,X)$ is identify algebraically and topologically with $L^q(0,T,X')$.*

## Some useful inequalities

In this section, we shall recall some inequalities which will be used in the subsequent chapters.

### Young's inequality

**Theorem 0.31.** *Let $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad a,b > 0.$$

**Theorem 0.32.** *(Young inequality with $\varepsilon$) Let $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \varepsilon\frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}}\frac{b^q}{q} \quad a,b > 0.$$

The Young's inequality has several variants in the following.

**Corollary 0.33.** *Let* $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. *Then*
*i)* $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.
*ii)* $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p\varepsilon^{\frac{1}{q}}} + \frac{b\varepsilon^{\frac{1}{p}}}{q} \qquad \forall \varepsilon > 0$.
*iii)* $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \qquad 0 < \alpha < 1$.

## Hölder's inequality

**Theorem 0.34.** *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then if* $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *we have*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} . \|g\|_{L^q(\Omega)}.$$

**Theorem 0.35.** *(Generalized Hölder inequality) Let* $1 \leq p_1, ...., p_m \leq \infty$,
$\frac{1}{p_1} + .... + \frac{1}{p_m} = 1$, *then if* $f_k \in L^{p_k}(\Omega)$ *for* $k = 1, ..., m$, *we have*

$$\int_{\Omega} |f_1....f_m| dx \leq \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\Omega)}.$$

**Remark 0.36.** We have the corresponding weighted Hölder inequality of the integral form. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $\omega(x) > 0$ on $\Omega$. Then

$$\int_{\Omega} |fg| \omega(x) dx \leq \left( \int_{\Omega} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |g(x)|^q \omega(x) dx \right)^{\frac{1}{q}}.$$

## Poincaré's inequality

**Theorem 0.37.** *Let* $\Omega$ *be a bounded domain in* $\mathbb{R}^n$ *and* $f \in H_0^1(\Omega)$. *Then there is a positive constant* $C$ *such that*

$$\|f\|_{L^2(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)}, \qquad \forall f \in H_0^1(\Omega).$$

**Theorem 0.38.** *Let* $\Omega$ *be a bounded domain of* $C^1$ *in* $\mathbb{R}^n$. *There is a positive constant* $C$, *such that for any* $f \in H^1(\Omega)$

$$\|f - \tilde{f}\|_{L^2(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)}.$$

*Where* $\tilde{f} = \frac{1}{|\Omega|}\int_{\Omega} f(x)dx$ *is the integral average of* $f$ *over* $\Omega$*, and* $|\Omega|$ *is the volume of* $\Omega$.

**Theorem 0.39.** *Under assumption of Theorem* (0.38) *for any* $f \in H^1(\Omega)$*, we have*

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\nabla f\|_{L^2(\Omega)} + |\int_{\Omega} f dx|\right).$$

### Cauchy-Schwarz inequality

**Theorem 0.40.** *Let* $f \in L^2(\Omega)$*,* $g \in L^2(\Omega)$*, we have*

$$\int_{\Omega} |fg|dx \leq \left(\int_{\Omega} |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\Omega} |g(x)|^2 dx\right)^{\frac{1}{2}}.$$

## Basic theory of semigroups

In this section, we recall some basic knowledge in semigroups,most of whiche will be used in the subsequent chapters. A general reference to this topic is [12] and [11].

### $C_0-$Semigroups of Linear Operators

**Definition 0.41.** (semigroups)

Let $X$ be a Banach space, the one-parametre family $S(t), 0 \leq t < \infty$ from $X$ to $X$ is called a semigroups if

(i) $S(0) = I$ ( $I$ is the identity operateur on $X$ ).

(ii) $S(t+s) = S(t).S(s)$ for every $t, s \geq 0$ (the semigroup property).

**Definition 0.42.** The linear operator $A$ defined by

$$D(A) = \left\{x \in X : \lim_{t \to 0^+} (S(t)x - x)/t \quad exists\right\},$$

and

$$Ax = \lim_{t \to 0^+} (S(t)x - x)/t = \frac{d(S(t)x)}{dt}|_{t=0} \quad \forall \quad x \in D(A),$$

is called the infinitesimal generator of the semigroup $S(t)$, $D(A)$ is called the domain of $A$.

**Definition 0.43.** ($C_0-$Semigroups).

A semigroup $S(t)$,$0 \leq t < \infty$, from $X$ to $X$ is called a strong continuous semigroup of bounded linear operators if

$$\lim_{t \to 0^+} S(t)x = x \quad \forall x \in X,$$

or

$$\lim_{t \to 0^+} \|S(t)x - x\| = 0 \quad \forall x \in X.$$

i.e $S(t)$ $C_0-$Semigroup.

**Definition 0.44.** A semigroup $S(t)$, $0 \leq t < \infty$ is called a semigroup of contraction if there exists a constant $\alpha > 0$ $(0 < \alpha < 1)$ such that for all $t > 0$,

$$\|S(t)x - S(t)y\| \leq \alpha\|x - y\|, \quad \forall x, y \in X.$$

## Hille-Yosida Theorem

**Definition 0.45.** An unbonded linear operator $A : D(A) \subset H \to H$[1] is said to be monotone[2]if it satisfies

$$\langle Av, v \rangle \geq 0 \ \ \forall v \in D(A).$$

It is called maximal monotone if, in addition, $R(I + A) = H$ i.e

$$\forall f \in H \ \ \exists u \in D(A) \quad \text{such that} \ \ u + Au = f.$$

**Proposition 0.46.** *Let $A$ be a maximal monotone operator. Then*

*1. $D(A)$ is dense in $H$.*

*2. $A$ is closed operator.*

*3. For every $\lambda > 0, (I + \lambda A)$ is bijective from $D(A)$ into $H$, $(I + \lambda A)^{-1}$ is a bounded operator, and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.*

**Theorem 0.47.** *(Hille-Yosida) Let $A$ be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exist a unique function*

$$u \in C^1([0, +\infty), H) \cap C([0, +\infty), D(A)),$$

[1] H denotes a Hilbert space

[2]Some authors say that $A$ is accretive or $-A$ is dissipative.

*satisfying*

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad \text{on } [0, +\infty), \\ u(0) = u_0. \end{cases}$$

**Theorem 0.48.** *(Lumer-Phillips theorem) Let $A$ be a linear operator defined on a linear subspace $D(A)$ of the Banach space $X$. Then $A$ generates a contraction semigroup if and only if*

1. *$D(A)$ is dense in $H$.*

2. *$A$ is dissipative.*

3. *$(A - \lambda_0 I)$ is surjective for some $\lambda_0 > 0$, where $I$ denotes the identity operator .*

*An operator satisfying the last two conditions is called maximally dissipative.*

We summarize our contribution as follows

• **In chapter 1:** we study the well-posedness and exponential decay of the system laminated beam with an internal constant delay term in the tranvers displacement. The remainder of this chapther is organized as follows. The first section contains the statement of the problem, in the second section we prove the well-posedness of solution of the system (1.7), and in the last section we discuss the exponential stability results.

• **In chapter 2:** Our goal of this chapter is to study the global well posedness and stability of systems (2.1)-(2.3). The main points are summarized as follows: (i) We prove the global well posedness of systems (2.1)-(2.3) by using Lumer-Philips theorem. The main result is presented in Theorem 2.1. (ii) We introduce a new stability number denoted by

$$\chi = \tau\delta^2 D - (D\rho - GI_\rho)\left(\frac{\tau\rho_3 D}{I_\rho} - 1\right), \tag{0.4}$$

and we show that the system is exponential stable when $\chi = 0$ nd polynomial stable when $\chi \neq 0$. The main results are presented in Theorems 2.2 and 2.3.

(iii) The proof of stability results is based on the multiplier method. Since the boundary conditions here we considered are different from those in Apalara [4], so the multipliers we will define are greatly different from the multipliers in Apalara [4]. It follows, from (2.1), that

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^1 \omega(x,t)\mathrm{d}x = 0, \\ \tau\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 q(x,t)\mathrm{d}x + \beta\int_0^1 q(x,t)\mathrm{d}x = 0. \end{aligned} \tag{0.5}$$

If we denote

$$\begin{aligned} \bar{\omega}(x,t) &= \omega(x,t) - \int_0^1 \omega_0(x) - t\int_0^1 \omega_1(x)\mathrm{d}x, \\ \bar{q}(x,t) &= q(x,t) - e^{-(\beta/\tau)t}\int_0^1 q_0(x)\mathrm{d}x, \end{aligned} \tag{0.6}$$

we easily verify that $(\bar{\omega}, \psi, s, \theta, \bar{q})$ satisfies (2.1) and in addition,

$$\begin{aligned} \int_0^1 \bar{\omega}(x,t)\mathrm{d}x = 0, \\ \int_0^1 \bar{q}(x,t)\mathrm{d}x = 0, \\ \forall t \geq 0. \end{aligned} \tag{0.7}$$

Hence, Poincare's inequality holds for $\bar{\omega}$. In the following, we work with $\bar{\omega}$ and $\bar{q}$ but write $\omega$ and $q$ for convenience.

• **In chapter 3:** This research aims to investigate the problem denoted as (3.1)-(3.2), wherein a delay term is incorporated within the control term of the third equation. This delay, represented by the time-varying term $\mu_2\varpi_t(x, t - \varsigma(t))$, introduces distinct characteristics to the problem due to the influence of thermal effects. Such factors set this problem apart from those previously addressed (see [6, 3, 34, 10, 30, 32, 23]).

The structure of this chapter in Section 2 is dedicated to establishing the well-posedness. Moving to Section 3, we substantiate the exponential stability result using the energy method and Lyapunov function. This results have been puplished in [21].

Chapter 1

# Analysis of laminated beam with constant delay term

## 1.1 Statement of the problem

In this chapter we consider a system of laminated beam with constant delay term acting on the transverse displacement:

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \mu w_t(x, t-\tau) = 0, & \text{in } (0,1) \times (0,\infty), \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, & \text{in } (0,1) \times (0,\infty), \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, & \text{in } (0,1) \times (0,\infty), \\ w_t(x,-t) = f_0(x,t), & \text{in } (0,1) \times (0,\tau), \\ w(x,0) = w_0, s(x,0) = s_0, \psi(x,0) = \psi_0, & \text{in } (0,1), \\ w_t(x,0) = w_1, s_t(x,0) = s_1, \psi_t(x,0) = \psi_1 & \text{in } (0,1), \\ w(0,t) = s_x(0,t) = \psi_x(0,t) = 0, & \text{in } (0,\infty), \\ w_x(1,t) = s(1,t) = \psi(1,t) = 0, & \text{in } (0,\infty), \end{cases} \tag{1.1}$$

where $w_0, w_1, \psi_0, \psi_1, s_0, s_1, f_0$ is the initial data which belongs to an appropriate space, $\tau > 0$ is time delay and the none zero real number $\mu$ is the weight of delay. With some restrictions on $\mu$ , we prove that the adhesive damping is strong enough to stabilize the system exponentially, even in presence of delay without any other additional damping or boundary controls, provided the assumption of equal wave propagation speed ($GI_\rho = \rho D$) holds.

## 1.2 Existence and uniqueness

In this section, we state the well-possedness results of problem (1.7). Firstly, we proceed by introducing the following new variable as in [37].

$$z(x,\sigma,t) = w_t(x, t-\tau\sigma) \quad \text{in } (0,1)\times(0,1)\times(0,\infty).$$

It follows directly that $z$ satisfies

$$\tau z_t(x,\sigma,t) + z_\sigma(x,\sigma,t) = 0 \quad \text{in } (0,1)\times(0,1)\times(0,\infty).$$

Consequently, the system (1.1) is equivalent to

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \mu z(x,1,t) = 0, & \text{in } (0,1)\times(0,\infty), \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, & \text{in } (0,1)\times(0,\infty), \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau z_t(x,\sigma,t) + z_\sigma(x,\sigma,t) = 0, & \text{in } (0,1)\times(0,1)\times(0,\infty), \\ z(x,0,t) = w_t(x,t) & \text{in } (0,1)\times(0,\infty), \\ z(x,\sigma,0) = f_0(x,\tau\sigma), & \text{in } (0,1)\times(0,1), \\ w_t(x,-t) = f_0(x,t), & \text{in } (0,1)\times(0,\tau), \\ w(x,0) = w_0, s(x,0) = s_0, \psi(x,0) = \psi_0, & \text{in } (0,1), \\ w_t(x,0) = w_1, s_t(x,0) = s_1, \psi_t(x,0) = \psi_1 & \text{in } (0,1), \\ w(0,t) = s_x(0,t) = \psi_x(0,t) = 0, & \text{in } (0,\infty), \\ w_x(1,t) = s(1,t) = \psi(1,t) = 0, & \text{in } (0,\infty). \end{cases} \tag{1.2}$$

At this step, in order to define the energy functional of system (1.2), we multiply the first four equations in the system (1.2) by $w_t$, $(3s_t - \psi_t)$, $s_t$, and $|\mu| z$, respectively, then integrate over $(0,1)$, we have

$$\int_0^1 \rho w_{tt} w_t dx + G\int_0^1 (\psi - w_x)_x w_t dx + \mu \int_0^1 z(x,1) w_t dx = 0,$$

$$I_\rho \int_0^1 (3s_t - \psi_t)_t (3s_t - w_t) dx - D\int_0^1 (3s_x - \psi_x)_x (3s_t - w_t) dx$$
$$-G\int_0^1 (\psi - w_x)(3s_t - \psi_t) dx = 0,$$

$$3I_\rho\int_0^1 s_{tt}s_t dx - 3D\int_0^1 s_{xx}s_t dx + 3G\int_0^1 (\psi - w_x)s_t\,dx + 4\gamma\int_0^1 s_t s\,dx + 4\beta\int_0^1 s_t^2 dx = 0,$$

$$\tau|\mu|\int_0^1\int_0^1 zz_t(x,\sigma,t)d\sigma dx + |\mu|\int_0^1\int_0^1 zz_\sigma(x,\sigma,t)d\sigma dx = 0.$$

Summing up, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \rho w_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 I_\rho(3s_t-\psi_t)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 3I_\rho s_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 4\gamma s^2 dx\\
&+\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1 \tau|\mu| z^2(x,\sigma)d\sigma dx + 3D\int_0^1 s_x s_{xt}\,dx + D\int_0^1 (3s_x-\psi_x)(3s_x-\psi_x)_t dx \quad (1.3)\\
&+G\int_0^1 (\psi-w_x)_x w_t dx - 3G\int_0^1(\psi-w_x)s_t dx + G\int_0^1(\psi-w_x)\psi_t dx\\
&+3G\int_0^1(\psi-w_x)s_t dx = -4\beta\int_0^1 s_t^2 dx - \mu\int_0^1 z(x,1)w_t dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx\\
&+\frac{|\mu|}{2}\int_0^1 w_t^2 dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2\right] dx\\
&+\frac{1}{2}\frac{d}{dt}\int_0^1\left[4\gamma s^2 + \int_0^1 \tau|\mu| z^2(x,\sigma)d\sigma\right]dx + G\int_0^1(\psi-w_x)(\psi-w_x)_t dx \qquad (1.4)\\
&= -4\beta\int_0^1 s_t^2 dx - \mu\int_0^1 z(x,1)w_t dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx.
\end{aligned}$$

We end up with,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2\right] dx\\
&+\frac{1}{2}\frac{d}{dt}\int_0^1\left[G(\psi-w_x)^2 + \int_0^1 \tau|\mu| z^2(x,\sigma)d\sigma\right]dx = -\mu\int_0^1 z(x,1)w_t dx \qquad (1.5)\\
&-4\beta\int_0^1 s_t^2 dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx.
\end{aligned}$$

Then, the energy of the solution to the system (1.2) is given by

$$\begin{aligned}
E(t) &= \frac{1}{2}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2\right] dx \qquad (1.6)\\
&+\frac{1}{2}\int_0^1\left[4\gamma s^2 + G(\psi-w_x)^2 + \tau|\mu|\int_0^1 z^2(x,\sigma)d\sigma\right]dx.
\end{aligned}$$

On the existence, uniqueness, and smoothness of solution of problem (1.2), we introduce the vector function $\Phi = (w, u, \xi, v, s, y, z)^T, u = w_t, \xi = 3s - \psi, v = \xi_t$, and $y = s_t$ and thereby transform system (1.2) to

$$\begin{cases} \dfrac{d}{dt}\Phi(t) = \mathcal{A}\Phi(t), \quad t > 0, \\ \Phi(0) = \Phi_0 = (w_0, w_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, f_0)^T. \end{cases} \tag{1.7}$$

Where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} u \\ -\dfrac{1}{\rho}\left(G(3s - \xi - w_x)_x + \mu z(x,1)\right) \\ v \\ \dfrac{1}{I_\rho}\left(D\xi_{xx} + G(3s - \xi - w_x)\right) \\ y \\ \dfrac{1}{I_\rho}\left(Ds_{xx} - G(3s - \xi - w_x) - \dfrac{4\gamma}{3}s - \dfrac{4\beta}{3}y\right) \\ -\dfrac{1}{\tau}z_\sigma(x,\sigma) \end{pmatrix}.$$

We consider the following spaces

$$H_a^1 = \left\{v : v \in H^1(0,1) : v(0) = 0\right\}, \quad H_b^1 = \left\{v : v \in H^1(0,1) : v(1) = 0\right\},$$

and let

$$\mathcal{H} := H_a^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1)$$

$$\times L^2((0,1) \times (0,1)).$$

Be the Hilbert space equipped with the following inner product

$$(\Phi, \tilde{\Phi})_{\mathcal{H}} = \rho \int_0^1 u\tilde{u}dx + G\int_0^1 (3s - \xi - w_x)(3\tilde{s} - \tilde{\xi} - \tilde{w}_x)dx + I_\rho \int_0^1 v\tilde{v}dx$$

$$+3I_\rho \int_0^1 y\tilde{y}dx + D\int_0^1 \xi_x\tilde{\xi}_x dx + 4\gamma\int_0^1 s\tilde{s}dx + 3D\int_0^1 s_x\tilde{s}_x dx$$

$$+\tau|\mu| \int_0^1\int_0^1 z(x,\sigma)\tilde{z}(x,\sigma)d\sigma dx.$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \Phi \in \mathcal{H} \,|w \in H^2(0,1) \cap H_a^1(0,1), \quad \xi, s \in H^2(0,1) \cap H_b^1(0,1), \\ u \in H_a^1(0,1), \quad v, y \in H_b^1(0,1), z, z_\sigma \in L^2((0,1) \times (0,1)), \\ w_x(1) = \xi_x(0) = s_x(0) = 0 \end{array} \right\}.$$

We observe that $D(\mathcal{A})$ is independent of time $t > 0$. Furthermore, it is obvious that $D(\mathcal{A})$ is dense in $\mathcal{H}$. obvious that D(A) is dense in H. Now, we present the theorem of existence and uniqueness of solution, then present their proof. To this end we will use the semigroup method and the Lumer-Philips theorem.

**Theorem 1.1.** *Let $\Phi_0 \in \mathcal{H}$, , then there exists a unique weak solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem (1.7). Moreover, if $\Phi_0 \in D(\mathcal{A})$, then*

$$\Phi \in C(\mathbb{R}^+, D(\mathcal{A}) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

*Proof.* for any $\Phi \in D(\mathcal{A})$, we use the inner product and the operator $\mathcal{A}$, we have

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} = & - \int_0^1 u \left[ G(3s - \xi - w_x)_x + \mu z(x,1) \right] dx \\ & + G \int_0^1 (3y - v - u_x)(3s - \xi - w_x) dx \\ & + \int_0^1 v \left[ D\xi_{xx} + G(3s - \xi - w_x) \right] dx \\ & + 3 \int_0^1 y \left[ Ds_{xx} - G(3s - \xi - w_x) - \frac{4\gamma s}{3} - \frac{4\beta y}{3} \right] dx \\ & + D \int_0^1 v_x \xi_x dx + 4\gamma \int_0^1 y s dx + 3D \int_0^1 y_x s_x dx \\ & - |\mu| \int_0^1 \int_0^1 z(x,\sigma,t) z_\sigma(x,\sigma,t) d\sigma dx. \end{aligned}$$

Therefore

$$
\begin{aligned}
\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = &-G\int_0^1 u(3s-\xi-w_x)_x dx - \mu\int_0^1 uz(x,1)dx \\
&+3G\int_0^1 y(3s-\xi-w_x)dx - G\int_0^1 v(3s-\xi-w_x)dx \\
&-G\int_0^1 u_x(3s-\xi-w_x)dx + D\int_0^1 v\xi_{xx}dx \\
&+G\int_0^1 v(3s-\xi-w_x)dx + 3D\int_0^1 ys_{xx}dx \\
&-3G\int_0^1 y(3s-\xi-w_x)dx - 4\gamma\int_0^1 sydx - 4\beta\int_0^1 y^2dx \\
&+D\int_0^1 v_x\xi_x dx + 4\gamma\int_0^1 ysdx + 3D\int_0^1 y_x s_x dx \\
&-|\mu|\int_0^1\int_0^1 z(x,\sigma,t)z_\sigma(x,\sigma,t)d\sigma dx.
\end{aligned}
$$

Then, using integration by parts, and boundary condition, we get

$$
\begin{aligned}
\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = &-4\beta\int_0^1 y^2dx - \mu\int_0^1 uz(x,1)dx \\
&+G\int_0^1 u(3s-\xi-w_x)_x dx - G\int_0^1 u(3s-\xi-w_x)_x dx \\
&-D\int_0^1 v_x\xi_x dx - 3D\int_0^1 y_x s_x dx + D\int_0^1 v_x\xi_x dx \\
&+3D\int_0^1 y_x s_x dx - \frac{|\mu|}{2}\int_0^1\int_0^1 \frac{d}{d\sigma}z^2(x,\sigma)d\sigma dx \\
= &-4\beta\int_0^1 y^2dx - \mu\int_0^1 uz(x,1)dx \\
&-\frac{|\mu|}{2}\int_0^1 \left[z^2(x,1)-u^2\right]dx.
\end{aligned}
\tag{1.8}
$$

Now, using Yong's inquality, we find

$$
-\mu\int_0^1 uz(x,1)dx \leq \frac{|\mu|}{2}\int_0^1 u^2dx + \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx. \tag{1.9}
$$

Substituting (2.2) in (1.8), we find for some $c_1 > 0$,

$$
\begin{aligned}
\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} &\leq -4\beta\int_0^1 y^2dx + |\mu|\int_0^1 u^2dx \\
&\leq c_1\langle \Phi, \Phi\rangle_{\mathcal{H}}.
\end{aligned}
$$

This shows that the operator $\mathcal{A} - c_1 I$ is dissipative .

We have to show now that for all $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathcal{H}$ there exists $\Phi \in$

$D(\mathcal{A})$ such that

$$(I-\mathcal{A})\Phi = F. \tag{1.10}$$

Then in terms of it's components, equation (1.10) becomes :

$$\begin{cases} w-u=f_1, \\ \rho u + G(3s-\xi-w_x)_x + \mu z(x,1) = \rho f_2, \\ \xi - v = f_3, \\ I_\rho v - D\xi_{xx} - G(3s-\xi-w_x) = I_\rho f_4, \\ s-y=f_5, \\ 3I_\rho y - 3Ds_{xx} + 3G(3s-\xi-w_x) + 4\gamma s + 4\beta y = 3I_\rho f_6, \\ \tau z + z_\sigma = \tau f_7. \end{cases} \tag{1.11}$$

Following the same arguments in Nicaise and Pingotti [37] it follows from equation $(1.11)_7$, that

$$\tau z + z_\sigma = \tau f_7.$$

First, we solve the homogeneous equation, we have :

$$\tau z + z_\sigma = \tau f_7 \Longrightarrow z(x,\sigma) = ke^{-\sigma\tau}.$$

Then, using the variation of constants, we have

$$\tau k(x,\sigma)e^{-\sigma\tau} + k'(x,\sigma)e^{-\sigma\tau} - \tau k(x,\sigma)e^{-\sigma\tau} = \tau f_7.$$

Then,

$$k'(x,\sigma) = \tau e^{\sigma\tau} f_7 \Longrightarrow k(x,\sigma) = \tau \int_0^\sigma e^{\eta\tau} f_7 d\eta + c_1, c \in \mathbb{R}.$$

Hence,

$$z(x,\sigma) = ce^{-\sigma\tau} + \tau e^{-\sigma\tau}\int_0^\sigma e^{\eta\tau} f_7 d\eta.$$

Using the boundary conditions

$$z(x,\sigma) = u(x)e^{-\sigma\tau} + \tau e^{-\sigma\tau}\int_0^\sigma e^{\eta\tau} f_7 d\eta.$$

It follows that

$$z(x,1) = u(x)e^{-\tau} + \tau e^{-\tau}\int_0^1 e^{\eta\tau} f_7 d\eta. \tag{1.12}$$

Now we insert (1.12), $u = w - f_1, v = \xi - f_3$, and $y = s - f_5$ into $(1.11)_2$, $(1.11)_4$ and $(1.11)_6$, we obtain

$$\begin{cases} (\mu e^{-\sigma\tau} + \rho)w + G(3s - \xi - w_x)_x = \rho f_1 + \mu e^{-\sigma\tau} f_1 + \rho f_2 - \tau e^{-\sigma\tau} \displaystyle\int_0^1 e^{\eta\tau} f_7 d\eta, \\ I_\rho \xi - D\xi_{xx} - G(3s - \xi - w_x) = I_\rho(f_3 + f_4), \\ (3I_\rho + 4\gamma + 4\beta)s - 3Ds_{xx} + 3G(3s - \xi - w_x) = (3I_\rho + 4\beta)f_5 + (3I_\rho)f_6. \end{cases} \tag{1.13}$$

Then , we multiply $(1.13)_1$, $(1.13)_2$ and $(1.13)_3$ by $\tilde{w}, \tilde{\xi}$, and $\tilde{s}$ respectively and integrate over (0,1), we find

$$\begin{cases} \displaystyle\int_0^1 (\mu e^{-\sigma\tau} + \rho) w\tilde{w} dx + G \int_0^1 (3s - \xi - w_x)_x \tilde{w} dx \\ = \displaystyle\int_0^1 \left[ (\rho + \mu e^{-\sigma\tau}) f_1 + \rho f_2 - \tau e^{-\sigma\tau} \int_0^1 e^{\eta\tau} f_7 d\eta \right] \tilde{w} dx, \\ I_\rho \displaystyle\int_0^1 \xi\tilde{\xi} dx - D \int_0^1 \xi_{xx} \tilde{\xi} dx - G \int_0^1 (3s - \xi - w_x)\tilde{\xi} dx = I_\rho \int_0^1 (f_3 + f_4)\tilde{\xi} dx, \\ (3I_\rho + 4\gamma + 4\beta) \displaystyle\int_0^1 s\tilde{s} dx - 3D \int_0^1 s_{xx}\tilde{s} dx + 3G \int_0^1 (3s - \xi - w_x)\tilde{s} dx \\ = \displaystyle\int_0^1 \left[ (3I_\rho + 4\beta) f_4 + 3I_\rho f_6 \right] \tilde{s} dx. \end{cases} \tag{1.14}$$

To solve (1.14), we introduce the following variational formulation

$$B\left( (w, \xi, s), (\tilde{w}, \tilde{\xi}, \tilde{s}) \right) = L(\tilde{w}, \tilde{\xi}, \tilde{s}), \quad \forall (\tilde{w}, \tilde{\xi}, \tilde{s}) \in W. \tag{1.15}$$

Where the Hilbert space $W = H^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)$ is equipped with the norm

$$\|(w, \xi, s)\|_W^2 = \|(3s - \xi - w_x)\|_2^2 + \|w\|_2^2 + \|\xi_x\|_2^2 + \|s_x\|_2^2.$$

$B : W \times W \longrightarrow \mathbb{R}$ is bilinear from defined by

$$B\left( (w, \xi, s), (\tilde{w}, \tilde{\xi}, \tilde{s}) \right) = (\mu e^{-\tau\sigma} + \rho) \int_0^1 w\tilde{w} dx + I_\rho \int_0^1 \xi\tilde{\xi} dx \tag{1.16}$$
$$+ (3I_\rho + 4\gamma + 4\beta) \int_0^1 s\tilde{s} dx + D \int_0^1 \xi_x \tilde{\xi}_x dx + 3D \int_0^1 s_x \tilde{s}_x dx$$
$$+ G \int_0^1 (3s - \xi - w_x)(3\tilde{s} - \tilde{\xi} - \tilde{w}_x) dx,$$

and $L : W \longrightarrow \mathbb{R}$ is the linear form given by

$$L(\tilde{w}, \tilde{\xi}, \tilde{s}) = (\rho + \mu e^{-\tau\sigma}) \int_0^1 f_1 \tilde{w} dx + \rho \int_0^1 f_2 \tilde{w} dx \tag{1.17}$$

$$-\tau e^{-\tau\sigma}\int_0^1 \tilde{w}\int_0^\sigma f_7 e^{\eta\tau}d\eta dx + I_\rho\int_0^1 (f_3+f_4)\tilde{\xi}dx$$
$$+\int_0^1 \left[(3I_\rho+4\beta)f_4 + 3If_6\right]\tilde{s}dx.$$

We easily show that $B$ and $L$ are continuous. In addition form the definition of $B$, we have

$$\begin{aligned} B\Big((w,\xi,s),(\tilde{w},\tilde{\xi},\tilde{s})\Big) &= (\rho+\mu e^{-\tau\sigma})\int_0^1 w^2dx + I_\rho\int_0^1 \xi^2dx \\ &+ (3I_\rho+4\beta+4\gamma)\int_0^1 s^2dx + D\int_0^1 \xi_x^2dx \\ &+ 3D\int_0^1 s_x^2dx + G\int_0^1 (3s-\xi-w_x)^2dx, \\ &\geq \alpha\|(w,\xi,s)\|_w^2,\ \alpha>0. \end{aligned} \tag{1.18}$$

Hence $B$ is coercive. Therefore, by using the Lax-Milgram theorem we can obtain that problem (1.15) has a unique solution $(w,\xi,s)\in W$. Now substituning $w,\ \xi$, and $s$ into $(1.11)_1, (1.11)_3, (1.11)_5$ we get $u\in H^1(0,1)$, $v\in H_0^1(0,1)$, and $y\in H_0^1(0,1)$. Then using the classical elliptic regularity it follows that $\Phi\in D(\mathcal{A})$ therfore $\mathcal{A}$ is surjective.

Finally, by using the Lumer-Philips theorem, the well-posedness results of problem (1.7) stated in Theorem 1.1 are established. This completes the proof. □

## 1.3 Stability results

In this section, we state the stability results and we prove them by using the energy method. We establish an exponential stability result for the considered system in case of equal wave speeds.

### 1.3.1 Technical lemmas

In this subsection, we state and prove some technical lemmas essential to the proof of our stability result. We use multiplier technique to establish stability results for the energy of the solution of system (1.2). This requires constructing a suitable Lyapunov functional equivalent to energy as we elaborate in the subsequent section.

**Lemma 1.2.** *If $(w, \psi, s, z)$ is a solution of (1.2), then the energy functional $E$ defined by (1.6) satisfies*

$$E'(t) \leq -4\beta \int_0^1 s_t^2 dx + |\mu| \int_0^1 w_t^2 dx, \qquad \forall t \geq 0. \tag{1.19}$$

*Proof.* Multiplying the first three equations in the system(1.2) by $w_t$, $(3s_t - \psi_t)$ and $s_t$ respectively, then integrating each by parts over $(0, 1)$ using the boundary conditions

$$\begin{cases} \rho \displaystyle\int_0^1 w_{tt} w_t dx + G \int_0^1 (\psi - w_x)_x w_t dx + \mu \int_0^1 z(x,1) w_t dx = 0, \\ I_\rho \displaystyle\int_0^1 (3s_t - \psi_t)_t (3s_t - \psi_t) dx - D \int_0^1 (3s_x - \psi_x)_x (3s_t - \psi_t) dx \\ -G \displaystyle\int_0^1 (\psi - w_x)(3s_t - \psi_t) dx = 0, \\ 3I_\rho \displaystyle\int_0^1 s_{tt} s_t dx - 3D \int_0^1 s_{xx} s_t dx + 3G \int_0^1 (\psi - w_x) s_t dx + 4\gamma \int_0^1 s_t s dx \\ +4\beta \displaystyle\int_0^1 s_t^2 dx = 0, \end{cases} \tag{1.20}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \rho w_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 I_\rho (3s_t - \psi_t)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 3I_\rho s_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 4\gamma s^2 dx \\ &+ 3D\int_0^1 s_x s_{xt} dx + D\int_0^1 (3s_x - \psi_x)(3s_x - \psi_x)_t dx + G\int_0^1 (\psi - w_x)_x w_t dx \\ &\qquad -3G\int_0^1 (\psi - w_x) s_t dx + G\int_0^1 (\psi - w_x)\psi_t dx + 3G\int_0^1 (\psi - w_x) s_t dx \\ &\qquad\qquad = -4\beta\int_0^1 s_t^2 dx - \mu\int_0^1 z(x,1) w_t dx, \end{aligned} \tag{1.21}$$

that is

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + 3I_\rho s_t^2 + 3D s_x^2\right] dx \\ &+ \frac{1}{2}\frac{d}{dt}\int_0^1 4\gamma s^2 dx + G\int_0^1 (\psi - w_x)(\psi - w_x)_t dx = -4\beta\int_0^1 s_t^2 dx \\ &\qquad -\mu\int_0^1 z(x,1) w_t dx, \end{aligned} \tag{1.22}$$

we finally end up with,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + 3I_\rho s_t^2 + 3D s_x^2 + 4\gamma s^2\right] dx \\ &+ \frac{1}{2}\frac{d}{dt}\int_0^1 \left[G(\psi - w_x)^2\right] dx = -\mu\int_0^1 z(x,1) w_t dx - 4\beta\int_0^1 s_t^2 dx. \end{aligned} \tag{1.23}$$

Similarly, multiplying $(1.2)_4$ by $|\mu|z$, followed by integrating the product over $(0,1)\times(0,1)$ and then using the substitution $z(x,0,t)=w_t$ ,

$$\begin{aligned}\frac{\tau|\mu|}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\sigma)d\sigma dx &= -|\mu|\int_0^1\int_0^1 zz_\sigma(x,\sigma,t)d\sigma dx,\\ &= -\frac{|\mu|}{2}\int_0^1\int_0^1 \frac{d}{d\sigma}z^2(x,\sigma)d\sigma dx,\\ &= -\frac{|\mu|}{2}\left[\int_0^1 z^2(x,1,t)dx-\int_0^1 z^2(x,0,t)dx\right],\\ &= -\frac{|\mu|}{2}\left[\int_0^1 z^2(x,1,t)dx-\int_0^1 w_t^2dx\right].\end{aligned}$$

We obtain,

$$\frac{\tau|\mu|}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\sigma)d\sigma dx=-\frac{|\mu|}{2}\int_0^1 z^2(x,1)dx+\frac{|\mu|}{2}\int_0^1 w_t^2dx. \tag{1.24}$$

Next, merging (1.23) and (1.24, we note that from (1.6),

$$E'(t)=-\mu\int_0^1 z(x,1)w_tdx-4\beta\int_0^1 s_t^2dx-\frac{|\mu|}{2}\int_0^1 z^2(x,1)dx+\frac{|\mu|}{2}\int_0^1 w_t^2dx. \tag{1.25}$$

We now exploit Young's inequality on the first term of (1.25) to obtain

$$-\mu\int_0^1 z(x,1)w_tdx\le\frac{|\mu|}{2}\int_0^1 z^2(x,1)dx+\frac{|\mu|}{2}\int_0^1 w_t^2dx. \tag{1.26}$$

Consequently, substituting (1.26) in (1.25) completes the proof of (1.19). □

**Lemma 1.3.** *If $(w,\psi,s,z)$ is a solution of (1.2), then the functional $F_1$ defined by*

$$F_1(t)=-\rho\int_0^1 ww_tdx+\rho\int_0^1 w_t\int_0^x\psi(y)dydx.$$

*For any $\varepsilon_1>0$, satisfies the estimate*

$$\begin{aligned}\frac{d}{dt}F_1(t)\le -\frac{\rho}{2}\int_0^1 w_t^2dx+\rho\int_0^1(3s_t-\psi_t)^2dx+\varepsilon_1\int_0^1 z^2(x,1)dx\\ +9\rho\int_0^1 s_t^2dx+\left(\frac{G}{2}+\frac{\mu^2}{4\varepsilon_1}\right)\int_0^1(\psi-w_x)^2dx.\end{aligned} \tag{1.27}$$

*Proof.* Differentiating$F_1$,using the first equation in(1.2), integrating by parts the term containing $(\psi - w_x)_x$ and exploiting the fact that $\psi_t = -(3s_t - \psi_t) + 3s_t$,

$$\begin{aligned}
\frac{d}{dt}F_1(t) = &-\rho\int_0^1 (w_t^2 + w_{tt}w)dx + \rho\int_0^1 w_{tt}\int_0^x \psi(y)dydx \\
&+\rho\int_0^1 w_t\int_0^x \psi_t(y)dydx, \\
&= -\rho\int_0^1 w_t^2 dx - \int_0^1 \rho w_{tt} dx + \rho\int_0^1 w_{tt}\int_0^x \psi(y)dydx \\
&+\rho\int_0^1 w_t\int_0^x \psi_t(y)dydx, \\
&= -\rho\int_0^1 w_t^2 dx + \int_0^1 \left[G(\psi - w_x)_x - \mu z(x,1)\right] w dx \\
&+\int_0^1 \left[-G(\psi - w_x)_x - \mu z(x,1)\right]\int_0^1 \psi(y)dydx \\
&+\rho\int_0^1 w_t\int_0^x ((3s_t - \psi_t) + 3s_t)(y)dydx, \\
&= -\rho\int_0^1 w_t^2 dx + \int_0^1 G(\psi - w_x)_x w dx - \int_0^1 \mu z(x,1) w dx \\
&-G\int_0^1 (\psi - w_x)_x\int_0^1 \psi(y)dydx - \mu\int_0^1 z(x,1)\int_0^1 \psi(y)dydx \\
&-\rho\int_0^1 w_t\int_0^x (3s_t - \psi_t)(y)dydx + 3\rho\int_0^1 w_t\int_0^x s_t(y)dydx.
\end{aligned}$$

It follows that,

$$\begin{aligned}
\frac{d}{dt}F_1(t) = &-\rho\int_0^1 w_t^2 dx - \rho\int_0^1 w_t\int_0^x (3s_t - \psi_t)(y)dydx \\
&+3\rho\int_0^1 w_t\int_0^x s_t(y)dydx - \mu\int_0^1 \left(\int_0^x (\psi(y)dy - w)z(x,1)\right)dx \\
&+G\int_0^1 (\psi - w_x)\psi dx - G\int_0^1 (\psi - w_x)w_x dx.
\end{aligned}$$

We deduce that,

$$\begin{aligned}
\frac{d}{dt}F_1(t) = &-\rho\int_0^1 w_t^2 dx + G\int_0^1 (\psi - w_x)^2 dx - \rho\int_0^1 w_t\int_0^x (3s_t - \psi_t)(y)dydx \\
&+3\rho\int_0^1 w_t\int_0^x s_t(y)dydx - \mu\int_0^1 \left(\int_0^x \psi(y)dy - w\right) z(x,1)dx. \qquad (1.28)
\end{aligned}$$

By Young's and Poincaré's inequalities, the last three terms of (1.28) give

$$-\rho\int_0^1 w_t\int_0^x (3s_t - \psi_t)(y)dydx \leq \rho\int_0^1 \left(\int_0^x (3s_t - \psi_t)dy\right)^2 dx + \frac{\rho}{4}\int_0^1 w_t^2 dx,$$

$$\leq \rho \int_0^1 (3s_t - \psi_t)^2 dx + \frac{\rho}{4}\int_0^1 w_t^2 dx,$$

$$3\rho \int_0^1 w_t \int_0^x s_t(y)dydx \leq 9\rho \int_0^1 \left(\int_0^x s_t(y)dy\right)^2 dx + \frac{\rho}{4}\int_0^1 w_t^2 dx,$$
$$\leq 9\rho \int_0^1 s_t^2 dx \frac{\rho}{4}\int_0^1 w_t^2 dx.$$

And,

$$-\mu \int_0^1 \left(\int_0^x \psi(y)dy - w\right) z(x,1)dx \leq \frac{\mu^2}{4\varepsilon_1}\int_0^1 \left(\int_0^x \psi(y)dy - w\right)^2 dx$$
$$+\varepsilon_1 \int_0^1 z^2(x,1)dx \leq \int_0^1 (\psi - w_x)^2 dx + \varepsilon_1 \int_0^1 z^2(x,1)dx. \tag{1.29}$$

The combination of (1.28)-(1.29) leads to (1.27). □

**Lemma 1.4.** *If $(w, \psi, s, z)$ is a solution of (1.2), then the functional $F_2$ defined by*

$$F_2(t) = -I_\rho \int_0^1 (3s_t - \psi_t)(3s - \psi)dx.$$

*Satisfies the estimate*

$$\frac{d}{dt}F_2(t) \leq -I_\rho \int_0^1 (3s_t - \psi_t)^2 dx + \frac{3D}{2}\int_0^1 (3s_x - \psi_x)^2 dx \tag{1.30}$$
$$+\frac{G^2}{2D}\int_0^1 (\psi - w_x)^2 dx.$$

*Proof.* By direct computations, using the second equation in (1.2), we find

$$\frac{d}{dt}F_3(t) = -I_\rho \int_0^1 (3s_{tt} - \psi_{tt})(3s - \psi)dx - I_\rho \int_0^1 (3s_t - \psi_t)^2 dx$$
$$= -\int_0^1 [D(3s_{xx} - \psi_{xx}) + G(\psi - w_x)](3s - \psi)dx - I_\rho \int_0^1 (3s_t - \psi_t)^2 dx.$$

Then,

$$\frac{d}{dt}F_3(t) = -D\int_0^1 (3s_{xx} - \psi_{xx})(3s - \psi)dx - G\int_0^1 (\psi - w_x)(3s - \psi)dx$$
$$- I_\rho \int_0^1 (3s_t - \psi_t)^2 dx.$$

Integrating by parts, we obtain,

$$\frac{d}{dt}F_2(t) = -I_\rho \int_0^1 (3s_t - \psi_t)^2 dx + D\int_0^1 (3s_x - \psi_x)^2 dx(3s - \psi)dx - G\int_0^1 (\psi - w_x)(3s - \psi)dx. \tag{1.31}$$

Exploiting Young's and Poincaré's inequalities, we estimate the last term of (1.31) as follows

$$-G\int_0^1(\psi-w_x)(3s-\psi)dx \le \frac{G^2}{2D}\int_0^1(\psi-w_x)^2dx+\frac{D}{2}\int_0^1(3s-\psi)^2dx$$
$$\le \frac{G^2}{2D}\int_0^1(\psi-w_x)^2dx+\frac{D}{2}\int_0^1(3s_x-\psi_x)^2dx. \tag{1.32}$$

Consequently, the relation (1.30) follows directly by substituting (1.32) into (1.31).

□

**Lemma 1.5.** *If $(w,\psi,s,z)$ is a solution of (1.2), then the functional $F_3$ defined by*

$$F_3(t)=3I_\rho\int_0^1 s_t s dx+2\beta\int_0^1 s^2dx+3\rho\int_0^1 w_t\int_0^x s(y)dydx.$$

*For any $\varepsilon_2>0$, satisfies the estimate*

$$\frac{d}{dt}F_3(t)\le -3D\int_0^1 s_x^2dx-3\gamma\int_0^1 s^2dx+\varepsilon_2\int_0^1 w_t^2dx+\frac{9\mu^2}{4\gamma}\int_0^1 z^2(x,1)dx \tag{1.33}$$
$$+\left(3I_\rho+\frac{9\rho^2}{4\varepsilon_2}\right)\int_0^1 s_t^2dx.$$

*Proof.* Differentiating $F_3$, we find

$$\frac{d}{dt}F_3(t)=3I_\rho\int_0^1(s_{tt}s+s_t^2)dx+4\beta\int_0^1 s_t s dx+3\rho\int_0^1 w_{tt}\int_0^x s(y)dydx$$
$$+3\rho\int_0^1 w_t\int_0^x s_t(y)dydx.$$

Using (1.2), we find

$$\begin{aligned}\frac{d}{dt}F_3(t)=&\int_0^1\left(3Ds_{xx}-3G(\psi-w_x)-4\gamma s-4\beta s_t\right)sdx\\&-3\int_0^1\left[G(\psi-w_x)_x+\mu z(x,1)\right]\int_0^x s(y)dydx\\&+3\rho\int_0^1 w_t\int_0^x s_t(y)dydx+4\beta\int_0^1 s_t s dx+3I_\rho\int_0^1 s_t^2dx,\\=&\,3D\int_0^1 s_{xx}sdx-3G\int_0^1(\psi-w_x)sdx-4\gamma\int_0^1 s^2dx\\&-4\beta\int_0^1 s_t s dx-3G\int_0^1(\psi-w_x)_x\int_0^x s(y)dydx\\&-3\mu\int_0^1 z(x,1)\int_0^x s(y)dydx+3\rho\int_0^1 w_t\int_0^x s_t(y)dydx\\&+4\beta\int_0^1 s_t s dx+3I_\rho\int_0^1 s_t^2dx.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}F_3(t) = &-3D\int_0^1 s_x^2dx - 4\gamma\int_0^1 s^2dx + 3\rho\int_0^1 w_t\int_0^x s_t(y)dydx\\ &+3I_\rho\int_0^1 s_t^2dx - 3\mu\int_0^1 z(x,1)\int_0^x s(y)dydx\\ &-3G\int_0^1(\psi-w_x)sdx + 3G\int_0^1(\psi-w_x)sdx + 4\beta\int_0^1 s_tsdx\\ &-4\beta\int_0^1 s_tsdx.\end{aligned}$$

We finally arrive at,

$$\frac{d}{dt}F_3(t) = -3D\int_0^1 s_x^2dx - 4\gamma\int_0^1 s^2dx + 3\rho\int_0^1 w_t\int_0^x s_t(y)dydx \tag{1.34}$$
$$+3I_\rho\int_0^1 s_t^2dx - 3\mu\int_0^1 z(x,1)\int_0^x s(y)dydx.$$

Using Young's, Poincaré's and Cauchy-Schwarz inequalities, we have

$$-3\mu\int_0^1 z(x,1)\int_0^x s(y)dydx \le \frac{9\mu^2}{4\gamma}\int_0^1 z^2(x,1)dx + \gamma\int_0^1\left(\int_0^x s(y)dy\right)^2dx,$$
$$\le \frac{9\mu^2}{4\gamma}\int_0^1 z^2(x,1)dx + \gamma\int_0^1 s^2dx, \tag{1.35}$$
$$3\rho\int_0^1 w_t\int_0^x s_t(y)dydx \le \varepsilon_2\int_0^1 w_t^2dx + +\frac{9\rho^2}{4\varepsilon_2}\int_0^1\left(\int_0^x s_t(y)dy\right)^2dx,$$
$$\le \varepsilon_2\int_0^1 w_t^2dx + +\frac{9\rho^2}{4\varepsilon_2}\int_0^1 s_t^2dx. \tag{1.36}$$

For any $\varepsilon_2 > 0$. Estimate (1.33) follows directly by virtue of (1.34)-(1.36). □

The assumption of equal wave speeds $GI_\rho = \rho D$ plays an important role in the next two lemmas.

**Lemma 1.6.** *If $(w,\psi,s,z)$ is a solution of (1.2), then the functional $F_4$ defined by*

$$F_4(t) = -\int_0^1(3s_t-\psi_t)w_xdx - \int_0^1(3s_x-\psi_x)w_tdx + 3\int_0^1(3s_t-\psi_t)sdx.$$

*For any $\varepsilon_3 > 0$, satisfies the estimate*

$$\frac{d}{dt}F_4(t) \le -\frac{D}{2I\rho}\int_0^1(3s_x-\psi_x)^2dx + \varepsilon_3\int_0^1(3s_t-\psi_t)^2dx + \frac{9}{\varepsilon_3}\int_0^1 s_t^2dx \tag{1.37}$$
$$+\frac{I_\rho\mu^2}{D\rho^2}\int_0^1 z^2(x,1)dx + \left(\frac{G}{I_\rho}+\frac{G^2}{DI_\rho}\right)\int_0^1(\psi-w_x)^2dx.$$

*Proof.* As in the previous Lemmas, we have

$$\begin{aligned}
\frac{d}{dt}F_4(t) = & -\int_0^1 (3s_{tt}-\psi_{tt})w_x dx - \int_0^1 (3s_t-\psi_t)w_{tx}dx \\
& -\int_0^1 (3s_x-\psi_x)_t w_t dx - \int_0^1 (3s_x-\psi_x)w_{tt}dx \\
& +3\int_0^1 (3s_{tt}-\psi_{tt})s dx + 3\int_0^1 (3s_t-\psi_t)s_t dx,
\end{aligned}$$

using system (1.2), we get

$$\begin{aligned}
\frac{d}{dt}F_4(t) = & -\int_0^1 \left[\frac{G}{I_\rho}(\psi-w_x)+\frac{D}{I_\rho}(3s_{xx}-\psi_{xx})\right] w_x dx - \int_0^1 (3s_t-\psi_t)w_{tx}dx \\
& +\int_0^1 (3s_x-\psi_x)\left[\frac{\mu}{\rho}z(x,1)+\frac{G}{\rho}(\psi-w_x)_x\right] dx \\
& +3\int_0^1 \left[\frac{G}{I_\rho}(\psi-w_x)+\frac{D}{I_\rho}(3s_{xx}-\psi_{xx})\right] s dx + 3\int_0^1 (3s_t-\psi_t)s_t dx, \\
= & -\frac{G}{I_\rho}\int_0^1 (\psi-w_x)w_x dx - \frac{D}{I_\rho}\int_0^1 (3s_{xx}-\psi_{xx})w_x dx \\
& -\int_0^1 (3s_t-\psi_t)w_{tx}dx - \int_0^1 (3s_x-\psi_x)_t w_t dx \\
& +\frac{\mu}{\rho}\int_0^1 (3s_x-\psi_x)z(x,1)dx + \frac{G}{\rho}\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x dx \\
& +\frac{3G}{I_\rho}\int_0^1 (\psi-w_x)s dx + \frac{3D}{I_\rho}\int_0^1 (3s_{xx}-\psi_{xx})s dx \\
& +3\int_0^1 (3s_t-\psi_t)s_t dx.
\end{aligned}$$

Integration by parts the term containing $3s_{xx}-\psi_{xx}$,

$$\begin{aligned}
\frac{d}{dt}F_4(t) = & -\frac{G}{I_\rho}\int_0^1 (\psi-w_x)w_x dx + \frac{D}{I_\rho}\int_0^1 (3s_x-\psi_x)w_{xx}dx \\
& -\int_0^1 (3s_t-\psi_t)w_{tx}dx - \int_0^1 (3s_x-\psi_x)_t w_t dx \\
& +\frac{\mu}{\rho}\int_0^1 (3s_x-\psi_x)z(x,1)dx + \frac{G}{\rho}\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x dx \\
& +\frac{3G}{I_\rho}\int_0^1 (\psi-w_x)s dx - \frac{3D}{I_\rho}\int_0^1 (3s_x-\psi_x)s_x dx + 3\int_0^1 (3s_t-\psi_t)s_t dx.
\end{aligned}$$

And exploiting the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$, we find

$$\begin{aligned}
\frac{d}{dt}F_4(t) = &\frac{G}{I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{G}{I_\rho}\int_0^1 (\psi - w_x)(3s-\psi)dx \\
&- \frac{3G}{I_\rho}\int_0^1 (\psi - w_x)s dx - \frac{D}{I_\rho}\int_0^1 (3s_x - \psi_x)(\psi - w_x)_x dx \\
&- \frac{D}{I_\rho}\int_0^1 (3s_x - \psi_x)^2 dx + \frac{3D}{I_\rho}\int_0^1 (3s_x - \psi_x)s_x dx \\
&+ \int_0^1 (3s_t - \psi_t)w_{tx} dx - \int_0^1 (3s_t - \psi_t)_x w_t dx \\
&+ \frac{\mu}{\rho}\int_0^1 (3s_x - \psi_x)z(x,1)dx + \frac{G}{\rho}\int_0^1 (3s_x - \psi_x)(\psi - w_x)_x dx \\
&+ \frac{3G}{I_\rho}\int_0^1 (\psi - w_x)s dx - \frac{3D}{I_\rho}\int_0^1 (3s_x - \psi_x)s_x dx \\
&+ 3\int_0^1 (3s_t - \psi_t)s_t dx.
\end{aligned}$$

We end up with,

$$\begin{aligned}
\frac{d}{dt}F_4(t) = &-\frac{D}{I_\rho}\int_0^1 (3s_x - \psi_x)^2 + \frac{G}{I_\rho}\int_0^1 (\psi - w_x)^2 dx + 3\int_0^1 (3s_t - \psi_t)s_t dx \\
&+ \frac{G}{I_\rho}\int_0^1 (\psi - w_x)(3s - \psi)dx + \frac{\mu}{\rho}\int_0^1 (3s_x - \psi_x)z(x,1)dx. \qquad (1.38)
\end{aligned}$$

Next, Young's and Poincaré's inequalities guarantee the relations

$$3\int_0^1 (3s_t - \psi_t)s_t dx \le \varepsilon_3 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{9}{\varepsilon_3}\int_0^1 s_t^2 dx, \qquad (1.39)$$

$$\begin{aligned}
\frac{G}{I_\rho}\int_0^1 (\psi - w_x)(3s - \psi)dx &\le \frac{G^2}{DI_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{D}{4I\rho}\int_0^1 (3s - \psi)^2 dx, \\
&\le \frac{G^2}{DI_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{D}{4I\rho}\int_0^1 (3s_x - \psi_x)^2 dx,
\end{aligned}$$

and,

$$\frac{\mu}{\rho}\int_0^1 (3s_x - \psi_x)z(x,1)dx \le \frac{D}{4I\rho}\int_0^1 (3s_x - \psi_x)^2 dx + \frac{I_\rho \mu^2}{D\rho^2}\int_0^1 z^2(x,1)dx, \qquad (1.40)$$

for any $\varepsilon_3 > 0$, estimate (1.37) follows directly by substituting (1.39)-(1.40) into (1.38). $\square$

**Lemma 1.7.** *If $(w, \psi, s, z)$ is a solution of (1.2), then the functional $F_5$ defined by*

$$F_5(t) = \int_0^1 (\psi - w_x)s_t dx - \int_0^1 w_t s_x dx,$$

*for any $\varepsilon_4, \varepsilon_5 > 0$, satisfies the estimate*

$$\frac{d}{dt}F_5(t) \leq -\frac{G}{2I_\rho}\int_0^1 (\psi - w_x)^2 dx + \varepsilon_4 \int_0^1 (3s_t - \psi_t)^2 dx + \varepsilon_5 \int_0^1 z^2(x,1)dx$$

$$+ \left(\frac{16\gamma^2}{9I_\rho} + \frac{\mu^2}{4\rho^2\epsilon_5}\right)\int_0^1 s_x^2 dx + \left(3 + \frac{1}{4\varepsilon_4} + \frac{16\beta^2}{9I_\rho}\right)\int_0^1 s_t^2 dx. \tag{1.41}$$

*Proof.* Differentiating $F_5$ then integrating by parts over $(0,1)$ the term containing $s_{xt}$ and using the substitution $w_{xt} = -(3s_t - \psi_t) - (\psi - w_x)_t + 3s_t$,

$$\begin{aligned}
\frac{d}{dt}F_5(t) &= \int_0^1 (\psi - w_x)s_{tt}dx - \int_0^1 w_{tt}s_x dx - \int_0^1 w_t s_{xt} dx + \int_0^1 (\psi - w_x)_t s_t dx,\\
&= \int_0^1 (\psi - w_x)s_{tt}dx - \int_0^1 w_{tt}s_x dx - \int_0^1 (3s_t - \psi_t)s_t dx\\
&- \int_0^1 (\psi - w_x)_t s_t dx + \int_0^1 (\psi - w_x)s_{tt}dx - \int_0^1 w_{tt}s_x dx\\
&+ 3\int_0^1 s_t dx + \int_0^1 (\psi - w_x)_t s_t dx.
\end{aligned}$$

We arrive at,

$$\frac{d}{dt}F_5(t) = \int_0^1 (\psi - w_x)s_{tt}dx - \int_0^1 w_{tt}s_x dx - \int_0^1 (3s_t - \psi_t)s_t dx + 3\int_0^1 s_t^2 dx. \tag{1.42}$$

Next, using (1.42), (1.2) and integrating by parts the term containing $s_x$, we end up with

$$\begin{aligned}
\frac{d}{dt}F_5(t) &= \int_0^1 (\psi - w_x)s_{tt}dx - \int_0^1 w_{tt}s_x dx - \int_0^1 (3s_t - \psi_t)s_t dx + 3\int_0^1 s_t^2 dx,\\
&= \int_0^1 (\psi - w_x)\left(-\frac{4\beta}{3I_\rho}s_t - \frac{4\gamma}{3I_\rho}s - \frac{3G}{3I_\rho} + \frac{D}{I_\rho}s_{xx}\right)dx\\
&+ \int_0^1 \left[\frac{\mu}{\rho}z(x,1)dx + \frac{G}{\rho}(\psi - w_x)_x\right]s_x dx - \int_0^1 (3s_t - \psi_t)s_t dx + 3\int_0^1 s_t^2 dx,
\end{aligned}$$

that is,

$$\begin{aligned}
\frac{d}{dt}F_5(t) = &-\frac{4\beta}{3I_\rho}\int_0^1(\psi-w_x)s_t dx-\frac{4\gamma}{3I_\rho}\int_0^1(\psi-w_x)s dx-\frac{G}{I_\rho}\int_0^1(\psi-w_x)^2dx\\
&+\frac{D}{I_\rho}\int_0^1(\psi-w_x)s_{xx}dx+\frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx+\frac{G}{\rho}\int_0^1(\psi-w_x)_x s_x dx\\
&-\int_0^1(3s_t-\psi_t)s_t dx+3\int_0^1 s_t^2 dx,\\
= &-\frac{4\beta}{3I_\rho}\int_0^1(\psi-w_x)s_t dx-\frac{4\gamma}{3I_\rho}\int_0^1(\psi-w_x)s dx-\frac{G}{I_\rho}\int_0^1(\psi-w_x)^2dx\\
&+\frac{D}{I_\rho}\int_0^1(\psi-w_x)s_{xx}dx+\frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx-\frac{G}{\rho}\int_0^1(\psi-w_x)s_{xx} dx\\
&-\int_0^1(3s_t-\psi_t)s_t dx+3\int_0^1 s_t^2 dx,
\end{aligned}$$

we arrive at

$$\begin{aligned}
\frac{d}{dt}F_5(t) = &-\frac{G}{I_\rho}\int_0^1(\psi-w_x)^2dx+3\int_0^1 s_t^2dx-\frac{4\gamma}{3I_\rho}\int_0^1 s(\psi-w_x)dx\\
-\frac{4\beta}{3I_\rho}\int_0^1 s_t(\psi-w_x)dx&-\int_0^1(3s_t-\psi_t)s_t dx+\frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx. \qquad (1.43)
\end{aligned}$$

Exploiting Young's and Poincaré's inequalities, the last four terms of (1.43) are estimated as follows

$$\begin{aligned}
-\frac{4\gamma}{3I_\rho}\int_0^1 s(\psi-w_x)dx &\le \frac{G}{4I_\rho}\int_0^1(\psi-w_x)^2dx+\frac{16\gamma^2}{9I_\rho}\int_0^1 s^2dx,\\
&\le \frac{G}{4I_\rho}\int_0^1(\psi-w_x)^2dx+\frac{16\gamma^2}{9I_\rho}\int_0^1 s_x^2dx, \qquad (1.44)\\
-\frac{4\beta}{3I_\rho}\int_0^1 s_t(\psi-w_x)dx &\le \frac{G}{4I_\rho}\int_0^1(\psi-w_x)^2dx+\frac{16\beta^2}{9I_\rho}\int_0^1 s_t^2dx,\\
-\int_0^1(3s_t-\psi_t)s_t dx &\le \varepsilon_4\int_0^1(3s_t-\psi_t)^2dx+\frac{1}{4\varepsilon_4}\int_0^1 s_t^2dx,\\
\frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx &\le \frac{\mu^2}{4\rho^2\varepsilon_5}\int_0^1 s_x^2dx+\varepsilon_5\int_0^1 z^2(x,1)dx, \qquad (1.45)
\end{aligned}$$

for any $\varepsilon_4, \varepsilon_5 > 0$, the assertion of the lemma follows from the estimates (1.44)-(1.45) and (1.43). □

**Lemma 1.8.** *If $(w,\psi,s,z)$ is a solution of (1.2), then the functional $F_6$ defined by*

$$F_6(t) = \tau\int_0^1\int_0^1 e^{-\sigma\tau}z^2(x,\sigma)d\sigma dx,$$

*satisfies, for $m_1 > 0$ the estimate*

$$\frac{d}{dt}F_6(t) \le -m_1 \int_0^1 z^2(x,1)dx - m_1\tau \int_0^1 \int_0^1 z^2(x,\sigma)d\sigma dx + \int_0^1 w_t^2 dx. \tag{1.46}$$

*Proof.* Differentiate $F_6$ and use the fourth equation in the system (1.2) and $z(x,0) = w_t$ as follows

$$\frac{d}{dt}F_6(t) = 2\tau \int_0^1 \int_0^1 e^{-\sigma\tau} z_t(x,\sigma)z(x,\sigma)d\sigma dx,$$

$$\begin{aligned}
\frac{d}{dt}F_6(t) &= -2 \int_0^1 \int_0^1 e^{-\sigma\tau} z(x,\sigma)z_\sigma(x,\sigma)d\sigma dx, \\
&= -\int_0^1 \int_0^1 \frac{d}{d\sigma}\left[e^{-\sigma\tau}z^2(x,\sigma)\right] d\sigma dx - \tau \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x,\sigma)d\sigma dx, \\
&= -\int_0^1 \left[e^{-\tau}z^2(x,1) - z^2(x,0)\right] dx - \tau \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x,\sigma)d\sigma dx, \\
&= -\int_0^1 e^{-\tau}z^2(x,1)dx + \int_0^1 w_t^2 dx - \tau \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x,\sigma)d\sigma dx.
\end{aligned}$$

Observe that, $\forall\sigma \in (0,1)$, the relation $e^{-\tau} \le e^{-\tau\sigma} \le 1$ holds. Therefore, for some $m_1 = e^{-\tau}$, we arrive at the estimate(1.46). □

### 1.3.2 Exponential stability

In this subsection, using the lemmas obtained in subsection 1.3.1, we state and prove our main stability results.

**Lemma 1.9.** *Let $N, N_k, k = 1, \cdots, 6$, be positive constants. The functional defined by*

$$\mathcal{L}(t) = NE(t) + \sum_{k=1}^{6} N_k F_k(t), \quad N_k > 0, k = 1, \cdots, 6, t \ge 0, \tag{1.47}$$

*satisfies the equivalence relation*

$$c_1 E(t) \le \mathcal{L} \le c_2 E(t), \quad \forall t \ge 0, \tag{1.48}$$

*for some positive constants $c_1$ and $c_2$.*

*Proof.* Let $\mathbf{L} = \sum_{k=1}^{6} N_k F_k(t)$,

$$|\mathbf{L}(t)| = |N_1F_1 + N_2F_2 + N_3F_3 + N_4F_4 + N_5F_5 + N_6F_6|,$$

$$
\begin{aligned}
|\mathbf{L}(t)| \leq \rho N_1 \int_0^1 |w w_t| dx + \rho N_1 \int_0^1 \left| w_t \int_0^x \psi(y) dy \right| dx \\
+ I_\rho N_2 \int_0^1 |(3s_x - \psi_x)(3s_t - \psi_t)| dx + 3 I_\rho N_3 \int_0^1 |s_t s| dx \\
+ 2\beta N_3 \int_0^1 s^2 dx + 3\rho N_3 \int_0^1 \left| w_t \int_0^x \psi(y) dy \right| dx \\
+ 3N_4 \int_0^1 |(3s_t - \psi_t) s| dx + N_4 \int_0^1 |(3s_t - \psi_t) w_x| dx \\
+ N_4 \int_0^1 |(3s_x - \psi_x) w_t| dx + N_5 \int_0^1 |(\psi - w_x) s_t| \, dx \\
+ N_5 \int_0^1 |w_t s_x| \, dx + \tau N_6 \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x,\sigma) d\sigma dx.
\end{aligned}
$$

Exploiting Young's, Poincaré's, Cauchy-Schwarz inequalities (1.6) accompanied with the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$ and $e^{-\sigma\tau} \leq 1$ for all $\sigma \in (0,1)$, we deduce that for some positive constant $\eta$.

$$
\begin{aligned}
|\mathbf{L}(t)| \leq \eta \int_0^1 \left[ w_t^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_t^2 + s_x^2 + s^2 + (\psi - w_x)^2 \right] dx \\
+ \eta \int_0^1 \int_0^1 z^2(x,\sigma) d\sigma dx \leq \eta E(t).
\end{aligned}
$$

It is easy to observe that, from (1.47) that $|\mathcal{L}(t) - N E(t)| \leq \eta E(t)$, which is equivalent to

$$
(N - \eta) E(t) \leq \mathcal{L}(t) \leq (N + \eta) E(t),
$$

and hence the relation (1.48) follows by taking $N$ large enough. □

At this point, we're in position to prove our stability result which reads as follows.

**Theorem 1.10.** *Let $(w, \psi, s, z)$ be a solution of (1.2), and suppose that $GI_\rho = \rho D$, there exist a positive number $\bar{\mu}$ such that if $|\mu| < \bar{\mu}$, then the energy $E(t)$of (1.2) defined by (1.6) vanishes exponentially as $t$ approaches infinity, i.e.*

$$
E(t) \leq a e^{-bt}, \quad t \geq 0, \tag{1.49}
$$

*for some positive constants $a$ and $b$.*

*Proof.* We proceed by differentiating (1.47), then substitute for functionals $F_1$ to $F_6$ using estimates (1.27), (1.30), (1.33), (1.37), (1.41) and (1.46) respectively. Setting $N_1 = 1, \ \ \varepsilon_1 = \varepsilon_5 = \mu^2, \ \ \varepsilon_2 = \dfrac{\rho}{4N_3}, \ \ \varepsilon_3 = \dfrac{I_\rho N_2}{2N_4}, \ \ N_6 = |\mu|,$

$$
\begin{aligned}
\mathcal{L}'(t) \leq & N E' + \sum_{k=1}^{6} N_k' F_k'(t), \\
\leq & -\left[4\beta N - 9\rho N_1 - \left(3I_\rho + \frac{9\rho^2}{4\varepsilon_2}\right) N_3 - \frac{9}{\varepsilon_3} N_4 \right. \\
& \left. - \left(3 + \frac{1}{4\varepsilon_4} + \frac{16\beta^2}{9I_\rho}\right)\right] \int_0^1 s_t^2 dx - 3\gamma N_3 \int_0^1 s^2 dx \\
& - \left[I_\rho N_3 - \rho N_1 - \varepsilon_3 N_4 - \varepsilon_4 N_5\right] \int_0^1 (3s_t - \psi_t)^2 dx \\
& - \left[\frac{DN_4}{2I_\rho} - \frac{3DN_2}{2}\right] \int_0^1 (3s_x - \psi_x)^2 dx \\
& - \left[3DN_3 - \left(\frac{16\gamma^2}{9I_\rho} + \frac{\mu^2}{4\rho^2\varepsilon_5}\right) N_5\right] \int_0^1 s_x^2 dx \\
& - \left[-\frac{G^2 N_2}{2D} - \left(\frac{G}{I_p} + \frac{G^2}{DI_p}\right) N_4 + \frac{GN_5}{2I_\rho}\right] \int_0^1 (\psi - w_x)^2 dx \\
& + \left[\varepsilon_1 N_1 + \frac{9\mu^2 N_3}{4\gamma} + \frac{I_\rho \mu^2 N_3}{D\rho^2} + \varepsilon_5 N_5 - m_1 N_6\right] \int_0^1 z^2(x,1) dx \\
& - m_1 \tau |\mu| \int_0^1 \int_0^1 z^2(x,\sigma) d\sigma dx - \left[\frac{\rho}{2} - |\mu| N - \varepsilon_2 N_3 - N_6\right] \int_0^1 w_t^2 dx,
\end{aligned}
$$

we end up with

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -\left[4\beta N - c_3 - c_3 N_3(1 + N_3) - \frac{cN_4^2}{N_2} - c_3 N_5 \left(1 + \frac{1}{\varepsilon_4}\right)\right] \int_0^1 s_t^2 dx \\
& - 3\gamma N_3 \int_0^1 s^2 dx - \left[\frac{I_\rho}{2} N_2 - \rho - \varepsilon_4 N_5\right] \int_0^1 (3s_t - \psi_t)^2 dx \\
& - \left[\frac{DN_4}{2I_\rho} - \frac{3DN_2}{2}\right] \int_0^1 (3s_x - \psi_x)^2 dx - \left[3DN_3 - c_3 N_5\right] \int_0^1 s_x^2 dx \\
& - \left[\frac{GN_5}{2I_\rho} - c_3 - c_3 N_2 - c_3 N_4\right] \int_0^1 (\psi - w_x)^2 dx \\
& - |\mu| \left[m_1 - |\mu| \left(1 + \frac{9N_3}{4\gamma} + \frac{I_\rho N_4}{D\rho^2} + N_5\right)\right] \int_0^1 z^2(x,1) dx \\
& - m_1 \tau |\mu| \int_0^1 \int_0^1 z^2(x,\sigma) d\sigma dx - \left[\frac{\rho}{4} - |\mu|(N+1)\right] \int_0^1 w_t^2 dx,
\end{aligned}
$$

for some $c_3 > 0$. Next, we choose $N_2$ large enough such that

$$
k = \frac{I_\rho}{2} N_2 - \rho > 0.
$$

Fixing $N_2$ permits to choose $N_4$ large enough such that

$$\frac{DN_4}{2I_\rho} - \frac{3DN_2}{2} > 0.$$

With $N_2$ and $N_4$ fixed, we can easily choose $N_5$ large enough such that

$$\frac{GN_5}{2I_\rho} - c_3 - c_3 N_2 - c_3 N_4 > 0.$$

We pick $\varepsilon_4$ adequately small and $N_3$ sufficiently large such that

$$k - \varepsilon_4 N_5 > 0 \quad \text{and} \quad 3DN_3 - c_3 N_5 > 0,$$

respectively. Next, we select N sufficiently large such that (1.48) remains valid and that

$$4\beta N - c_3 - c_3 N_3(1+N_3) - \frac{cN_4^2}{N_2} - c_3 N_5 \left(1 + \frac{1}{\varepsilon_4}\right) > 0.$$

Finally, pick

$$\bar{\mu} = \min\left\{ \frac{m_1}{\left(1 + \frac{9N_3}{4\gamma} + \frac{I_\rho N_4}{D\rho^2} + N_5\right)}, \frac{\rho}{4(N+1)} \right\},$$

to we end up with

$$\begin{aligned} \mathcal{L}'(t) \leq & -\alpha \int_0^1 \left[w_t^2 + s_t^2 + (3s_t - \psi_t)^2 + (s_x - \psi_x)^2 + s_x^2 + s^2\right] dx \\ & - \alpha \int_0^1 \left[(\psi - w_x)^2 + z^2(x,1) + \int_0^1 z^2(x,\sigma)d\sigma\right] dx, \end{aligned}$$

for some $\alpha > 0$. By the virtue of (1.6), it is clear that for some $\alpha_0 > 0$

$$\mathcal{L}'(t) \leq -\alpha_0 E(t), \quad \forall \geq 0. \tag{1.50}$$

It then follows directly from (1.48) and (1.50) that

$$\mathcal{L}'(t) \leq -b\mathcal{L}(t), \quad \forall \geq 0, \tag{1.51}$$

where $b = \dfrac{\alpha_0}{c_2}$. A simple integration of (1.51) over $(0,t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-bt}, \quad \forall \geq 0. \tag{1.52}$$

Consequently, the assertion of the relation (1.49) follows from (1.52) and (1.48) with $a = c_2 E(0)/c_1$. $\square$

Chapter 2

# Analysis of thermoelastic laminated Timoshenko beam with second sound

## 2.1 Position of the problem

In this chapter, we consider the following thermoelastic laminated Timoshenko beam in $(0,1)\times(0,\infty)$:

$$
\begin{cases}
\rho\omega_{tt} + G\left(\psi - \omega_x\right)_x + \delta\theta_x = 0,\\
I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G\left(\psi-\omega_x\right) = 0,\\
I_\rho s_{tt} - Ds_{xx} + G\left(\psi-\omega_x\right) + \dfrac{4}{3}\gamma s + \dfrac{4}{3}\beta s_t + \dfrac{4}{3} = 0,\\
\rho_3\theta_t + q_x + \delta\omega_{tx} = 0,\\
\tau q_t + \alpha q + \theta_x = 0,
\end{cases}
\tag{2.1}
$$

we have the following boundary conditions and initial condition

$$
\begin{cases}
\omega_x(0,t) = \psi(0,t) = s(0,t) = \theta(0,t) = 0, \quad t\in(0,\infty),\\
\omega_x(1,t) = \psi(1,t) = s(1,t) = \theta(1,t) = 0, \quad t\in(0,\infty),
\end{cases}
\tag{2.2}
$$

$$
\begin{cases}
\omega(x,0) = \omega_0(x), \psi(x,0) = \psi_0(x), s(x,0) = s_0(x), \theta(x,0) = \theta_0(x), \quad x\in(0,1),\\
q(x,0) = q_0(x), \omega_t(x,0) = \omega_1(x), \psi_t(x,0) = \psi_1(x), s_t(x,0) = s_1(x), \quad x\in(0,1),
\end{cases}
\tag{2.3}
$$

where $\rho, G, I_\rho, D, \gamma, \beta, \rho_3, \delta, \tau$, and $\alpha$ are positive constants. $\theta(x,t)$ represents the difference temperature and $q(x,t)$ is the heat flux.

## 2.2 Existence and uniqueness

We start by denoting the vector-valued function by $U$ :

$$\begin{aligned}
&U = (\omega, \Phi, 3s-\psi, 3\Lambda-\Psi, s, \Lambda, \theta, q)^T,\\
&\quad\text{with } \Phi = \omega_t,\\
&\Psi = \psi_t, \text{ and } \Lambda = s_t.
\end{aligned} \tag{2.4}$$

Then, systems (2.1)-(2.3) can be written as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}U(t) = \mathcal{A}U, \quad t>0,\\ U(0) = U_0 = (\omega_0, \omega_1, 3s_0-\psi_0, 3s_1-\psi_1, s_0, s_1, \theta_0, q_0)^T. \end{cases} \tag{2.5}$$

Where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} \Phi \\ -\dfrac{G}{\rho}(\psi-\omega_x)_x - \dfrac{\delta}{\rho}\theta_x \\ 3\Lambda - \Psi \\ \dfrac{D}{I_\rho}(3s-\psi)_{xx} + \dfrac{G}{I_\rho}(\psi-\omega_x) \\ \Lambda \\ \dfrac{D}{I_\rho}s_{xx} - \dfrac{G}{I_\rho}(\psi-\omega_x) - \dfrac{4\gamma}{3I_\rho}s - \dfrac{4\beta}{3I_\rho}\Lambda \\ -\dfrac{1}{\rho_3}q_x - \dfrac{\delta}{\rho_3}\Phi_x \\ -\dfrac{\alpha}{\tau}q - \dfrac{1}{\tau}\theta_x \end{pmatrix}. \tag{2.6}$$

We have the following spaces

$$\begin{aligned}
&L^2_*(0,1) = \left\{ v \in L^2(0,1) : \int_0^1 v(x)\mathrm{d}x = 0 \right\},\\
&H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1),\\
&H^2_*(0,1) = \left\{ v \in H^2(0,1) : v_x(0) = v_x(1) = 0 \right\}.
\end{aligned} \tag{2.7}$$

Let

$$\begin{aligned}
\mathcal{H} = &H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1)\\
&\times L^2(0,1) \times L^2(0,1) \times L^2_*(0,1).
\end{aligned} \tag{2.8}$$

We have the inner product

$$
\begin{aligned}
(U,\widetilde{U})_{\mathcal{H}} =&\rho \int_0^1 \Phi\widetilde{\Phi}dx + I_\rho \int_0^1 (3\Lambda-\Psi)(3\widetilde{\Lambda}-\widetilde{\Psi})\mathrm{d}x + 3I_\rho \int_0^1 \Lambda\widetilde{\Lambda}dx \\
&+ \rho_3 \int_0^1 \theta\widetilde{\theta}\mathrm{d}x + \tau \int_0^1 q\widetilde{q}\,\mathrm{d}x + 4\gamma \int_0^1 \widetilde{s}\tilde{d}x + D\int_0^1 (3s-\psi)_x(3\widetilde{s}-\widetilde{\psi})_x\,\mathrm{d}x \\
&+ G\int_0^1 (\psi-\omega_x)\left(\widetilde{\psi}-\widetilde{\omega}_x\right)\mathrm{d}x + 3D\int_0^1 s_x\widetilde{s}_x\,\mathrm{d}x.
\end{aligned}
\tag{2.9}
$$

The domain of $\mathcal{A}$ is given by

$$
D\mathcal{A} = \left\{ U\in\mathcal{H} \;\middle|\; \begin{array}{c} \omega \in H^2_*0,1\cap H^1_*0,1, 3s-\psi, s\in H^20,1\cap H^1_00,1, \\ \Phi, q\in H^1_*0,1, 3\Lambda-\Psi, \Lambda, \theta\in H^1_00,1 \end{array} \right\}. \tag{2.10}
$$

The well posedness result can be stated in the following theorem.

**Theorem 2.1.** *Let $U_0 \in \mathcal{H}$ then problems (2.1)-(2.3) admit a unique weak solution $U\in C\left(\mathbb{R}^+,\mathcal{H}\right)$. In addition, if $U_0\in D(\mathcal{A})$, then $U\in C\left(\mathbb{R}^+, D(\mathcal{A})\right)\cap C^1\left(\mathbb{R}^+,\mathcal{H}\right)$.*

*Proof.* It is easy to obtain that, for any $U=(\omega,\Phi,3s-\psi,3\Lambda-\Psi,s,\Lambda,\theta,q)^T\in D(\mathcal{A})$.

$$
(\mathcal{A}U,U)_{\mathcal{H}} = -4\beta\int_0^1 \Lambda^2\,\mathrm{d}x - \alpha\int_0^1 q^2\,\mathrm{d}x \le 0, \tag{2.11}
$$

which implies the operator $\mathcal{A}$ is a dissipative operator. In what follows, we shall show the operator $Id-\mathcal{A}$ is surjective. In other words, given $F=(f_1,f_2,f_3,f_4,$ $f_5,f_6,f_7,f_8)\in\mathcal{H}$, we will seek a solution $V=(v_1,v_2,v_3,\ v_4,v_5,v_6,v_7,v_8)\in D(\mathcal{A})$ of

$$
(Id-\mathcal{A})V = F. \tag{2.12}
$$

We rewrite (2.12) as

$$
\begin{cases}
v_1 - v_2 = f_1, \\
\rho v_2 - Gv_{1xx} - Gv_{3x} + 3Gv_{5x} + \delta v_{7x} = \rho f_2, \\
v_3 - v_4 = f_3, \\
I_\rho v_4 - Dv_{3xx} - 3Gv_5 + Gv_3 + Gv_{1x} = I_\rho f_4, \\
v_5 - v_6 = v_5, \\
\left(I_\rho + \dfrac{4}{3}\beta\right)v_6 - Dv_{5xx} - Gv_3 - Gv_{1x} + \left(3G+\dfrac{4}{3}\gamma\right)v_5 = I_\rho f_6, \\
\rho_3 v_7 + v_{8x} + \delta v_{2x} = \rho_3 v_7, \\
(\tau+\alpha)v_8 + v_{7x} = \tau f_8.
\end{cases}
\tag{2.13}
$$

Which implies that

$$v_2 = v_1 - f_1, \tag{2.14}$$

$$v_4 = v_3 - v_3, \tag{2.15}$$

$$v_6 = v_5 - f_5, \tag{2.16}$$

$$v_{7x} = -(\tau + \beta)v_8 + \tau f_8. \tag{2.17}$$

We infer from (2.17) that

$$v_7 = -(\tau + \alpha)\int_0^x v_8(y)\mathrm{d}y + \tau\int_0^x f_8(y)\mathrm{d}y. \tag{2.18}$$

Replacing (2.14) – (2.16) and (2.18) in (2.13), we see that

$$\begin{cases} \rho v_1 - Gv_{1xx} - Gv_{3x} + 3Gv_{5x} - \delta(\tau+\alpha)v_8 = \rho\left(f_1 + f_2\right) - \tau f_8, \\ I_\rho v_3 - Dv_{3xx} - 3Gv_5 + Gv_3 + Gv_{1x} = I_\rho\left(f_3 + f_4\right), \\ \left(I_\rho + 3G + \frac{4}{3}\beta + \frac{4}{3}\gamma\right)v_5 - Dv_{5xx} - Gv_3 - Gv_{1x} = I_\rho\left(f_5 + f_6\right) + \frac{4}{3}\beta f_5, \\ -\rho_3(\tau+\alpha)\int_0^x v_8(y)\mathrm{d}y + v_{8x} + \delta v_{1x} = \rho_3 f_7 - \rho_3\tau\int_0^x f_8(y)\mathrm{d}y + \delta f_{1x}. \end{cases} \tag{2.19}$$

We multiply (2.19) by $\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5$, and $(\tau+\alpha)\int_0^x \widetilde{v}_8(y)\mathrm{d}y$, respectively, and integrate their sum over $(0,1)$ to get the following variational formulation

$$\mathcal{B}\left((v_1, v_3, v_5, v_8), (\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8)\right) = \mathcal{L}\left(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8\right), \tag{2.20}$$

where the bilinear form $\mathcal{B} : \left[H^1{}_*(0,1) \times H_0^1(0,1) \times \ H_0^1(0,1) \times L_*^2(0,1)\right]^2 \longrightarrow \mathbb{R}$ is given by

$$\begin{aligned} &\mathcal{B}\left((v_1, v_3, v_5, v_8), (\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8)\right) \\ &= G\int_0^1 \left(-v_{1x} - v_3 + 3v_5\right)\left(-\widetilde{v}_{1x} - \widetilde{v}_3 + 3\widetilde{v}_5\right)\mathrm{d}x + \rho\int_0^1 v_1\widetilde{v}_1\,\mathrm{d}x + I_\rho\int_0^1 v_3\widetilde{v}_3\,\mathrm{d}x \\ &+ (3I_\rho + 4\beta + 4\gamma)\int_0^1 v_5\widetilde{v}_5\,\mathrm{d}x + (\tau+\alpha)\int_0^1 v_8\widetilde{v}_8\,\mathrm{d}x + D\int_0^1 v_{3x}\widetilde{v}_{3x}\,\mathrm{d}x \\ &+ 3D\int_0^1 v_{5x}\widetilde{v}_{5x}\,\mathrm{d}x + \rho_3(\tau+\beta)^2\int_0^1\left(\int_0^x v_8(y)\mathrm{d}y\right)\left(\int_0^x \widetilde{v}_8(y)\mathrm{d}y\right)\mathrm{d}x, \end{aligned} \tag{2.21}$$

and the linear form $\mathcal{L} : \left[H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times \ L_*^2(0,1)\right] \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{L}\left(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8\right) = &\int_0^1 \left(\rho f_1 + \rho f_2 - \tau\delta f_8\right)\widetilde{v}_1 \,\mathrm{d}x + I_\rho \int_0^1 \left(f_3 + f_4\right)\widetilde{u}_3 \,\mathrm{d}x \\ &+ \int_0^1 \left[\left(3I_\rho + 4\beta\right) f_5 + 3I_\rho f_6\right]\widetilde{v}_5 \,\mathrm{d}x + \delta(\tau+\alpha)\int_0^1 f_1\widetilde{v}_8 \,\mathrm{d}x \\ &+ \int_0^1 \left[\rho_3\tau(\tau+\alpha)\int_0^x f_8(y)\mathrm{d}y - \rho_3(\tau+\alpha) f_7\right]\left(\int_0^x \widetilde{v}_8(y)\mathrm{d}y\right)\mathrm{d}x.\end{aligned} \tag{2.22}$$

We denote the Hilbert space $V$ by

$$V = H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times L_*^2(0,1), \tag{2.23}$$

equipped with the norm

$$\|(v_1, v_3, v_5, v_8)\|_V^2 = \|-v_{1x} - v_3 + 3v_5\|_2^2 + \|v_1\|_2^2 + \|v_8\|_2^2 + \|v_{3x}\|_2^2 + \|v_{5x}\|_2^2. \tag{2.24}$$

It is easy to get that $\mathcal{B}(\cdot,\cdot)$ and $\mathcal{L}(\cdot)$ are bounded. Moreover there exist a positive constant $m$ such that

$$\begin{aligned}&\mathcal{B}\left(\left(v_1, v_3, v_5, v_8\right), \left(v_1, v_3, v_5, v_8\right)\right) \\ &= G\int_0^1 \left(-v_{1x} - v_3 + 3v_5\right)^2 \,\mathrm{d}x + \rho\int_0^1 v_1^2 \,\mathrm{d}x + I_\rho\int_0^1 v_3^2 \,\mathrm{d}x \\ &+ \left(3I_\rho + 4\beta + 4\gamma\right)\int_0^1 v_5^2 \,\mathrm{d}x + (\tau+\alpha)\int_0^1 v_8^2 \,\mathrm{d}x + D\int_0^1 v_{3x}^2 \,\mathrm{d}x \\ &+ 3D\int_0^1 v_{5x}^2 \,\mathrm{d}x + \rho_3(\tau+\alpha)^2\int_0^1 \left(\int_0^x v_8(y)\mathrm{d}y\right)^2 \,\mathrm{d}x, \\ &\geq G\int_0^1 \left(-v_{1x} - v_3 + 3v_5\right)^2 \,\mathrm{d}x + \rho\int_0^1 v_1^2 \,\mathrm{d}x \\ &+ (\tau+\alpha)\int_0^1 v_8^2 \,\mathrm{d}x + D\int_0^1 v_{3x}^2 \,\mathrm{d}x + 3D\int_0^1 v_{5x}^2 \,\mathrm{d}x, \\ &\geq m\left\|\left(v_1, v_3, v_5, v_8\right)\right\|_V^2.\end{aligned} \tag{2.25}$$

Thus, $\mathcal{B}$ is coercive on $V \times V$. Consequently, using Lax-Milgram theorem, we conclude that (2.20) has a unique solution

$$\begin{aligned}v_1 &\in H_*^1(0,1), \\ v_3, v_5 &\in H_0^1(0,1), \\ v_8 &\in L_*^2(0,1).\end{aligned} \tag{2.26}$$

Substituting $v_1, v_3, v_5$, and $v_8$ into (2.14) – (2.16) and (2.18), respectively, we have

$$\begin{aligned} v_2 &\in H_*^1(0,1), \\ v_4, v_6 &\in H_0^1(0,1), \\ v_7 &\in H_0^1(0,1). \end{aligned} \tag{2.27}$$

Let $\widetilde{v}_1 \in H_0^1(0,1)$ and denote

$$\widehat{\widetilde{v}}_1(x) = \widetilde{v}_1(x) - \int_0^1 \widetilde{v}_1(s)\mathrm{d}s, \tag{2.28}$$

which gives us $\widehat{\widetilde{v}}_1 \in H_*^1(0,1)$. Now we replace $(\widetilde{v}_1, \widetilde{v}_3, \widetilde{v}_5, \widetilde{v}_8)$ by $\left(\widehat{\widetilde{v}}_1, 0, 0, 0\right)$ in (2.25) we obtain

$$\begin{aligned} G\int_0^1 (-v_{1x} - v_3 + 3v_5)\left(-\widehat{\widetilde{v}}_{1x}\right)\mathrm{d}x + \rho\int_0^1 v_1\widehat{\widetilde{v}}_1 dx \\ = \int_0^1 (\rho f_1 + \rho f_2 - \tau\delta f_8)\,\widehat{\widetilde{v}}_1\,\mathrm{d}x, \end{aligned} \tag{2.29}$$

so

$$\begin{aligned} G\int_0^1 v_{1xx}\widehat{\widetilde{v}}_1\,\mathrm{d}x =& \rho\int_0^1 v_1\widehat{\widetilde{v}}_1\,\mathrm{d}x - G\int_0^1 v_{3x}\widehat{\widetilde{v}}_1\,\mathrm{d}x + 3v\int_0^1 v_{5x}\widehat{\widetilde{v}}_1\,\mathrm{d}x \\ & - \int_0^1 (\rho f_1 + \rho f_2 - \tau\delta f_8)\,\widehat{\widetilde{v}}_1\,\mathrm{d}x, \quad \forall \widetilde{v}_1 \in H_0^1(0,1), \end{aligned} \tag{2.30}$$

which yields

$$Gv_{1xx} = \rho v_1 - Gv_{3x} + 3Gv_{5x} - (\rho f_1 + \rho f_2 - \tau\delta f_8) \in L^2(0,1). \tag{2.31}$$

Thus

$$v_1 \in H^2(0,1). \tag{2.32}$$

Moreover, (2.29) also holds for any $\omega \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$\begin{aligned} & Gv_{1x}(1)\phi(1) - Gv_{1x}(0)\phi(0) - G\int_0^1 v_{1xx}\phi\mathrm{d}x \\ & + \rho\int_0^1 v_1\phi\mathrm{d}x - G\int_0^1 v_{3x}\phi\mathrm{d}x \\ & + 3G\int_0^1 v_{5x}\phi\mathrm{d}x - \int_0^1 (\rho f_1 + \rho f_2 - \tau\delta f_8)\,\phi\mathrm{d}x = 0. \end{aligned} \tag{2.33}$$

Then, we get for any $\phi \in C^1([0,1])$,

$$Gv_{1x}(1)\phi(1) - Gv_{1x}(0)\phi(0) = 0. \tag{2.34}$$

From (2.18), we obtain

$$v_7(0) = v_7(1) = 0, \tag{2.35}$$

since $\phi$ is arbitrary, we get that $v_{1x}(0) = v_{1x}(1) = 0$. Hence, $v_1 \in H^2_*(0,1)$. Using similar arguments as above, we can obtain

$$\begin{aligned} v_3, v_5 &\in H^2(0,1) \cap H^1_0(0,1), \\ v_7 &\in H^1_0(0,1), \\ v_8 &\in H^1_*(0,1). \end{aligned} \tag{2.36}$$

Thus, $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8) \in D(\mathcal{A})$ and $\mathcal{A}$ is maximal. By using Lumer-Philips theorem, see for example, Liu and Zheng [29] and Pazy [39], we end the proof of the theorem. $\square$

## 2.3 Stability results

In this section, we study the stability of systems $(2.1) - (2.3)$. More precisely, we establish exponential and polynomial decay results depending on $\chi$ defined by

$$\chi = \tau\delta^2 D - (D\rho - GI_\rho)\left(\frac{\tau\rho_3 D}{I_\rho} - 1\right). \tag{2.37}$$

The energy functional of systems $(2.1) - (2.3)$ is defined by

$$\begin{aligned} E(t) =& E(\omega, \psi, s, \theta, q), \\ =& \frac{1}{2}\int_0^1 \left[\rho\omega_t^2 + I_\rho\left[(3s-\psi)_t\right]^2 + 3I_\rho s_t^2 + \rho_3\theta^2\right. \\ & \left. + \tau q^2 + 4\gamma s^2 + D\left[(3s-\psi)_x\right]^2 + G\left(\psi - \omega_x\right)^2 + 3Ds_x^2\right] \mathrm{d}x. \end{aligned} \tag{2.38}$$

Now we give our stability results.

**Theorem 2.2.** *(exponential decay). Suppose that $\chi = 0$. for any initial data $U_0 \in \mathcal{H}$, there exist two positive constants $\mu$ and $\eta$ such that the energy functional (2.38) satisfies*

$$E(t) \le \mu e^{-\eta t} \quad \forall t \ge 0. \tag{2.39}$$

**Theorem 2.3.** *(polynomial decay). Suppose that $\chi \neq 0$, for any initial data $U_0 \in D(\mathcal{A})$, there exist positive constant $\mu_0$ such that the energy functional (2.38) satisfies*

$$E(t) \leq \frac{\mu_0}{t} \quad \forall t > 0. \tag{2.40}$$

To prove Theorems 2.2 and 2.3, we need the following technical lemmas.

### 2.3.1 Technical Lemmas

**Lemma 2.4.** *It holds that the energy functional $E(t)$ is non increasing and satisfies*

$$E'(t) = -4\beta \int_0^1 s_t^2 \, \mathrm{d}x - \alpha \int_0^1 q^2 \, \mathrm{d}x \leq 0. \tag{2.41}$$

*Proof.* Multiplying (2.1) by $\omega_t, (3s-\psi)_t, s_t, \theta$, and $q$, respectively, integrating the results by parts and using boundary condition (2.2), we easily get (2.41). $\square$

**Lemma 2.5.** *Define the functional $F_1(t)$ by*

$$F_1(t) = I_\rho \int_0^1 (3s-\psi)_t(3s-\psi)\mathrm{d}x - \rho \int_0^1 \omega_t \int_0^x (3s-\psi)(y)\mathrm{d}y \, \mathrm{d}x. \tag{2.42}$$

*Then, we have for any $\varepsilon_1 > 0$,*

$$\begin{aligned} F_1'(t) \leq & -\frac{G}{2} \int_0^1 \left[(3s-\psi)_x\right]^2 \, \mathrm{d}x \\ & + \varepsilon_1 \int_0^1 \omega_t^2 \, \mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \left[(3s-\psi)_t\right]^2 \, \mathrm{d}x \\ & + \frac{\delta^2 c_*^2}{2D} \int_0^1 \theta^2 \, \mathrm{d}x. \end{aligned} \tag{2.43}$$

*Where $c_* > 0$ is the Poincaré constant.*

*Proof.* It follows from (2.1) that

$$\begin{aligned} F_1'(t) = & G \int_0^1 (3s-\psi)_{xx}(3s-\psi)\mathrm{d}x \\ & + G \int_0^1 (\psi - \omega_x)(3s-\psi)\mathrm{d}x + I_\rho \int_0^1 \left[(3s-\psi)_t\right]^2 \, \mathrm{d}x \\ & + G \int_0^1 (\psi - \omega_x) \int_0^x (3s-\psi)(y)\mathrm{d}y \, \mathrm{d}x \\ & + \delta \int_0^1 \theta_x \int_0^x (3s-\psi)(y)\mathrm{d}y \, \mathrm{d}x \\ & - \rho \int_0^1 \omega_t \int_0^x (3s-\psi)_t(y)\mathrm{d}y \, \mathrm{d}x. \end{aligned} \tag{2.44}$$

Using integration by parts and boundary condition (2.2), we arrive at

$$\begin{aligned} F_1'(t) = & -D\int_0^1 \left[(3s-\psi)_x\right]^2 \,\mathrm{d}x + I_\rho \int_0^1 \left[(3r-\psi)_t\right]^2 \,\mathrm{d}x \\ & -\delta\int_0^1 \theta(3s-\psi)\mathrm{d}x \\ & -\rho\int_0^1 \omega_t \int_0^x (3s-\psi)_t(y)\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{2.45}$$

So, by using Hölder's, Young's and Poincaré's inequalities, we can get (2.43). □

**Lemma 2.6.** *The functional* $F_2(t)$ *defined by*

$$F_2(t) = \rho\int_0^1 (\psi-\omega_x)\int_0^x \omega_t(y)\mathrm{d}y\,\mathrm{d}x, \tag{2.46}$$

*satisfies for any* $\varepsilon_2 > 0$*,*

$$\begin{aligned} F_2'(t) \leq & -\frac{G}{2}\int_0^1 (\psi-\omega_x)^2 \,\mathrm{d}x + \varepsilon_2\int_0^1 \psi_t^2 \,\mathrm{d}x + c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1 \omega_t^2 \,\mathrm{d}x \\ & + \frac{\delta^2}{2G}\int_0^1 \theta^2 \,\mathrm{d}x. \end{aligned} \tag{2.47}$$

*Proof.* Differentiating $F_2(t)$ with respect to $t$ and using (2.2), we see that

$$\begin{aligned} F_2'(t) = & \rho\int_0^1 \psi_t \int_0^x \omega_t(y)\mathrm{d}y\,\mathrm{d}x - \rho\int_0^1 \omega_{xt}\int_0^x \omega_t(y)\mathrm{d}y\,\mathrm{d}x \\ & -G\int_0^1 (\psi-\omega_x)\int_0^x (\psi-\omega_y)_y \,\mathrm{d}y\,\mathrm{d}x \\ & -\delta\int_0^1 (\psi-\omega_x)\int_0^x \theta_y \,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{2.48}$$

Using integration by parts, we obtain

$$\begin{aligned} F_2'(t) = & \rho\int_0^1 \psi_t \int_0^x \omega_t(y)\mathrm{d}y\,\mathrm{d}x + \rho\int_0^1 \omega_t^2 \,\mathrm{d}x \\ & -G\int_0^1 (\psi-\omega_x)^2 \,\mathrm{d}x - \delta\int_0^1 \theta\left(\psi-\omega_x\right)\mathrm{d}x. \end{aligned} \tag{2.49}$$

Then, by using Young's inequality and Hölder's inequality, we can get (2.47). □

**Lemma 2.7.** *Define the functional* $F_3(t)$ *by*

$$F_3(t) = \tau\rho_3\int_0^1 \theta\int_0^x q(y)\mathrm{d}y\,\mathrm{d}x. \tag{2.50}$$

*Then, we can get for any* $\varepsilon_3 > 0$*,*

$$F_3'(t) \leq -\frac{\rho_3}{2}\int_0^1 \theta^2 \,\mathrm{d}x + \varepsilon_3\int_0^1 \omega_t^2 \,\mathrm{d}x + c\left(1+\frac{1}{\varepsilon_3}\right)\int_0^1 q^2 \,\mathrm{d}x. \tag{2.51}$$

*Proof.* Differentiating $F_3$ with respect to $t$ and using (2.2), we obtain

$$\begin{aligned} F_3'(t) =& -\tau \int_0^1 q_x \int_0^x q(y)\mathrm{d}y\, \mathrm{d}x - \tau\delta \int_0^1 \omega_{xt} \int_0^x q(y)\mathrm{d}y\, \mathrm{d}x \\ & - \rho_3\beta \int_0^1 \theta \int_0^x q\, \mathrm{d}y\, \mathrm{d}x - \rho_3 \int_0^1 \theta \int_0^x \theta_y\, \mathrm{d}y\, \mathrm{d}x. \end{aligned} \tag{2.52}$$

Integration by parts gives us

$$\begin{aligned} F_3'(t) =& \tau \int_0^1 q^2\, \mathrm{d}y + \tau\delta \int_0^1 \omega_t q\, \mathrm{d}x - \rho_3\beta \int_0^1 \theta \int_0^x q\, \mathrm{d}y\, \mathrm{d}x \\ & - \rho_3 \int_0^1 \theta^2\, \mathrm{d}x. \end{aligned} \tag{2.53}$$

By using Young's inequality and Hölder's inequality, we can get (2.51). □

**Lemma 2.8.** *The functional $F_4(t)$ defined by*

$$F_4(t) = -\rho\rho_3 \int_0^1 \theta \int_0^x \omega_t(y)\mathrm{d}y\, \mathrm{d}x, \tag{2.54}$$

*satisfies for any $\varepsilon_4 > 0$,*

$$\begin{aligned} F_4'(t) \le & -\frac{\rho\delta}{2} \int_0^1 \omega_t^2\, \mathrm{d}x + \varepsilon_4 \int_0^1 (\psi - \omega_x)^2\, \mathrm{d}x + c\left(1 + \frac{1}{\varepsilon_4}\right) \int_0^1 \theta^2\, \mathrm{d}x \\ & + \frac{\rho}{2\delta} \int_0^1 q^2\, \mathrm{d}x. \end{aligned} \tag{2.55}$$

*Proof.* We take the derivative of $F_4$ and use (2.2) and integrate by parts to obtain

$$\begin{aligned} F_4'(t) =& \rho \int_0^1 q_x \int_0^x \omega_t(y)\mathrm{d}y\, \mathrm{d}x + \rho\delta \int_0^1 \omega_{xt} \int_0^x \omega_t(y)\mathrm{d}y\, \mathrm{d}x \\ & + \rho_3 G \int_0^1 \theta \int_0^x (\psi - \omega_y)_y (y)\mathrm{d}y\, \mathrm{d}x \\ & + \rho_3\delta \int_0^1 \theta \int_0^x \theta_y(y), \\ =& -\rho \int_0^1 q\omega_t\, \mathrm{d}x - \rho\delta \int_0^1 \omega_t^2\, \mathrm{d}x \\ & + \rho_3 G \int_0^1 \theta (\psi - \omega_x)\, \mathrm{d}x + \rho_3\delta \int_0^1 \theta^2\, \mathrm{d}x. \end{aligned} \tag{2.56}$$

Then, using Young's inequality, we can get (2.55). □

**Lemma 2.9.** *Define the functional* $F_5(t)$ *by*

$$
\begin{aligned}
F_5(t) =& \tau G\delta I_\rho \int_0^1 (3s-\psi)_t\,(\psi-\omega_x)\,\mathrm{d}x \\
&- \tau\delta D\rho \int_0^1 \omega_t (3s-\psi)_x\,\mathrm{d}x \\
&+ \tau\rho_3\,(D\rho - GI_\rho)\int_0^1 \theta(3s-\psi)_t\,\mathrm{d}x \\
&- \tau\,(D\rho - GI_\rho)\int_0^1 q(3s-\psi)_x\,\mathrm{d}x.
\end{aligned}
\tag{2.57}
$$

*Then, we have for any* $\varepsilon_5 > 0$*,*

$$
\begin{aligned}
F_5'(t) \leq & -\frac{\tau G\delta I_\rho}{2}\int_0^1 \left[(3s-\psi)_t\right]^2\,\mathrm{d}x + c_1\int_0^1 s_t^2\,\mathrm{d}x \\
&+ c_2\int_0^1 \theta^2\,\mathrm{d}x + c_3\int_0^1 (\psi-\omega_x)^2\,\mathrm{d}x \\
&+ \varepsilon_5\int_0^1 \left[(3s-\psi)_x\right]^2\,\mathrm{d}x + C_{\varepsilon_5}\int_0^1 q^2\,\mathrm{d}x \\
&+ \chi\int_0^1 \theta_x(3s-\psi)_x\,\mathrm{d}x,
\end{aligned}
\tag{2.58}
$$

*where* $c_i (i = 1, 2, 3)$ *are positive constants.*

*Proof.* By differentiating $F_5$ with respect to $t$, we have

$$
\begin{aligned}
F_5'(t) =& \underbrace{\tau G\delta I_\rho \int_0^1 (3s-\psi)_{tt}\,(\psi-\omega_x)\,\mathrm{d}x}_{:=I_2} + \tau G\delta I_\rho \int_0^1 (3s-\psi)_t\,(\psi-\omega_x)_t\,\mathrm{d}x \\
& \underbrace{- \tau\delta D\rho \int_0^1 \omega_{tt}(3r-\psi)_x\,\mathrm{d}x - \tau\delta D\rho\int_0^1 \omega_t(3s-\psi)_{xt}\,\mathrm{d}x}_{:=I_3} \\
& \underbrace{+ \tau\rho_3\,(D\rho - GI_\rho)\int_0^1 \theta_t(3s-\psi)_t\,\mathrm{d}x}_{:=I_5} + \underbrace{\tau\rho_3\,(D\rho - GI_\rho)\int_0^1 \theta(3s-\psi)_{tt}\,\mathrm{d}x}_{:=I_6} \\
& \underbrace{- \tau\,(D\rho - GI_\rho)\int_0^1 q_t(3s-\psi)_x\,\mathrm{d}x} - \underbrace{-\tau\,(D\rho - GI_\rho)\int_0^1 q(3s-\psi)_{xt}\,\mathrm{d}x}.
\end{aligned}
\tag{2.59}
$$

Using equation (2.2) and integrating by parts, we see that

$$
\begin{aligned}
I_1 =& -\tau G\delta D\int_0^1 (3s-\psi)_x\,(\psi-\omega_x)_x\,\mathrm{d}x \\
&+ \tau G^2\delta\int_0^1 (\psi-\omega_x)^2\,\mathrm{d}x,
\end{aligned}
\tag{2.60}
$$

$$\begin{aligned} I_2 =& \tau\delta DG \int_0^1 (\psi-\omega_x)_x (3s-\psi)_x \,\mathrm{d}x \\ &+ \tau\delta^2 D \int_0^1 \theta_x(3s-\psi)_x \,\mathrm{d}x, \end{aligned} \tag{2.61}$$

$$\begin{aligned} I_3 =& -\tau\left(D\rho - GI_\rho\right) \int_0^1 q_x(3s-\psi)_t \,\mathrm{d}x \\ &- \tau\delta\left(D\rho - GI_\rho\right) \int_0^1 \omega_{xt}(3s-\psi)_t \,\mathrm{d}x, \end{aligned} \tag{2.62}$$

$$\begin{aligned} I_4 =& -\frac{\tau\rho_3 D}{I_\rho}\left(D\rho - GI_\rho\right) \int_0^1 \theta_x(3r-\psi)_x \,\mathrm{d}x \\ &+ \frac{\tau\rho_3 G}{I_\rho}\left(D\rho - GI_\rho\right) \int_0^1 \theta\left(\psi-\omega_x\right) \mathrm{d}x, \end{aligned} \tag{2.63}$$

$$\begin{aligned} I_5 =& \beta\left(D\rho - GI_\rho\right) \int_0^1 q(3s-\psi)_x \,\mathrm{d}x \\ &+ \left(D\rho - GI_\rho\right) \int_0^1 \theta_x(3s-\psi)_x \,\mathrm{d}x, \end{aligned} \tag{2.64}$$

$$I_6 = \tau\left(D\rho - GI_\rho\right) \int_0^1 q_x(3s-\psi)_t \,\mathrm{d}x. \tag{2.65}$$

Inserting (2.60)-(2.65) into (2.59), we can obtain

$$\begin{aligned} F_5'(t) =& \tau G^2\delta \int_0^1 \left(\psi-\omega_x\right)^2 \,\mathrm{d}x + \tau G\delta I_\rho \int_0^1 \psi_t(3s-\psi)_t \,\mathrm{d}x \\ &+ \frac{\tau\rho_3 G}{I_\rho}\left(D\rho - GI_\rho\right) \int_0^1 \theta\left(\psi-\omega_x\right) \mathrm{d}x \\ &+ \beta\left(D\rho - GI_\rho\right) \int_0^1 q(3s-\psi)_x \,\mathrm{d}x \\ &+ \chi \int_0^1 \theta_x(3s-\psi)_x \,\mathrm{d}x. \end{aligned} \tag{2.66}$$

Recalling $\psi = (\psi - 3s) + 3s$ and using Young's inequality, we conclude that

$$\begin{aligned} \tau G\delta I_\rho \int_0^1 \psi_t(3s-\psi)_t \,\mathrm{d}x =& -\tau G\delta I_\rho \int_0^1 \left[(3s-\psi)_t\right]^2 \,\mathrm{d}x + 3\tau G\delta I_\rho \int_0^1 r_t(3s-\psi)_t \,\mathrm{d}x, \\ \le& -\frac{\tau G\delta I_\rho}{2} \int_0^1 \left[(3s-\psi)_t\right]^2 \,\mathrm{d}x + c_1 \int_0^1 s_t^2 \,\mathrm{d}x, \end{aligned} \tag{2.67}$$

$$\frac{\tau\rho_3 G}{j_\rho}\left(D\rho - GI_\rho\right) \int_0^1 \theta\left(\psi-\omega_x\right) \mathrm{d}x \le c_2 \int_0^1 \theta^2 \,\mathrm{d}x + c_3 \int_0^1 \left(\psi-\omega_x\right)^2 \,\mathrm{d}x, \tag{2.68}$$

and for any $\varepsilon_5 > 0$,

$$\gamma\left(D\rho - GI_\rho\right) \int_0^1 q(3ss-\psi)_x \,\mathrm{d}x \le \varepsilon_5 \int_0^1 \left[(3s-\psi)_x\right]^2 \,\mathrm{d}x + C_{\varepsilon_5} \int_0^1 q^2 \,\mathrm{d}x, \tag{2.69}$$

which, together with (2.66)-(2.68), gives us (2.58). □

**Lemma 2.10.** *The functional $F_6(t)$ defined by*

$$F_6(t) = 3I_\rho \int_0^1 s_t r \,\mathrm{d}x + 2\beta \int_0^1 s^2 \,\mathrm{d}x, \tag{2.70}$$

*satisfies*

$$\begin{aligned} F_6'(t) \leq & -3\gamma \int_0^1 s^2 \,\mathrm{d}x - 3D \int_0^1 s_x^2 \,\mathrm{d}x \\ & + c_4 \int_0^1 (\psi - \omega_x)^2 \,\mathrm{d}x + 3I_\rho \int_0^1 s_t^2 \,\mathrm{d}x, \end{aligned} \tag{2.71}$$

*where $c_4$ is a positive constant.*

*Proof.* Follows from (2.2) that

$$\begin{aligned} F_6'(t) = & -3D \int_0^1 s_x^2 \,\mathrm{d}x - 3G \int_0^1 s\,(\psi - \omega_x)\,\mathrm{d}x \\ & - 4\gamma \int_0^1 s^2 \,\mathrm{d}x + 3I_\rho \int_0^1 s_t^2 \,\mathrm{d}x. \end{aligned} \tag{2.72}$$

Young's inequality gives us (2.71). □

### 2.3.2 Exponential Stability: Proof of Theorem 2.1

*Proof.* We define the functional $\mathcal{L}(t)$ by

$$\begin{aligned} \mathcal{L}(t) = & NE(t) + G_1(t) + N_2G_2(t) + N_3G_3(t) \\ & + N_4G_4(t) + N_5G_5(t) + G_6(t), \end{aligned} \tag{2.73}$$

where $N$ and $N_i (i = 2, 3, 4, 5)$ are positive constants that will be chosen later.

Note that

$$\begin{aligned} \int_0^1 \psi_t^2 \,\mathrm{d}x = & \int_0^1 [(3s - \psi) - 3s]_t^2 \,\mathrm{d}x \\ \leq & \, 2 \int_0^1 [(3s - \psi)_t]^2 \,\mathrm{d}x + 18 \int_0^1 s_t^2 \,\mathrm{d}x. \end{aligned} \tag{2.74}$$

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -\left(4\beta N - c_1 N_5 - 18\varepsilon_2 N_2 - 3I_\rho\right) \int_0^1 s_t^2 \, \mathrm{d}x \\
& -\left(\frac{D}{2} - \varepsilon_5 N_5\right) \int_0^1 \left[(3s-\psi)_x\right]^2 \, \mathrm{d}x \\
& -\left[\gamma N - cN_3\left(1+\frac{1}{\varepsilon_3}\right) - \frac{\rho}{2\delta}N_4 - C_{\varepsilon_5}N_5\right] \int_0^1 q^2 \, \mathrm{d}x \\
& -\left[\frac{\rho\delta}{2}N_4 - \varepsilon_1 - \varepsilon_3 N_3 - cN_2\left(1+\frac{1}{\varepsilon_2}\right)\right] \int_0^1 \omega_t^2 \, \mathrm{d}x \\
& -\left[\frac{\tau G\delta I_\rho}{2}N_5 - 2\varepsilon_2 N_2 - c\left(1+\frac{1}{\varepsilon_1}\right)\right] \int_0^1 \left[(3s-\psi)_t\right]^2 \, \mathrm{d}x \\
& -\left(\frac{G}{2}N_2 - \varepsilon_4 N_4 - c_3 N_5 - c_4\right) \int_0^1 (\psi-\omega_x)^2 \, \mathrm{d}x \\
& -\left[\frac{\rho_3}{2}N_3 - \frac{\delta^2 c_*^2}{2D} - \frac{\delta^2}{2G}N_2 - cN_4\left(1+\frac{1}{\varepsilon_4}\right) - c_2 N_5\right] \int_0^1 \theta^2 \, \mathrm{d}x \\
& - 3D\int_0^1 r_x^2 \, \mathrm{d}x - 3\gamma \int_0^1 r^2 \, \mathrm{d}x.
\end{aligned}
\tag{2.75}
$$

Taking

$$
\varepsilon_1 = 1, \varepsilon_2 = \frac{\tau G\delta I_\rho}{8N_2}N_5, \varepsilon_3 = \frac{1}{N_3}, \varepsilon_4 = \frac{G}{4N_4}N_2, \varepsilon_5 = \frac{D}{4N_5}, \tag{2.76}
$$

we obtain

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -\left(4\beta N - c_1 N_5 - \frac{9}{4}\tau G\delta I_\rho N_5 - 3I_\rho\right) \int_0^1 s_t^2 \, \mathrm{d}x \\
& -\frac{D}{4} \int_0^1 \left[(3s-\psi)_x\right]^2 \, \mathrm{d}x \\
& -\left[\gamma N - cN_3\left(1+N_3\right) - \frac{\rho}{2\delta}N_4 - C_{\varepsilon_5}N_5\right] \int_0^1 q^2 \, \mathrm{d}x \\
& -\left[\frac{\rho\delta}{2}N_4 - 2 - cN_2\left(1+\frac{8N_2}{N_5\tau G\delta I_\rho}\right)\right] \int_0^1 \omega_t^2 \, \mathrm{d}x \\
& -\left(\frac{\tau G\delta I_\rho}{4}N_5 - 2c\right) \int_0^1 \left[(3s-\psi)_t\right]^2 \, \mathrm{d}x \\
& -\left(\frac{G}{4}N_2 - c_3 N_5 - c_4\right) \int_0^1 (\psi-\omega_x)^2 \, \mathrm{d}x \\
& -\left[\frac{\rho_3}{2}N_3 - \frac{\delta^2 c_*^2}{2D} - \frac{\delta^2}{2G}N_2\right. \\
& \left. -cN_4\left(1+\frac{4N_4}{GN_2}\right) - c_2 N_5\right] \int_0^1 \theta^2 \, \mathrm{d}x \\
& - 3D\int_0^1 r_x^2 \, \mathrm{d}x - 3\gamma \int_0^1 s^2 \, \mathrm{d}x.
\end{aligned}
\tag{2.77}
$$

At this point, we first choose $N_5 > 0$ large enough such that

$$
\frac{\tau G\delta I_\rho}{4}N_5 - 2c > 0, \tag{2.78}
$$

for fixed $N_5$, we take $N_2 > 0$ so large that

$$\frac{G}{4}N_2 - c_3N_5 - c_4 > 0. \tag{2.79}$$

Then, we pick $N_4 > 0$ large so that

$$\frac{\rho\delta}{2}N_4 - 2 - cN_2\left(1 + \frac{8N_2}{N_5\tau G\delta I_\rho}\right) > 0. \tag{2.80}$$

And then we choose $N_3$ so large that

$$\frac{\rho_3}{2}N_3 - \frac{\delta^2 c_*^2}{2D} - \frac{\delta^2}{2G}N_2 - cN_4\left(1 + \frac{4N_4}{GN_2}\right) - c_2N_5 > 0. \tag{2.81}$$

At last, we take $N > 0$ large enough so that the functional $\mathcal{L}(t)$ is equivalent to the energy functional $E(t)$, i.e., there exist two positive constants

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \tag{2.82}$$

and further so that

$$\begin{aligned} \gamma N - cN_3\left(1 + N_3\right) - \frac{\rho}{2\delta}N_4 - C_{\varepsilon_5}N_5 > 0. \\ 4\beta N - c_1N_5 - \frac{9}{4}\tau G\delta I_\rho N_5 - 3I_\rho > 0. \end{aligned} \tag{2.83}$$

Recalling (2.38), we infer that there exists a positive constant $\beta_3$ such that, for any $t > 0$,

$$\mathcal{L}'(t) \leq -\beta_3 E(t), \tag{2.84}$$

which, along with (2.82), implies

$$\mathcal{L}'(t) \leq -\frac{\beta_3}{\beta_2}\mathcal{L}(t). \tag{2.85}$$

Integrating (2.85) over $(0, t)$, we have, for any $t > 0$,

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-(\beta_3/\beta_2)t}, \tag{2.86}$$

which using (2.85) again, gives us (2.39). The proof of Theorem 2.1 is done. □

### 2.3.3 Polynomial Stability: Proof of Theorem 2.2

In this subsection, we consider the case $\chi \neq 0$ to prove Theorem 2.2 . Differentiating system (2.1) with respect to time, we obtain the following system:

$$\begin{cases} \rho\omega_{ttt} + G\left(\psi - \omega_x\right)_{xt} + \delta\theta_{xt} = 0, \\ I_\rho(3s-\psi)_{ttt} - D(3s-\psi)_{xxt} - G\left(\psi - \omega_x\right)_t = 0, \\ I_\rho s_{ttt} - Ds_{xxt} + G\left(\psi - \omega_x\right)_t + \dfrac{4}{3}\gamma s_t + \dfrac{4}{3}\beta s_{tt} = 0, \\ \rho_3\theta_{tt} + q_{xt} + \delta\omega_{xtt} = 0, \\ \tau q_{tt} + \gamma q_t + \theta_{xt} = 0. \end{cases} \tag{2.87}$$

Which subject to the following boundary conditions

$$\begin{cases} \omega_{xt}(0,t) = \psi_t(0,t) = s_t(0,t) = \theta_t(0,t) = 0, \quad t \in (0,\infty), \\ \omega_{xt}(1,t) = \psi_t(1,t) = s_t(1,t) = \theta_t(1,t) = 0, \quad t \in (0,\infty). \end{cases} \tag{2.88}$$

For any initial data $U_0 \in D(\mathcal{A})$, system (2.87) is well posed. Next, we introduce second-order energy functional $\widetilde{E}(t)$ by

$$\begin{aligned} \widetilde{E}(t) =& E\left(\omega_t, \psi_t, s_t, \theta_t, q_t\right) \\ =& \frac{1}{2}\int_0^1 \left[\rho\omega_{tt}^2 + I_\rho\left[(3s-\psi)_{tt}\right]^2 + 3I_\rho s_{tt}^2 + \rho_3\theta_t^2 + \tau q_t^2 + 4\gamma s_t^2\right. \\ & \left. + D\left[(3s-\psi)_{xt}\right]^2 + G\left[\left(\psi-\omega_x\right)_t\right]^2 + 3Ds_{xt}^2\right] \mathrm{d}x. \end{aligned} \tag{2.89}$$

By using the same arguments as in Lemma (2.6), we can get the second-order energy $\widetilde{E}(t)$ defined by (2.89) is nonincreasing and satisfies

$$\widetilde{E}'(t) = -4\beta\int_0^1 s_{tt}^2\,\mathrm{d}x - \gamma\int_0^1 q_t^2\,\mathrm{d}x \leq 0. \tag{2.90}$$

In Lemma (2.9), we have proved that, for any $\varepsilon_5 > 0$,

$$\begin{aligned} F_5'(t) \leq & -\frac{\tau G\delta I_\rho}{2}\int_0^1 \left[(3s-\psi_t)^2\,\mathrm{d}x\right] + c_1\int_0^1 s_t^2\,\mathrm{d}x \\ & + c_2\int_0^1 \theta^2\,\mathrm{d}x + c_3\int_0^1 \left(\psi-\omega_x\right)^2\,\mathrm{d}x \\ & + \varepsilon_5\int_0^1 \left[(3s-\psi)_x\right]^2\,\mathrm{d}x + C_{\varepsilon_5}\int_0^1 q^2\,\mathrm{d}x \\ & + \chi\int_0^1 \theta_x(3s-\psi)_x\,\mathrm{d}x. \end{aligned} \tag{2.91}$$

Thanks to (2.1) and Young's inequality, we derive that

$$\int_0^1 \theta_x^2 \, dx \leq c \int_0^1 q^2 \, dx + c \int_0^1 q_t^2 \, dx. \tag{2.92}$$

Then, for any $\varepsilon_5 > 0$,

$$\begin{aligned} x \int_0^1 \theta_x (3s-\psi)_x \, dx &\leq \varepsilon_5 \int_0^1 \left[(3s-\psi)_x\right]^2 \, dx + C_{\varepsilon_5} \int_0^1 \theta_x^2 \, dx, \\ &\leq \varepsilon_5 \int_0^1 \left[(3s-\psi)_x\right]^2 \, dx + C_{\varepsilon_5} \int_0^1 q^2 \, dx \\ &\quad + C_{\varepsilon 5} \int_0^1 q_t^2 \, dx. \end{aligned} \tag{2.93}$$

Therefore, the derivative of $F_5$ satisfies

$$\begin{aligned} F_5'(t) \leq &- \frac{\tau G \delta I_\rho}{2} \int_0^1 \left[(3s-\psi)_t\right]^2 \, dx + c_1 \int_0^1 s_t^2 \, dx \\ &+ c_2 \int_0^1 \theta^2 \, dx + c_3 \int_0^1 (\psi - \omega_x)^2 \, dx \\ &+ 2\varepsilon_5 \int_0^1 \left[(3s-\psi)_x\right]^2 \, dx + C_{\varepsilon_5} \int_0^1 q^2 \, dx + C_{\varepsilon_5} \int_0^1 q_t^2 \, dx. \end{aligned} \tag{2.94}$$

*Proof.* We define the functional $\tilde{\mathcal{L}}(t)$ by

$$\begin{aligned} \widetilde{\mathcal{L}}(t) =& N(E(t) + \widetilde{E}(t)) + G_1(t) + N_2 G_2(t) \\ &+ N_3 G_3(t) + N_4 G_4(t) + N_5 G_5(t) + G_6(t). \end{aligned} \tag{2.95}$$

It follows from (2.41)-(2.43), (2.47)-(2.55), and (2.90)-(2.94) that

$$\begin{aligned} \tilde{\mathcal{L}}'(t) \leq &- (4\beta N - c_1 N_5 - 18\varepsilon_2 N_2 - 3I_\rho) \int_0^1 s_t^2 \, dx - \left(\frac{D}{2} - 2\varepsilon_5 N_5\right) \int_0^1 \left[(3s-\psi)_x\right]^2 \, dx \\ &- \left[\beta N - cN_3 \left(1 + \frac{1}{\varepsilon_3}\right) - \frac{\rho}{2\delta} N_4 - C_{\varepsilon_5} N_5\right] \int_0^1 q^2 \, dx \\ &- \left[\frac{\rho\delta}{2} N_4 - \varepsilon_1 - \varepsilon_3 N_3 - cN_2 \left(1 + \frac{1}{\varepsilon_2}\right)\right] \int_0^1 \omega_t^2 \, dx \\ &- \left[\frac{\tau G \delta I_\rho}{2} N_5 - 2\varepsilon_2 N_2 - c\left(1 + \frac{1}{\varepsilon_1}\right)\right] \int_0^1 \left[(3s-\psi)_t\right]^2 \, dx \\ &- \left(\frac{G}{2} N_2 - \varepsilon_4 N_4 - c_3 N_5 - c_4\right) \int_0^1 (\psi - \omega_x)^2 \, dx \\ &- \left[\frac{\rho_3}{2} N_3 - \frac{\delta^2 c_*^2}{2D} - \frac{\delta^2}{2G} N_2 - cN_4 \left(1 + \frac{1}{\varepsilon_4}\right) - c_2 N_5\right] \int_0^1 \theta^2 \, dx \\ &- 3D \int_0^1 s_x^2 \, dx - 3\gamma \int_0^1 s^2 \, dx - (\gamma N - C_{\varepsilon_5}) \int_0^1 q_t^2 \, dx. \end{aligned} \tag{2.96}$$

With the same choice of constants as in subsection 2.3.2, we further take $N > 0$ so large that

$$\gamma M - C_{\varepsilon_5} > 0. \tag{2.97}$$

Noting that (2.38), we know that there exists a positive constant $\mu_1$ such that, for any $t > 0$,

$$\tilde{\mathcal{L}}'(t) \leq -\mu_1 E(t). \tag{2.98}$$

Since the energy functional $E(t)$ is positive and non increasing, we infer (2.98) that, for any $t > 0$,

$$tE(t) \leq \int_0^t E(s)\mathrm{d}s \leq \frac{1}{\mu_1}(\tilde{\mathcal{L}}(0)t - n\tilde{\mathcal{L}}q(t)) \leq \frac{\tilde{\mathcal{L}}(0)}{\mu_1}, \tag{2.99}$$

which gives us

$$E(t) \leq \frac{\mu_0}{t}, \quad \forall t > 0. \tag{2.100}$$

Here, $\mu_0 = \left(\tilde{\mathcal{L}}(0)t/\mu_1\right) = \left(E(0) + \widetilde{E}(0)/\mu_1\right)$. The proof is complete. $\square$

Chapter 3

# *Analysis of thermoelastic laminated Timoshenko beam with time-varying delay*

## 3.1 Statement of the problem

In this chapter we investigate the subsequent problem with time-varying delay

$$\begin{cases} \rho\nu_{tt} + G(r-\nu_x)_x + \delta\vartheta_x = 0, \\ I_\rho(3\varpi - r)_{tt} - D(3\varpi - r)_{xx} - G(r-\nu_x) = 0, \\ 3I_\rho\varpi_{tt} - 3D\varpi_{xx} + 3G(r-\nu_x) + 4\gamma\varpi + \mu_1\varpi_t + \mu_2\varpi_t(x, t-\varsigma(t)) = 0, \\ \rho_3\vartheta_t + \mathcal{S}_x + \delta\nu_{tx} = 0, \\ \rho_4\mathcal{S}_t + \alpha\mathcal{S} + \vartheta_x = 0, \end{cases} \tag{3.1}$$

where

$$(x,t) \in (0,1) \times (0,\infty),$$

and

$$\begin{cases} \nu_x(0,t) = \nu_x(1,t) = r(0,t) = r(1,t) = 0, t \geq 0, \\ \varpi(1,t) = \varpi(0,t) = \vartheta(0,t) = \vartheta(1,t) = 0, t \geq 0, \\ \nu(x,0) = \nu_0(x), \nu_t(x,0) = \nu_1(x), r(x,0) = r_0(x), \quad x \in (0,1), \\ r_t(x,0) = r_1(x), \varpi(x,0) = \varpi_0(x), \varpi_t(x,0) = \varpi_1(x), \quad x \in (0,1), \\ \vartheta(x,0) = \vartheta_0(x), \mathcal{S}(x,0) = \mathcal{S}_0(x), \varpi_t(x, t-\varsigma(0)) = f_0(x, t-\varsigma(0)) \quad x \in (0,1). \end{cases} \tag{3.2}$$

Here, $\vartheta(x,t)$ define the difference temperature and $\mathcal{S}(x,t)$ is the heat flux, $\rho$, $G$, $I_\rho$, $D$, $\gamma$, $\mu_1$, $\delta$, $\rho_3$, $\alpha$ and $\rho_4$ on the other hand, are positive constants, $\mu_2$ is a real number, and $\varsigma(t) > 0$ represents the time-varying delay, where

$$\begin{cases} |\mu_2| \leq \sqrt{1-d}\mu_1, \\ \varsigma'(t) \leq d < 1, & \forall t > 0, \\ \varsigma \in W^{2,\infty}([0,T]), & \forall T > 0, \\ 0 < \varsigma_0 < \varsigma(t) < \bar{\varsigma} & \forall t > 0. \end{cases} \tag{3.3}$$

## 3.2 Existence and uniqueness

To accomplish our aim which is proving the existence and uniqueness of system (3.1)-(3.2) solutions, we employ the semigroup theory, so, we consider as in [38]

$$\mathcal{K}(x,\rho,t) = \varpi_t(x, t - \varsigma(t)\rho),$$

thereby, we find

$$\begin{cases} \varsigma(t)\mathcal{K}_t(x,\rho,t) + (1-\varsigma'(t)\rho)\mathcal{K}_\rho(x,\rho,t) = 0, \\ \mathcal{K}(x,0,t) = \varpi_t(x,t). \end{cases}$$

As a result we rewrite problem (3.1) as:

$$\begin{cases} \rho\nu_{tt} + G(r-\nu_x)_x + \delta\vartheta_x = 0, \\ I_\rho(3\varpi - r)_{tt} - D(3\varpi - r)_{xx} - G(r-\nu_x) = 0, \\ 3I_\rho\varpi_{tt} - 3D\varpi_{xx} + 3G(r-\nu_x) + 4\gamma\varpi + \mu_1\varpi_t + \mu_2\mathcal{K}(x,1,t) = 0, \\ \rho_3\vartheta_t + \mathcal{S}_x + \delta\nu_{tx} = 0, \\ \rho_4\mathcal{S}_t + \alpha\mathcal{S} + \vartheta_x = 0, \\ \varsigma(t)\mathcal{K}_t(x,\rho,t) + (1-\varsigma'(t)\rho)\mathcal{K}_\rho(x,\rho,t) = 0, \end{cases} \tag{3.4}$$

where

$$(x,\rho,t) \in (0,1) \times (0,1) \times (0,\infty),$$

problem (3.4) is subject to the following initial and boundary condition

$$\begin{aligned}
&\nu_x(0,t) = \nu_x(1,t) = r(0,t) = r(1,t) = 0, t \geq 0,\\
&\varpi(1,t) = \varpi(0,t) = \vartheta(0,t) = \vartheta(1,t) = 0, t \geq 0,\\
&\nu(x,0) = \nu_0(x), \nu_t(x,0) = \nu_1(x), r(x,0) = r_0(x),\\
&r_t(x,0) = r_1(x), \varpi(x,0) = \varpi_0(x), \varpi_t(x,0) = \varpi_1(x),\\
&\vartheta(x,0) = \vartheta_0(x), \mathcal{S}(x,0) = \mathcal{S}_0(x),\\
\mathcal{K}(x,\rho,0) = &f_0(x, -\rho\varsigma(0))/x \in (0,1), \rho \in (0,1).
\end{aligned}$$

Then, using (3.1) and (3.2), we establish

$$\frac{d^2}{dt^2}\int_0^1 \nu(x,t)dx = 0. \tag{3.5}$$

We then solve (3.5) and include the initial data of $\nu$ to obtain

$$\int_0^1 \nu(x,t)dx = t\int_0^1 \nu_1(x)dx + \int_0^1 \nu_0(x)dx.$$

Let us consider

$$\bar{\nu}(x,t) = \nu(x,t) - t\int_0^1 \nu_1(x)dx - \int_0^1 \nu_0(x)dx, \tag{3.6}$$

therefore,

$$\int_0^1 \bar{\nu}(x,t)dx = 0, \ \forall t \geq 0,$$

on the other hand, we see that

$$\rho_4 \frac{d}{dt}\int_0^1 \mathcal{S}(x,t)dx + \alpha\int_0^1 \mathcal{S}(x,t)dx = 0, \tag{3.7}$$

then, we take

$$\bar{\mathcal{S}}(x,t) = \mathcal{S}(x,t) - e^{\left(\frac{\alpha}{\rho_4}\right)t}\int_0^1 \mathcal{S}_0(x)dx, \tag{3.8}$$

which gives

$$\int_0^1 \bar{\mathcal{S}}(x,t)dx = 0, \ \forall t \geq 0,$$

this proves that we can use Poincaré inequality for $\bar{\nu}, \bar{\mathcal{S}}$, then, if we replace in system (3.1), we find that $(\bar{\nu}, r, \varpi, \vartheta, \bar{\mathcal{S}})$ fulfils the system, so, we will use $\bar{\nu}, \bar{\mathcal{S}}$ in place of

$\nu,\ \mathcal{S}$, and write $\nu,\ \mathcal{S}$ for the purpose of simplicity. Let $Y$ be the vector function given by

$$Y = (\nu, \nu_t, 3\varpi - r, (3\varpi - r)_t, \varpi, \varpi_t, \vartheta, \mathcal{S}, \mathcal{K})^T,$$

and let $v = \nu_t, u = r_t, \varphi = \varpi_t$, then we can rewrite the system (3.4) in the form:

$$\begin{cases} Y_t = \mathcal{Z}(t)Y, \\ Y(0) = Y_0 = (\nu_0, \nu_1, 3\varpi_0 - r_0, 3\varpi_1 - r_1, \varpi_0, \varpi_1, \vartheta_0, \mathcal{S}_0, f_0)^T, \end{cases} \tag{3.9}$$

where, $\mathcal{Z}(t) : D(\mathcal{Z}(t)) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is given by

$$\mathcal{Z}(t)Y = \begin{pmatrix} v \\ -\dfrac{1}{\rho}[G(r-\nu_x)_x + \delta\vartheta_x] \\ 3\varphi - u \\ \dfrac{1}{I_\rho}[D(3\varpi - r)_{xx} + G(r-\nu_x)] \\ \varphi \\ \dfrac{1}{I_\rho}[D\varpi_{xx} - G(r-\nu_x) - \dfrac{4}{3}\gamma\varpi - \dfrac{1}{3}\mu_1\varphi - \dfrac{1}{3}\mu_2\mathcal{K}(x,1,t)] \\ -\dfrac{1}{\rho_3}[\mathcal{S}_x + \delta v_x] \\ -\dfrac{1}{\rho_4}[\alpha\mathcal{S} + \vartheta_x] \\ -\dfrac{1-\varsigma'(t)\rho}{\varsigma(t)}\mathcal{K}_\rho \end{pmatrix}, \tag{3.10}$$

and where the energy space $\mathcal{H}$ is:

$$\mathcal{H} = H_*^1 \times L_*^2(0,1) \times H_0^1 \times L^2(0,1) \times H_0^1 \times L^2(0,1) \times L^2(0,1) \times L_*^2(0,1) \times L^2((0,1)\times(0,1)),$$

with

$$\begin{aligned} L_*^2(0,1) &= \{\phi \in L^2(0,1) / \int_0^1 \phi(x)dx = 0\}, \\ H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1), \\ H_*^2(0,1) &= \{\phi \in H^2(0,1) / \phi_x(1) = \phi_x(0) = 0\}. \end{aligned}$$

We consider over $\mathcal{H}$ the inner product

$$\begin{aligned}\langle Y, \hat{Y}\rangle_{\mathcal{H}} = \rho \int_0^1 v\hat{v}dx + I_\rho \int_0^1 (3\varphi - u)(3\hat{\varphi} - \hat{u})dx + 3I_\rho \int_0^1 \varphi\hat{\varphi}dx \\ + G\int_0^1 (r - \nu_x)(\hat{r} - \hat{\nu_x})dx + \rho_3 \int_0^1 \vartheta\hat{\vartheta}dx + \rho_4 \int_0^1 \mathcal{S}\hat{\mathcal{S}}dx \\ + D\int_0^1 (3\varpi - r)_x(3\hat{\varpi} - \hat{r})_x dx + 4\gamma \int_0^1 \varpi\hat{\varpi}dx + 3D\int_0^1 \varpi_x\hat{\varpi}_x dx \\ + \int_0^1\int_0^1 \mathcal{K}\hat{\mathcal{K}}\, d\rho dx,\end{aligned} \tag{3.11}$$

for all

$$Y = (\nu, v, 3\varpi - r, 3\varphi - u, \varpi, \varphi, \vartheta, \mathcal{S}, \mathcal{K})^T \in \mathcal{H},$$

$$\hat{Y} = (\hat{\nu}, \hat{v}, 3\hat{\varpi} - \hat{r}, 3\hat{\varphi} - \hat{u}, \hat{\varpi}, \hat{\varphi}, \hat{\vartheta}, \hat{\mathcal{S}}, \hat{\mathcal{K}})^T \in \mathcal{H}.$$

Next, we give $D(\mathcal{Z}(t))$ by

$$D(\mathcal{Z}(t)) = \left\{ \begin{array}{l} Y \in \mathcal{H}/\nu \in H^2_* \cap H^1_*, 3\varpi - r, \varpi \in H^2 \cap H^1_*, \\ v, \mathcal{S} \in H^1_*, 3\varphi - u, \varphi, \vartheta \in H^1_0(0,1), \\ \mathcal{K}, \mathcal{K}_\rho \in L^2((0,1)\times(0,1)), \mathcal{K}(x,0,t) = \varphi. \end{array} \right\},$$

we clearly see that $D(\mathcal{Z}(t))$ is independent of $t$, and dense in $\mathcal{H}$. Let us now announce the existence and uniqueness theorem.

**Theorem 3.1.** *Let $Y_0 \in \mathcal{H}$ and suppose that (3.3) is verified . Then, there exists a unique solution $Y \in C(\mathbb{R}_+, \mathcal{H})$ of the problem (3.9). Moreover, if $Y_0 \in D(\mathcal{Z}(0))$, then*

$$Y \in C(\mathbb{R}_+, D(\mathcal{Z}(0))) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

We will apply the variable norm approach established by Kato in [24] to show Theorem 3.1.

**Theorem 3.2.** *Let:*

*1. $D(\mathcal{Z}(0))$ be dense in $\mathcal{H}$;*

*2. $D(\mathcal{Z}(t)) = D(\mathcal{Z}(0)), \quad \forall t > 0$;*

*3. for all $t \in [0, T]$, $\mathcal{Z}(t)$ generates a strongly continuous semi-group on $H$, and the family $\mathcal{Z} = \{\mathcal{Z}(t) : t \in [0, T]\}$ is stable with stability constants $C$ and $m$ independent of $t$. i.e. the semi-group $(S_t(s))_{s\geq 0}$ generated by $\mathcal{Z}(t)$ satisfies*

$$\|S_t(s)(y)\|_{\mathcal{H}} \leq Ce^{ms}\|y\|_{\mathcal{H}}, \quad u \in \mathcal{H}, \quad s \geq 0;$$

*4.* $\frac{d}{dt}\mathcal{Z}(t) \in L^{\infty}_{*}([0,T], B(D(\mathcal{Z}(0)),\mathcal{H}))$, *where* $L^{\infty}_{*}([0,T], B(D(\mathcal{Z}(0)),\mathcal{H}))$ *is the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0,T]$ *into the set* $B(D(\mathcal{Z}(0)),\mathcal{H})$ *of bounded operators from* $D(\mathcal{Z}(0))$ *into* $\mathcal{H}$*. Then, problem (3.9) has a unique solution*

$$Y \in C([0,T), D(\mathcal{Z}(0))) \cap C^1([0,T),\mathcal{H}),$$

*for any initial datum in* $D(\mathcal{Z}(0))$.

*Proof.* Our proof uses the same procedure as [38].

$(i)$ The fact that $D(\mathcal{Z}(0))$ is a dense in $\mathcal{H}$ is evident.

$(ii)$ Because $D(\mathcal{Z}(t))$ does not depend on $t$, we obtain $D(\mathcal{Z}(t)) = D(\mathcal{Z}(0)), \quad \forall t > 0.$

$(iii)$ In this step, we establish that $\mathcal{Z}(t)$ generates a $C_0$-semi-group in $\mathcal{H}$ for a fixed $t$. For this goal, we introduce the time-dependent inner-product on $\mathcal{H}$, (which is equivalent to the classical inner product).

$$\begin{aligned}\langle Y,\hat{Y}\rangle_t = {} & \rho\int_0^1 v\hat{v}dx + I_\rho\int_0^1(3\varphi-u)(3\hat{\varphi}-\hat{u})dx + 3I_\rho\int_0^1\varphi\hat{\varphi}dx\\ & + G\int_0^1(r-\nu_x)(\hat{r}-\hat{\nu_x})dx + \rho_3\int_0^1\vartheta\hat{\vartheta}dx + \rho_4\int_0^1\mathcal{S}\hat{\mathcal{S}}dx\\ & + D\int_0^1(3\varpi-r)_x(3\hat{\varpi}-\hat{r})_x dx + 4\gamma\int_0^1\varpi\hat{\varpi}dx + 3D\int_0^1\varpi_x\hat{\varpi}_x dx\\ & + \kappa\varsigma(t)\int_0^1\int_0^1\mathcal{K}\hat{\mathcal{K}}\,d\rho dx,\end{aligned} \tag{3.12}$$

where the positive constant $\kappa$ satisfies by $(3.3)_1$ the inequality below

$$\frac{|\mu_2|}{\sqrt{1-d}} \le \kappa \le 2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}}. \tag{3.13}$$

We start by putting

$$\tau(t) = \frac{(\varsigma'(t)^2+1)^{\frac{1}{2}}}{2\varsigma(t)}. \tag{3.14}$$

Now, we show that the new operator $\tilde{\mathcal{Z}}(t) = \mathcal{Z}(t) - \tau(t)I$ is dissipative. When $t$ is fixed and $Y = (\nu, v, 3\varpi - r, 3\varphi - u, \varpi, \varphi, \vartheta, \mathcal{S}, \mathcal{K})^T \in D(\mathcal{Z}(t))$, we find

$$\begin{aligned}\langle\mathcal{Z}(t)Y,Y\rangle_t = {} & -\mu_1\int_0^1\varphi^2dx - \mu_2\int_0^1\varphi\mathcal{K}(x,1,t)dx\\ & - \kappa\int_0^1\int_0^1(1-\varsigma'(t)\rho)\mathcal{K}_\rho\mathcal{K}d\rho dx - \alpha\int_0^1\mathcal{S}^2dx.\end{aligned} \tag{3.15}$$

On the other hand, we have

$$\begin{aligned} -\int_0^1\int_0^1 (1-\varsigma'(t)\rho)\mathcal{K}_\rho\mathcal{K}d\rho dx &= -\frac{1}{2}\int_0^1\int_0^1 \frac{d}{d\rho}(1-\varsigma'(t)\rho)\mathcal{K}^2 d\rho dx \\ &= -\frac{\varsigma'(t)}{2}\int_0^1\int_0^1 \mathcal{K}^2(x,\rho,t)d\rho dx \\ &\quad -\frac{1}{2}\int_0^1 \left\{(1-\varsigma'(t))\mathcal{K}^2(x,1,t)dx - \mathcal{K}^2(x,0,t)\right\} dx. \end{aligned} \tag{3.16}$$

Thereby,

$$\begin{aligned} \langle \mathcal{Z}(t)Y,Y\rangle_t = -\mu_1\int_0^1 \varphi^2 dx - \mu_2\int_0^1 \varphi\mathcal{K}(x,1,t)dx - \frac{\kappa\varsigma'(t)}{2}\int_0^1\int_0^1 \mathcal{K}^2(x,\rho,t)d\rho dx \\ -\frac{\kappa}{2}\int_0^1 \left\{(1-\varsigma'(t))\mathcal{K}^2(x,1,t)dx - \mathcal{K}^2(x,0,t)\right\} dx - \alpha\int_0^1 \mathcal{S}^2 dx. \end{aligned} \tag{3.17}$$

Combining $\mathcal{K}(x,0,t) = \varphi(x,t)$, Young's, Cauchy-Schwartz inequalities, and relation (3.13), we obtain

$$\begin{aligned} \langle \mathcal{Z}(t)Y,Y\rangle_t &\le (-\mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}} + \frac{\kappa}{2})\int_0^1 \varphi^2 dx \\ &\quad + \left(\frac{|\mu_2|\sqrt{1-d}}{2} - \kappa\frac{(1-d)}{2}\right)\int_0^1 \mathcal{K}^2(x,1,t)dx \\ &\quad - \alpha\int_0^1 \mathcal{S}^2 dx + \tau(t)\langle Y,Y\rangle_t, \end{aligned} \tag{3.18}$$

therefore,

$$\langle \mathcal{Z}(t)Y,Y\rangle_t - \tau(t)\langle Y,Y\rangle_t \le 0,$$

which proves the dissipativity of $\tilde{\mathcal{Z}}(t)$.

We proceed and show that the operator $\mathcal{Z}(t)$ is surjective by establishing that $(Id - \mathcal{Z}(t))$ is surjective. In fact, for $G^* = (g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8,g_9)^T \in \mathcal{H}$, we establish the existence of a unique $\chi = (\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_6,\chi_7,\chi_8,\mathcal{K}) \in D(\mathcal{Z}(t))$ such that

$$(Id - \mathcal{Z}(t))\chi = G^*, \tag{3.19}$$

that is

$$\begin{cases} \chi_1 - \chi_2 = g_1, \\ \rho\chi_2 - G\chi_{1xx} - G\chi_{3x} + 3G\chi_{5x} + \delta\chi_{7x} = \rho g_2, \\ \chi_3 - \chi_4 = g_3, \\ I_\rho\chi_4 - D\chi_{3xx} - 3G\chi_5 + G\chi_3 + G\chi_{1x} = I_\rho g_4, \\ \chi_5 - \chi_6 = g_5, \\ \left(I_\rho + \frac{1}{3}\mu_1\right)\chi_6 - D\chi_{5xx} - G\chi_3 - G\chi_{1x} + \left(3G + \frac{4}{3}\gamma\right)\chi_5 + \frac{1}{3}\mu_2\mathcal{K}(x,1,t) = I_\rho g_6, \\ \rho_3\chi_7 + \chi_{8x} + \delta\chi_{2x} = \rho_3 g_7, \\ (\rho_4 + \alpha)\chi_8 + \chi_{7x} = \rho_4 g_8, \\ \varsigma(t)\mathcal{K}_t(x,\rho,t) + (1 - \varsigma'(t)\rho)\mathcal{K}_\rho(x,\rho,t) = \varsigma(t)g_9. \end{cases} \tag{3.20}$$

Check that the solution of (3.20)$_9$ is provided by

$$\begin{cases} \mathcal{K}(x,\rho,t) = \chi_5 e^{-\rho\varsigma(t)} + \varsigma(t)e^{-\varsigma(t)\rho}\displaystyle\int_0^\rho e^{\varsigma(t)\sigma}g_9(x,\sigma,t)d\sigma, & \text{if } \varsigma'(t) = 0, \\ \mathcal{K}(x,\rho,t) = \chi_5 e^{\lambda_\rho(t)} + e^{\lambda_\rho(t)}\displaystyle\int_0^\rho \frac{\varsigma(t)}{1 - \varsigma'(t)\sigma}g_9(x,\sigma,t)e^{-\lambda_\sigma(t)}d\sigma, & \text{if } \varsigma'(t) \neq 0, \end{cases} \tag{3.21}$$

with, $\lambda_\rho(t) = \dfrac{\varsigma(t)}{\varsigma'(t)}ln(1 - \varsigma'(t)\rho)$, thus,

$$\mathcal{K}(x,1,t) = \begin{cases} \chi_5 e^{-\varsigma(t)} + \mathcal{K}_0(x), & \text{if } \varsigma'(t) = 0, \\ \chi_5 e^{\lambda_1(t)} + \mathcal{K}_0(x), & \text{if } \varsigma'(t) \neq 0, \end{cases} \tag{3.22}$$

where,

$$\mathcal{K}_0(x) = \begin{cases} \varsigma(t)e^{-\varsigma(t)}\displaystyle\int_0^1 e^{\varsigma(t)\sigma}g_9(x,\sigma,t)d\sigma, & \text{if } \varsigma'(t) = 0, \\ e^{\lambda_1(t)}\displaystyle\int_0^1 \frac{\varsigma(t)}{1 - \varsigma'(t)\sigma}g_9(x,\sigma,t)e^{-\lambda_\sigma(t)}d\sigma, & \text{if } \varsigma'(t) \neq 0. \end{cases} \tag{3.23}$$

From (3.20)$_8$ we deduce that

$$\chi_7 = -(\rho_4 + \alpha)\int_0^x \chi_8(\sigma)d\sigma + \rho_4\int_0^x g_8(\sigma)d\sigma. \tag{3.24}$$

Meanwhile, we see that

$$\chi_2 = \chi_1 - g_1,\ \chi_4 = \chi_3 - g_3,\ \chi_6 = \chi_5 - g_5. \tag{3.25}$$

If we substitute (3.22), (3.24) and (3.25), into $(3.20)_2, (3.20)_4, (3.20)_6$ and $(3.20)_7$, we obtain

$$\begin{cases} \rho\chi_1 - G\chi_{1xx} - G\chi_{3x} + 3G\chi_{5x} - \delta(\rho_4+\alpha)\chi_8 = \ell_1, \\ I_\rho\chi_3 - D\chi_{3xx} - 3G\chi_5 + G\chi_3 + 3G\chi_{1x} = \ell_2, \\ \mu_3\chi_5 - D\chi_{5xx} - G\chi_3 - G\chi_{1x} = \ell_3, \\ -\rho_3(\rho_4+\alpha)\displaystyle\int_0^x \chi_8(\sigma)d\sigma\sigma + \chi_{8x} + \delta\chi_{1x} = \ell_4, \end{cases} \tag{3.26}$$

where,

$$\begin{cases} \mu_3 = I_\rho + 3G + \dfrac{1}{3}\mu_1 + \dfrac{4}{3}\gamma + \dfrac{1}{3}\mu_2 e^{-\varsigma(t)}, \\ \ell_1 = \rho(g_1+g_2) - \rho_4 g_8, \\ \ell_2 = I_\rho(g_3+g_4), \\ \ell_3 = I_\rho(g_5+g_6) + \dfrac{1}{3}\mu_1 g_5 - \dfrac{1}{3}\mu_2\mathcal{K}_0(x), \\ \ell_4 = \rho_3 g_7 - \rho_3\rho_4\displaystyle\int_0^x g_8(\sigma)d\sigma + \delta g_{1x}, \end{cases} \quad \text{if } \varsigma'(t) = 0, \tag{3.27}$$

and,

$$\begin{cases} \mu_3 = I_\rho + 3G + \dfrac{1}{3}\mu_1 + \dfrac{4}{3}\gamma + \dfrac{1}{3}\mu_2 e^{\lambda_1(t)}, \\ \ell_1 = \rho(g_1+g_2) - \rho_4 g_8, \\ \ell_2 = I_\rho(g_3+g_4), \\ \ell_3 = I_\rho(g_5+g_6) + \dfrac{1}{3}\mu_1 g_5 - \dfrac{1}{3}\mu_2\mathcal{K}_0(x), \\ \ell_4 = \rho_3 g_7 - \rho_3\rho_4\displaystyle\int_0^x g_8(\sigma)d\sigma + \delta g_{1x}, \end{cases} \quad \text{if } \varsigma'(t) \neq 0. \tag{3.28}$$

Multiplying each case by $\hat{\chi_1}, \hat{\chi_3}, \hat{\chi_5},$ and $(\rho_4+\alpha)\displaystyle\int_0^x \hat{\chi_8}(\sigma)d\sigma$, respectively, summing up, and then, integrating over $(0,1)$, we end up with the following variational formulation

$$\mathcal{B}((\chi_1,\chi_3,\chi_5,\chi_8),(\hat{\chi_1},\hat{\chi_3},\hat{\chi_5},\hat{\chi_8})) = \Gamma(\hat{\chi_1},\hat{\chi_3},\hat{\chi_5},\hat{\chi_8}). \tag{3.29}$$

Where, we define the bilinear form

$$\mathcal{B} : (H_*^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times L_*^2(0,1))^2 \longrightarrow \mathbb{R},$$

by

$$\begin{aligned}\mathcal{B}((\chi_1,\chi_3,\chi_5,\chi_8),(\hat{\chi_1},\hat{\chi_3},\hat{\chi_5},\hat{\chi_8})) =&\mu_4\int_0^1 \chi_5\hat{\chi_5}dx+\rho\int_0^1 \chi_1\hat{\chi_1}dx\\ &+I_\rho\int_0^1 \chi_3\hat{\chi_3}dx+(\rho_4+\alpha)\int_0^1 \chi_8\hat{\chi_8}dx\\ &+D\int_0^1 \chi_{3x}\hat{\chi_{3x}}dx+3D\int_0^1 \chi_{5x}\hat{\chi_{5x}}dx\\ &+G\int_0^1(-\chi_{1x}-\chi_3+3\chi_5)(-\hat{\chi_{1x}}-\hat{\chi_3}+3\hat{\chi_5})dx\\ &+\rho_3(\rho_4+\alpha)^2\int_0^1(\int_0^x \chi_8(\sigma)d\sigma)(\int_0^x \hat{\chi_8}(\sigma)d\sigma)dx,\end{aligned} \tag{3.30}$$

where,

$$\begin{cases}\mu_4 = 3I_\rho+\mu_1+4\gamma+\mu_2 e^{-\varsigma(t)}, & \text{if } \varsigma'(t)=0,\\ \mu_4 = 3I_\rho+\mu_1+4\gamma+\mu_2 e^{\lambda_1(t)}, & \text{if } \varsigma'(t)\neq 0,\end{cases}$$

and the linear functional $\Gamma$

$$\Gamma : (H^1_*(0,1)\times H^1_0(0,1)\times H^1_0(0,1)\times L^2_*(0,1)) \longrightarrow \mathbb{R},$$

is defined by

$$\begin{aligned}\Gamma(\hat{\chi_1},\hat{\chi_3},\hat{\chi_5},\hat{\chi_8}) = &\int_0^1 \ell_1\hat{\chi_1}dx+\int_0^1 \ell_2\hat{\chi_3}dx+3\int_0^1 \ell_3\hat{\chi_5}dx\\ &+\int_0^1 \ell_4(-(\rho_4+\alpha)\int_0^1 \hat{\chi_8}(\sigma)d\sigma)dx.\end{aligned} \tag{3.31}$$

Next, we consider the following norm over $V = H^1_*(0,1)\times H^1_0(0,1)\times H^1_0(0,1)\times L^2_*(0,1)$,

$$\|(\chi_1,\chi_3,\chi_5,\chi_8)\|^2_\chi = \|(-\chi_{1x}-\chi_3+3\chi_5)\|^2_2+\|\chi_1\|^2_2+\|\chi_8\|^2_2+\|\chi_{3x}\|^2_2+\|\chi_{5x}\|^2_2.$$

We obtain as a result

$$\begin{aligned}
\mathcal{B}((\chi_1,\chi_3,\chi_5,\chi_8),(\chi_1,\chi_3,\chi_5,\chi_8)) &= \mu_4\int_0^1 \chi_5^2dx + \rho\int_0^1 \chi_1^2dx \\
&\quad + I_\rho\int_0^1 \chi_3^2dx + (\rho_4+\alpha)\int_0^1 \chi_8^2dx \\
&\quad + D\int_0^1 \chi_{3x}^2dx + 3D\int_0^1 \chi_{5x}^2dx \\
&\quad + G\int_0^1 (-\chi_{1x}-\chi_3+3\chi_5)^2dx \\
&\quad + \rho_3(\rho_4+\alpha)^2\int_0^1 (\int_0^x \chi_8(\sigma)d\sigma)^2dx, \qquad (3.32)\\
&\geq \rho\int_0^1 \chi_1^2dx + (\rho_4+\alpha)\int_0^1 \chi_8^2dx \\
&\quad + D\int_0^1 \chi_{3x}^2dx + 3D\int_0^1 \chi_{5x}^2dx \\
&\quad + G\int_0^1 (-\chi_{1x}-\chi_3+3\chi_5)^2dx, \\
&\geq M_0\|(\chi_1,\chi_3,\chi_5,\chi_8)\|_{\mathcal{X}}^2, \quad M_0>0.
\end{aligned}$$

Hence, $\mathcal{B}$ is coercive, and we deduce that (3.4) admits a unique solution by the Lax-Milgram theorem, thereby

$$\begin{aligned}
&\chi_1 \in H_*^1(0,1),\\
&\chi_3,\chi_5 \in H_0^1(0,1), \qquad (3.33)\\
&\chi_8 \in L_*^2(0,1).
\end{aligned}$$

We insert $\chi_1,\chi_3,\chi_5$ and $\chi_8$ in (3.22), (3.24) and (3.25), respectively to establish

$$\begin{aligned}
&\chi_2 \in H_*^1(0,1),\\
&\chi_4,\chi_6 \in H_0^1(0,1),\\
&\chi_7 \in H_*^1(0,1), \qquad (3.34)\\
&\mathcal{K},\mathcal{K}_\rho \in L^2((0,1)\times(0,1)).
\end{aligned}$$

If we take $\hat{\chi}_1 \in H_0^1(0,1)$ and

$$\hat{\chi}_1 = \hat{\chi}_1 - \int_0^1 \hat{\chi}_1(\kappa)d\kappa, \qquad (3.35)$$

we find $\hat{\chi}_1 \in H_*^1(0,1)$. Taking now $(\hat{\chi}_1,\hat{\chi}_3,\hat{\chi}_5,\hat{\chi}_8)=(\hat{\chi}_1,0,0,0)$ in (3.29), we get

$$G\int_0^1 (-\chi_{1x}-\chi_3+3\chi_5)(-\hat{\chi}_{1x})dx + \rho\int_0^1 \chi_1\hat{\chi}_1dx = \int_0^1 \ell_1\hat{\chi}_1dx,$$

then,

$$\begin{aligned} G\int_0^1 \chi_{1xx}\hat{\chi}_1 dx = \rho\int_0^1 \chi_1\hat{\chi}_1 dx - G\int_0^1 \chi_{3x}\hat{\chi}_1 dx \\ + G\int_0^1 \chi_{5x}\hat{\chi}_1 dx - \int_0^1 \ell_1\hat{\chi}_1 dx, \qquad \forall\hat{\chi}_1 \in H_0^1(0,1), \end{aligned} \tag{3.36}$$

which implies that

$$G\chi_{1xx} = \rho\chi_1 - G\chi_{3x} + G\chi_{5x} - \ell_1 \ \in L^2(0,1). \tag{3.37}$$

Hence,

$$\chi_1 \in H^2(0,1). \tag{3.38}$$

Additionally, for all $\phi \in C^1([0,1])$ we get by integrating by parts:

$$\begin{aligned} G\chi_{1x}(1)\phi(1) - G\chi_{1x}(0)\phi(0) - G\int_0^1 \chi_{1x}\phi_x dx + \rho\int_0^1 \chi_1\phi dx - G\int_0^1 \chi_{3x}\phi dx \\ + G\int_0^1 \chi_{5x}\phi dx - \int_0^1 \ell_1\phi dx = 0. \end{aligned} \tag{3.39}$$

Therefore,

$$G\chi_{1x}(1)\phi(1) - G\chi_{1x}(0)\phi(0) = 0, \quad \forall\phi \in C^1([0,1]). \tag{3.40}$$

The fact that $\phi$ is arbitrary implies that $\chi_{1x}(0) = \chi_{1x}(1) = 0$, that is $\chi_1 \in H_*^2(0,1)$, following the same arguments, we find

$$\begin{aligned} &\chi_3, \chi_5 \in H^2(0,1) \cap H_0^1(0,1), \\ &\chi_7 \in H_0^1(0,1), \\ &\chi_8 \in H_*^1(0,1). \end{aligned} \tag{3.41}$$

Lastly, applying a result of regularity theory to linear elliptic equations guarantees the existence of a unique $\chi \in D(\mathcal{Z}(t))$ satisfying (3.19).

Hence, for any fixed $t$ the operator $I - \mathcal{Z}(t)$ is surjective.

$$(I - \tilde{\mathcal{Z}(t)}) = (1 + \tau(t))\, I - \mathcal{Z}(t), \qquad \text{for any fixed } t > 0.$$

Because $\tau(t) > 0$, we determine the surjectivity of $(I - \tilde{A}(t))$.

Let us now prove that

$$\frac{\|\Phi\|_t}{\|\Phi\|_s} \leqslant e^{\frac{\varrho}{2\varsigma_0}|t-s|}, \quad \text{for any } t, s \in [0, T], \tag{3.42}$$

where $\phi = (\nu, \nu_t, 3\varpi - r, (3\varpi - r)_t, \varpi, \varpi_t, \vartheta, \mathcal{S}, \mathcal{K})^T$, with the help of (3.11), we get

$$\begin{aligned}\|\Phi\|_t^2 = \langle \Phi, \Phi\rangle_t = \int_0^1 &(\rho v^2 + I_\rho(3\varphi - u)^2 + 3I_\rho\varphi^2 + G(r-\nu_x)^2 + \rho_3\vartheta^2 \\ &+ \rho_4\mathcal{S}^2 + D(3\varpi - r)_x^2 + 4\gamma\varpi^2 + 3D\varpi_x^2 + \kappa\varsigma(t)\int_0^1 \mathcal{K}^2 d\rho)dx,\end{aligned}$$

and,

$$\begin{aligned}\|\Phi\|_s^2 = \langle \Phi, \Phi\rangle_s = \int_0^1 &(\rho v^2 + I_\rho(3\varphi - u)^2 + 3I_\rho\varphi^2 + G(r-\nu_x)^2 + \rho_3\vartheta^2 \\ &+ \rho_4\mathcal{S}^2 + D(3\varpi - r)_x^2 + 4\gamma\varpi^2 + 3D\varpi_x^2 + \kappa\varsigma(s)\int_0^1 \mathcal{K}^2 d\rho)dx,\end{aligned}$$

so,

$$\begin{aligned}\|\Phi\|_t^2 - \|\Phi\|_s^2 e^{\frac{\varrho}{2\varsigma_0}|t-s|} = &\left(1 - e^{\frac{\varrho}{2\varsigma_0}|t-s|}\right)\int_0^1 (\rho v^2 + I_\rho(3\varphi - u)^2 + 3I_\rho\varphi^2 + G(r-\nu_x)^2 + \rho_3\vartheta^2 \\ &+ \rho_4\mathcal{S}^2 + D(3\varpi - r)_x^2 + 4\gamma\varpi^2 + 3D\varpi_x^2)dx \\ &+ \kappa\left(\varsigma(t) - \varsigma(s)e^{\frac{\varrho}{2\varsigma_0}|t-s|}\right)\int_0^1\int_0^1 \mathcal{K}^2(x,p)d\rho\, dx.\end{aligned} \tag{3.43}$$

After that, we have to show that $\varsigma(t) - \varsigma(s)e^{\frac{\varrho}{2\varsigma_0}|t-s|} \leqslant 0$, for some $\varrho > 0$, to this end, we note that

$$\varsigma(t) = \varsigma(s) + \varsigma'(a)(t-s), \qquad \text{with } a \in (s,t),$$

then,

$$\frac{\varsigma(t)}{\varsigma(s)} \leqslant 1 + \frac{|\varsigma'(a)|}{\varsigma(s)}|t-s|.$$

By utilizing $\varsigma \in W^{2,\infty}([0,T])$, we infer that

$$\frac{\varsigma(t)}{\varsigma(s)} \leqslant 1 + \frac{\varrho}{\varsigma_0}|t-s| \leqslant e^{\frac{\varrho}{2\varsigma_0}|t-s|},$$

and (3.42) is therefore proved.

(iv) It is simple to demonstrate that

$$\frac{d}{dt}\mathcal{Z}(t)Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{\left(\varsigma''(t)\varsigma(t)p - \varsigma'(t)\left(\varsigma'(t)p - 1\right)\right)}{\varsigma^2(t)}\mathcal{K}_\rho \end{pmatrix}.$$

By utilizing relation $(3.3)_4$, we infer that

$$\frac{d}{dt}\mathcal{Z}(t) \in L^\infty_*([0,T], B(D(\mathcal{Z}(0),\mathcal{H})).$$

As in [38], (iv) holds. As a result, problem

$$\begin{cases} \tilde{Y}_t = \tilde{\mathcal{Z}}(t)\,\tilde{Y}, \\ \tilde{Y}(0) = Y_0, \end{cases} \tag{3.44}$$

admits a unique solution $\tilde{Y} \in C\left([0,\infty),\mathcal{H}\right)$, moreover if $Y_0 \in D(\mathcal{Z}(0))$, then

$$\tilde{Y} \in C([0,\infty), D(\mathcal{Z}(0))) \cap C^1([0,\infty),\mathcal{H}).$$

Finally, we let

$$Y(t) = e^{\omega(t)}\tilde{Y}(t),$$

where $\omega(t) = \displaystyle\int_0^t \tau(s)ds$, so by (3.44), we infer

$$\begin{aligned} Y_t(t) =& \tau(t)e^{\omega(t)}\tilde{Y}(t) + e^{\omega(t)}\tilde{Y}_t(t), \\ =& \tau(t)e^{\omega(t)}\tilde{Y}(t) + e^{\omega(t)}\tilde{\mathcal{Z}}(t)\tilde{Y}(t), \\ =& e^{\omega(t)}\left(\tau(t)\tilde{Y}(t) + \tilde{\mathcal{Z}}(t)\tilde{Y}(t)\right), \\ =& e^{\omega(t)}\mathcal{Z}(t)\tilde{Y}(t), \\ =& \mathcal{Z}(t)e^{\omega(t)}\tilde{Y}(t), \\ =& \mathcal{Z}(t)Y(t). \end{aligned}$$

Hence, we deduce that problem (3.9) unique solution is $Y(t)$. $\square$

## 3.3 Stability results

In this part, we provide and demonstrate our stability finding ,for this purpose, the following lemmas are required, and the positive generic constant $c$ will be used.

**Lemma 3.3.** *We define the energy functional $E$ by*

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^1 \{\rho\nu_t^2 + I_\rho(3\varpi - r)_t^2 + 3I_\rho\varpi_t^2 + 3D\varpi_x^2 + 4\gamma\varpi^2 + D(3\varpi - r)_x^2 \\ & + D(r-\nu_x)^2 + \rho_3\vartheta^2 + \rho_4\mathcal{S}^2 + \frac{\kappa}{2}\int_0^1 \varsigma(t)\mathcal{K}^2(x,\rho,t)d\rho\}dx, \end{aligned} \tag{3.45}$$

*then, it satisfies*

$$E'(t) \le -\lambda_0(\int_0^1 \varpi_t^2 dx + \int_0^1 \mathcal{K}^2(x,1,t)dx) - \alpha\int_0^1 \mathcal{S}^2 dx \le 0, \quad \text{where } \lambda_0 > 0. \tag{3.46}$$

*Proof.* To prove (3.46), we multiply $(3.4)_1$, $(3.4)_2$, $(3.4)_3$, $(3.4)_4$, $(3.4)_5$, and $(3.4)_6$ by $\nu_t$, $(3\varpi - r)_t$, $\varpi_t$, $\vartheta$, $\mathcal{S}$, and $\kappa, \mathcal{K}$ respectively, integrate over $(0,1)$, sum up, and utilize the integration by parts, to obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_0^1 \{\rho\nu_t^2 + I_\rho(3\varpi - r)_t^2 + 3I_\rho\varpi_t^2 + 3D\varpi_x^2 + 4\gamma\varpi^2 + D(3\varpi - r)_x^2 \\ & + D(r-\nu_x)^2 + \rho_3\vartheta^2 + \rho_4\mathcal{S}^2 + \frac{\kappa}{2}\int_0^1 \varsigma(t)\mathcal{K}^2(x,\rho,t)d\rho\}dx \\ & + \mu_1\int_0^1 \varpi_t^2 dx + \alpha\int_0^1 \mathcal{S}^2 dx + \mu_2\int_0^1 \varpi_t\mathcal{K}(x,1,t)dx \\ & - \frac{\kappa}{2}\int_0^1 \{\mathcal{K}^2(x,0,t) - \mathcal{K}^2(x,1,t)\}dx - \frac{\kappa\varsigma'(t)}{2}\int_0^1 \mathcal{K}^2(x,1,t)dx = 0. \end{aligned} \tag{3.47}$$

Then, we employ Young's inequality to get

$$-\mu_2\int_0^1 \varpi_t\mathcal{K}(x,1,t)dx \le \frac{|\mu_2|}{2\sqrt{1-d}}\int_0^1 \varpi_t^2 dx + \frac{|\mu_2|\sqrt{1-d}}{2}\int_0^1 \mathcal{K}^2(x,1,t)dx. \tag{3.48}$$

From $\mathcal{K}(x,0,t) = \varpi_t(x,t)$, (3.47), and (3.48), we find

$$\begin{aligned} E'(t) \le & -\alpha\int_0^1 \mathcal{S}^2 dx - \left(\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\kappa}{2}\right)\int_0^1 \varpi_t^2 dx \\ & + \left(\frac{\kappa}{2}(\varsigma'(t) - 1) + \frac{|\mu_2|\sqrt{1-d}}{2}\right)\int_0^1 \mathcal{K}^2(x,1,t)dx, \end{aligned} \tag{3.49}$$

thus, using (3.3) we conclude the existence of a positive constant $\lambda_0$ in a way that

$$E'(t) \leq -\lambda_0(\int_0^1 \varpi_t^2 dx + \int_0^1 \mathcal{K}^2(x,1,t)dx) - \alpha \int_0^1 \mathcal{S}^2 dx \leq 0. \tag{3.50}$$

□

**Lemma 3.4.** *The functional*

$$\mathcal{G}_1(t) = -\rho \int_0^1 \nu_t \left( \int_0^x (3\varpi - r)(y)dy \right) dx + I_\rho \int_0^1 (3\varpi - r)(3\varpi - r)_t dx, \tag{3.51}$$

*satisfies*

$$\begin{aligned} \mathcal{G}_1'(t) \leq & -\frac{D}{2} \int_0^1 (3\varpi - r)_x^2 dx + \epsilon_1 \int_0^1 \nu_t^2 dx \\ & + c\left(1 + \frac{1}{\epsilon_1}\right) \int_0^1 (3\varpi - r)_t^2 dx + c \int_0^1 \vartheta^2 dx. \end{aligned} \tag{3.52}$$

*Proof.* Multiplying the first and the second equation of the system (3.4) by $\int_0^x (3\varpi - r)(y,t)dy$ and $(3\varpi - r)$ respectively, integrating over $(0,1)$, and using (3.2), we obtain

$$\begin{aligned} \mathcal{G}_1'(t) =& I_\rho \int_0^1 (3\varpi - r)_t^2 dx - D \int_0^1 (3\varpi - r)_x^2 dx \\ & + \delta \int_0^1 \vartheta_x \left( \int_0^x (3\varpi - r)(y)dy \right) dx \\ & - \rho \int_0^1 \nu_t \left( \int_0^x (3\varpi - r)_t(y)dy \right) dx. \end{aligned} \tag{3.53}$$

With the use of Young's, Cauchy-schwarz's, and Poincaré's inequalities, estimate (3.52) is proved. □

**Lemma 3.5.** *The functional*

$$\mathcal{G}_2(t) = \rho \int_0^1 (r - \nu_x) \left( \int_0^x \nu_t(y)dy \right) dx, \tag{3.54}$$

*satisfies*

$$\begin{aligned} \mathcal{G}_2'(t) \leq & -\frac{G}{2} \int_0^1 (r - \nu_x)^2 dx + \epsilon_2 \int_0^1 (3\varpi - r)_t^2 dx \\ & + c\left(1 + \frac{1}{\epsilon_2}\right) \int_0^1 \nu_t^2 dx + c \int_0^1 \vartheta^2 dx + c \int_0^1 \varpi_t^2 dx. \end{aligned} \tag{3.55}$$

*Proof.* We take the derivative of $\mathcal{G}_2(t)$, integrate by parts, and exploit the first equation of system (3.4) to find

$$\begin{aligned}\mathcal{G}_2'(t) =&\rho \int_0^1 r_t \left(\int_0^x \nu_t(y)dy\right) dx + \rho \int_0^1 \nu_t^2 dx \\ &- G\int_0^1 (r-\nu_x)^2 dx - \delta \int_0^1 (r-\nu_x)\vartheta dx.\end{aligned} \tag{3.56}$$

Noticing that $r_t = 3\varpi_t - (3\varpi - r)_t$, and using Young's and Cauchy-schwarz inequalities, estimate (3.55) is proved. □

**Lemma 3.6.** *The functional*

$$\mathcal{G}_3(t) = \rho_4\rho_3 \int_0^1 \vartheta \left(\int_0^x \mathcal{S}(y)dy\right) dx, \tag{3.57}$$

*satisfies*

$$\mathcal{G}_3'(t) \leq -\frac{\rho_3}{2}\int_0^1 \vartheta^2 dx + \epsilon_3 \int_0^1 \nu_t^2 dx + c\left(1+\frac{1}{\epsilon_3}\right)\int_0^1 \mathcal{S}^2 dx. \tag{3.58}$$

*Proof.* We take the derivative of $\mathcal{G}_3(t)$, exploit system (3.4), and benefit of the integration by parts, to end up with

$$\mathcal{G}_3'(t) =\rho_4 \int_0^1 \mathcal{S}^2 dx + \rho_4\delta \int_0^1 \nu_t\mathcal{S} dx - \rho_3\alpha \int_0^1 \vartheta\left(\int_0^x \mathcal{S}(y)dy\right) dx - \rho_3 \int_0^1 \vartheta^2 dx, \tag{3.59}$$

then, (3.58) results by an application of both Young's and Cauchy-schwarz inequalities. □

**Lemma 3.7.** *The functional*

$$\mathcal{G}_4(t) = -\rho\rho_3 \int_0^1 \vartheta \left(\int_0^x \nu_t(y)dy\right) dx, \tag{3.60}$$

*satisfies,*

$$\mathcal{G}_4'(t) \leq -\frac{\rho\delta}{2}\int_0^1 \nu_t^2 dx + \epsilon_4 \int_0^1 (r-\nu_x)^2 dx + c\left(1+\frac{1}{\epsilon_4}\right)\int_0^1 \vartheta^2 dx + c\int_0^1 \mathcal{S}^2 dx. \tag{3.61}$$

*Proof.* Multiplying the fourth equation of the system (3.4) by $\int_0^x \nu_t(y,t)dy$, and integrating over $(0,1)$, using integration by parts , and the first equation of the system (3.4), we find

$$\mathcal{G}_4'(t) = -\rho\int_0^1 \mathcal{S}\nu_t dx - \rho\delta\int_0^1 \nu_t^2 dx + \rho_3 G\int_0^1 \vartheta(r-\nu_x)dx + \rho_3\delta\int_0^1 \vartheta^2 dx. \tag{3.62}$$

An application of Young's, Cauchy-schwarz and Poincaré's inequalities all together results in (3.61). □

**Lemma 3.8.** *The functional*

$$\begin{aligned}\mathcal{G}_5(t) =& \rho_4\delta G I_\rho\int_0^1 (3\varpi - r)_t(r-\nu_x)dx - \rho_4\delta D\rho\int_0^1 \nu_t(3\varpi - r)_x dx\\ &+ \rho_3\rho_4(D\rho - GI_\rho)\int_0^1 \vartheta(3\varpi - r)_t dx\\ &- \rho_4(D\rho - GI_\rho)\int_0^1 \mathcal{S}(3\varpi - r)_x dx,\end{aligned} \tag{3.63}$$

*satisfies,*

$$\begin{aligned}\mathcal{G}_5'(t) \le& -\frac{\rho_4\delta GI_\rho}{2}\int_0^1 (3\varpi - r)_t^2 dx + \epsilon_5\int_0^1 (3\varpi - r)_x^2 dx\\ &+ \epsilon_6\int_0^1 \vartheta^2 dx + c\left(1+\frac{1}{\epsilon_6}\right)\int_0^1 (r-\nu_x)^2 dx + \frac{c}{\epsilon_5}\int_0^1 \mathcal{S}^2 dx\\ &+ c\int_0^1 \varpi_t^2 dx + \chi\int_0^1 \vartheta_x(3\varpi - r)_x dx,\end{aligned} \tag{3.64}$$

*where*

$$\chi = \rho_4\delta^2 D - (D\rho - GI_\rho)\left(\frac{\rho_4\rho_3 D}{I_\rho} - 1\right). \tag{3.65}$$

*Proof.* We first differentiate $\mathcal{G}_5$ with respect to $t$, then, we apply the integration by parts to end up with

$$\begin{aligned}\mathcal{G}_5'(t) =& \rho_4 G^2\delta\int_0^1 (r-\nu_x)^2 dx + \rho_4 G\delta I_\rho\int_0^1 (3\varpi - r)_t r_t dx\\ &+ \delta^2\rho_4\int_0^1 \vartheta_x(3\varpi - r)_x dx + \frac{\rho_4\rho_3 D}{I_\rho}(D\rho - GI_\rho)\int_0^1 \vartheta_x(3\varpi - r)_x dx\\ &+ \frac{\rho_4\rho_3}{I_\rho}(D\rho - GI_\rho)\int_0^1 \vartheta(r-\nu_x)dx + \alpha(D\rho - GI_\rho)\int_0^1 \mathcal{S}(3\varpi - r)_x dx\\ &+ (D\rho - GI_\rho)\int_0^1 \vartheta_x(3\varpi - r)_x dx.\end{aligned} \tag{3.66}$$

The application of Young's inequality along with $r_t = 3\varpi_t - (3\varpi - r)_t$ leads to estimate (3.64). □

**Lemma 3.9.** *The functional*

$$\mathcal{G}_6(t) = 3I_\rho \int_0^1 \varpi\varpi_t dx + \frac{\mu_1}{2}\int_0^1 \varpi^2 dx, \tag{3.67}$$

*satisfies,*

$$\begin{aligned}\mathcal{G}_6'(t) \leq & -3D\int_0^1 \varpi_x^2 dx - 2\gamma\int_0^1 \varpi^2 dx + c\int_0^1 (r-\nu_x)^2 dx \\ & + c\int_0^1 \varpi_t^2 dx + c\int_0^1 \mathcal{K}^2(x,1,t)dx.\end{aligned} \tag{3.68}$$

*Proof.* Multiplying the third equation of the system (3.4) by $\varpi$, integrating over $(0,1)$, and using integration by parts and (3.2), we get

$$\begin{aligned}\mathcal{G}_6'(t) =& 3I_\rho\int_0^1 \varpi_t^2 dx + 3I_\rho\int_0^1 \varpi\varpi_{tt} dx + \mu_1\int_0^1 \varpi\varpi_t dx \\ =& 3I_\rho\int_0^1 \varpi_t^2 dx - 3D\int_0^1 \varpi_x^2 dx - 3G\int_0^1 \varpi(r-\nu_x)dx \\ & - 4\gamma\int_0^1 \varpi^2 dx - \mu_2\int_0^1 \varpi\mathcal{K}(x,1,t)dx.\end{aligned} \tag{3.69}$$

The application of Young's, Cauchy-schwarz, and Poincaré's inequalities results in (3.68). □

**Lemma 3.10.** *The functional*

$$\mathcal{G}_7(t) = \kappa\varsigma(t)\int_0^1\int_0^1 e^{-2\varsigma(t)\rho}\mathcal{K}^2(x,\rho,t)d\rho dx, \tag{3.70}$$

*satisfies,*

$$\mathcal{G}_7'(t) \leq -2\mathcal{G}_7(t) + \kappa\int_0^1 \varpi_t^2 dx. \tag{3.71}$$

*Proof.* We directly differentiate $\mathcal{G}_7$, to obtain

$$\begin{aligned}\mathcal{G}_7'(t) =& \kappa\varsigma'(t)\int_0^1\int_0^1 e^{-2\varsigma(t)\rho}\mathcal{K}^2(x,\rho,t)d\rho dx \\ & + 2\kappa\varsigma(t)\int_0^1\int_0^1 \{-\varsigma'(t)\rho e^{-2\varsigma(t)\rho}\mathcal{K}^2 + e^{-2\varsigma(t)\rho}\mathcal{K}_t\mathcal{K}\}d\rho dx,\end{aligned} \tag{3.72}$$

after that, we exploit (3.4)$_6$, to establish

$$\begin{aligned}\varsigma(t)\int_0^1\int_0^1 e^{-2\varsigma(t)\rho}\mathcal{K}_t\mathcal{K}d\rho dx =& \int_0^1\int_0^1(\varsigma'(t)\rho-1)e^{-2\varsigma(t)\rho}\mathcal{K}_\rho\mathcal{K}d\rho dx\\ =& \frac{1}{2}\int_0^1\int_0^1\frac{d}{d\rho}\{(\varsigma'(t)\rho-1)e^{-2\varsigma(t)\rho}\mathcal{K}^2\}d\rho dx\\ &+\varsigma(t)\int_0^1\int_0^1\{(\varsigma'(t)\rho-1)e^{-2\varsigma(t)\rho}\mathcal{K}^2\}d\rho dx\\ &-\frac{\varsigma'(t)}{2}\int_0^1\int_0^1 e^{-2\varsigma(t)\rho}\mathcal{K}^2d\rho dx,\end{aligned}\tag{3.73}$$

using (3.72) and (3.73), we obtain

$$\begin{aligned}\mathcal{G}_7'(t)=&-2\kappa\varsigma(t)\int_0^1\int_0^1 e^{-2\varsigma(t)\rho}\mathcal{K}^2dx+\kappa\int_0^1\varpi_t^2dx\\ &-\kappa(1-\varsigma'(t))e^{-2\varsigma(t)}\int_0^1\mathcal{K}^2(x,1,t)dx,\end{aligned}\tag{3.74}$$

therefore, we obtain (3.71). □

**Theorem 3.11.** *Suppose that (3.3) holds, there exist positive constants $\beta_1$ and $\beta_2$ such that the energy functional giving by (3.45) satisfies*

$$E(t)\le\beta_2e^{-\beta_1t},\ \forall t\ge0.\tag{3.75}$$

*Proof.* Let us introduce a Lyapunov functional:

$$\mathcal{L}(t)=NE(t)+\mathcal{G}_1(t)+\sum_{i=2}^{i=6}N_i\mathcal{G}_i+\mathcal{G}_7,\tag{3.76}$$

where $N$ and $N_i$, $i=2...6$, are positive constants to chose later.

Hence, we have

$$\begin{aligned}|\mathcal{L}(t)-NE(t)|\le& I_\rho\int_0^1|(3\varpi-r)(3\varpi-r)_t|dx+\rho\int_0^1|\nu_t\left(\int_0^x(3\varpi-r)(y)dy\right)|dx\\ &+N_2\rho\int_0^1|(r-\nu_x)(\int_0^x\nu_t(y)dy)|dx+N_2\rho_4\rho_3\int_0^1|\vartheta(\int_0^x\mathcal{S}(y)dy)|dx\\ &+N_4\rho\rho_3\int_0^1|\vartheta(\int_0^x\nu_t(y)dy)|dx+N_5\rho_4\delta GI_\rho\int_0^1|(3\varpi-r)_t(r-\nu_x)|dx\\ &+N_5\rho_4\delta D\rho\int_0^1|\nu_t(3\varpi-r)_x|dx+N_5\rho_3\rho_4|D\rho-GI_\rho|\int_0^1|\vartheta(3\varpi-r)_t|dx\\ &+N_5\rho_4|(D\rho-GI_\rho)|\int_0^1|\mathcal{S}(3\varpi-r)_x|dx+3I_\rho N_6\int_0^1|\varpi\varpi_t|dx\\ &+\frac{\mu_1}{2}N_6\int_0^1\varpi^2dx+\kappa\varsigma(t)\int_0^1\int_0^1e^{-2\varsigma(t)\rho}\mathcal{K}^2(x,\rho,t)d\rho dx.\end{aligned}$$

We derive by the application of Young's, Cauchy-shwarz, and Poincaré's inequalities

$$\begin{aligned}|\mathcal{L}(t) - NE(t)| \leq & c\int_0^1 \{ \nu_t^2 + (3\varpi - r)_t^2 + (3\varpi - r)_x^2 + (r-\nu_x)^2 + \varpi^2 + \varpi_x^2 + \varpi_t^2 + \\ & \int_0^1 \mathcal{K}^2(x,\rho,t)d\rho + \vartheta^2 + \mathcal{S}^2\}dx \\ & \leq cE(t).\end{aligned}$$

Which means that

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t). \tag{3.77}$$

Now, if we differentiate (3.76), and exploit (3.46),(3.52),(3.55),(3.58) ,(3.61),(3.64),(3.68), and (3.71),we get

$$\begin{aligned}\mathcal{L}'(t) \leq & -[\frac{D}{2} - \epsilon_5 N_5]\int_0^1 (3\varpi - r)_x^2 dx - [\lambda_0 N - cN_2 - cN_5 - cN_6 - \kappa]\int_0^1 \varpi_t^2 dx \\ & -[\frac{\rho_4\sigma GI_\rho}{2}N_5 - c(1+\frac{1}{\epsilon_1}) - \epsilon_2 N_2]\int_0^1 (3\varpi - r)_t^2 dx \\ & -[\frac{G}{2}N_2 - \epsilon_4 N_4 - c(1+\frac{1}{\epsilon_6})N_5 - cN_6]\int_0^1 (r-\nu_x)^2 dx \\ & -[\frac{\rho\sigma}{2}N_4 - \epsilon_1 - c(1+\frac{1}{\epsilon_2}) - \epsilon_3 N_3]\int_0^1 \nu_t^2 dx \\ & -[\frac{\rho_3}{2}N_3 - \epsilon_6 N_5 - c(1+\frac{1}{\epsilon_4})N_4 - cN_2 - c]\int_0^1 \vartheta^2 dx \\ & -[\alpha N - c(1+\frac{1}{\epsilon_3})N_3 - cN_4 - \frac{c}{\epsilon_5}N_5]\int_0^1 \mathcal{S}^2 dx \\ & -[2\gamma N_6]\int_0^1 \varpi^2 dx - [3DN_6]\int_0^1 \varpi_x^2 dx \\ & -[\lambda_0 N - cN_6]\int_0^1 \mathcal{K}^2(x,1,t)dx - 2\mathcal{G}_7(t).\end{aligned} \tag{3.78}$$

We then set

$$\epsilon_1 = 1, \epsilon_2 = \frac{\rho_4 G\delta I_\rho N_5}{4N_2}, \epsilon_3 = \frac{\rho\delta N_4}{4N_3}, \epsilon_4 = \frac{GN_2}{4N_4}, \epsilon_5 = \frac{D}{4N_5}, \epsilon_6 = \frac{\rho_3 N_3}{4N_5},$$

to find

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -\left(\frac{D}{4}\right)\int_0^1 (3\varpi - r)_x^2 dx - [\lambda_0 N - cN_2 - cN_3 - cN_6 - \kappa]\int_0^1 \varpi_t^2 dx \\
& - \left[\frac{\rho_4 G\delta I_\rho}{4} N_5 - 2c\right]\int_0^1 (3\varpi - r)_t^2 dx \\
& - \left[\frac{G}{4} N_2 - c\,(1+N_5)\,N_5 - cN_6\right]\int_0^1 (r-\nu_x)^2 dx \\
& - \left[\frac{\rho\delta}{4} N_4 - c\left(1+\frac{N_2}{N_5}\right)N_2 - 1\right]\int_0^1 \nu_t^2 dx \\
& - \left[\frac{\rho_3}{4} N_3 - c\left(1+\frac{N_4}{N_2}\right)N_4 - cN_2 - c\right]\int_0^1 \vartheta^2 dx \\
& - \left[\alpha N - cN_4 - c\,(1+N_3)\,N_3 - cN_5^2\right]\int_0^1 \mathcal{S}^2 dx \\
& - [2\gamma N_6]\int_0^1 \varpi^2 dx - [3DN_6]\int_0^1 \varpi_x^2 dx \\
& - [\lambda_0 N - cN_6]\int_0^1 \mathcal{K}^2(x,1,t)dx - 2\mathcal{G}_7(t).
\end{aligned}
\tag{3.79}
$$

The constants are then specifically selected to make he terms in parentheses positive. So let us first fix $N_6$, and take $N_5,\ N_2$ big enough to get

$$
\begin{cases}
\dfrac{\rho_4 G\delta I_\rho}{4} N_5 - 2c > 0, \\
\dfrac{G}{4} N_2 - c\,(1+N_5)\,N_5 - cN_6 > 0,
\end{cases}
$$

we continue and take $N_4,\ N_3$ sufficiently large such that

$$
\begin{cases}
\dfrac{\rho\delta}{4} N_4 - c\left(1+\dfrac{N_2}{N_5}\right) - 1 > 0, \\
\dfrac{\rho_3}{4} N_3 - c\left(1+\dfrac{N_4}{N_2}\right)N_4 - cN_2 - c > 0.
\end{cases}
$$

Finally, we select $N$ sufficiently large such that (3.77) holds and

$$
\begin{cases}
\lambda_0 N - cN_6 > 0, \\
\lambda_0 N - cN_2 - cN_3 - cN_6 - \kappa > 0, \\
\alpha N - cN_4 - c\left(1+\dfrac{N_3}{N_4}\right)N_3 - cN_5^2 > 0,
\end{cases}
$$

thus, there exists $C > 0$, such that

$$\begin{aligned}
\mathcal{L}'(t) \leq & - C\int_0^1 \{(3\varpi - r)_x^2 + \varpi_t^2 + (3\varpi - r)_t^2 + (r - \nu_x)^2 + \nu_t^2 \\
& + \vartheta^2 + \mathcal{S}^2 + \varpi^2 + \varpi_x^2 + \mathcal{K}^2(x,1,t) + \int_0^1 \mathcal{K}^2(x,\rho,t)d\rho\}dx, \\
& \leq -CE(t),
\end{aligned} \tag{3.80}$$

relation (3.77) and the choice of $N$ leads to

$$k_1E(t) \leq \mathcal{L}(t) \leq k_2E(t), \tag{3.81}$$

where, $k_1,\ k_2$ are positive constants, the latter along with (3.80) gives

$$\mathcal{L}'(t) \leq -\beta_1\mathcal{L}(t), \quad \beta_1 = \frac{C}{k_2}.$$

Lastly, we integrate (3.82) and exploit (3.81) to establish (3.75) which completes our proof.

$\square$

# Conclusion and future work

In this thesis, we analyzed several evolution systems arising in the context of laminated thermoelastic and Timoshenko-type beams with various delay terms.

Our contributions include:

- Proving the global existence and uniqueness of solutions.
- Establishing asymptotic stability and decay estimates under constant and time-varying delays.
- Extending classical models by incorporating second sound and non-Fourier heat conduction.

These results contribute to a deeper understanding of dynamic behaviors in thermoelastic structures with memory and delay and can be useful in engineering applications involving smart material, vibration control and thermal stress analysis.
We suggest possible directions for future research, such as:

1. Considering nonlinear effects or more general boundary conditions.
2. Investigating numerical simulations for the derived models.
3. Studying control problems or inverse problems in delayed beam systems.

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**Abstract :** In this thesis we considered some evolution systems with the presence of deferent of dissipation.
In chapter one, we consider a system of laminated beam with an internal constant delay term in the transverse displacement. Under suitable assumptions, we prove the global well-posedness and exponential stability result.
Chapter two is related to the linear thermoelastic laminated Timoshenko beam with second sound. We prove the polynomial and exponential stabilities results.
In chapter three, we consider a thermoelastic laminated Timoshenko beam with time varying delay. We prove the existence, uniqueness and stability results.

**Keywords:** Timoshenko, thermoelastic laminated, well-posedness, exponential decay, polynomial decay, delay term, second sound, time varying delay.

**ملخص:** في هذه الأطروحة، ندرس بعض الأنظمة التطورية في وجود أنواع مختلفة من التخميد.
في الفصل الأول، ندرس نظامًا لشعاع مرصوص يحتوي على تأخير زمني ثابت داخلي في الإزاحة العرضية، وبافتراضات مناسبة نثبت الوجود، الوحدانية والاستقرار الأسي للنتائج.
يرتبط الفصل الثاني بشعاع تيموشينكو Timoshenko المرصوص الخطي الحراري المرن مع الصوت الثاني، حيث نثبت نتائج الاستقرار من نوع كثير الحدود والاستقرار الأسي.
في الفصل الثالث، ندرس شعاع تيموشينكو-Timoshenko الحراري المرن المرصوص مع تأخير زمني متغير، حيث نثبت نتائج الوجود، الوحدانية والاستقرار.

**الكلمات المفتاحية:** تيموشينكو-Timoshenko، شعاع حراري مرن مرصوص، الوجود، الوحدانية، الانحلال الأسي، الانحلال متعدد الحدود، مصطلح التأخير، الصوت الثاني، تأخير زمني متغير.

**Résumé :** Dans cette thèse, nous avons étudié certains systèmes d'évolution en présence de différents types de dissipation.
Dans le premier chapitre, nous considérons un système de poutre laminée avec un terme de retard constant interne dans le déplacement transversal. Sous des hypothèses appropriées, nous prouvons l'existence globale et la stabilité exponentielle du resultas.
Le deuxième chapitre concerne une poutre thermoélastique laminée de Timoshenko linéaire avec le second son. Nous prouvons des résultats de stabilité polynomiale et exponentielle.
Dans le troisième chapitre, nous étudions une poutre thermoélastique laminée de Timoshenko avec un retard variable dans le temps. Nous prouvons l'existence, l'unicité et la stabilité des solutions.

**Mots-clés** : Timoshenko, poutre thermoélastique laminée, existence globale, décroissance exponentielle, décroissance polynomiale, terme de retard, second son, retard variable dans le temps.