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COURSE HANDOUT

Fundamentals of Operations Research

موجهة لطلبة السنة الثانية جذع مشترك علوم المالية والمحاسبة

Second-Year Undergraduate Students

Financial and Accounting Sciences

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Preface

In today's increasingly complex and competitive world, decision-making has become a critical skill across all domains of economics, business, management, and engineering. Organizations are constantly faced with the challenge of allocating limited resources — such as time, labor, materials, or capital — in the most efficient and effective way. To address these challenges, the field of **Operations Research (OR)** has emerged as a powerful discipline that applies mathematical modeling, analytical thinking, and optimization techniques to support rational decision-making.

Operations Research is not only a theoretical field but also a highly practical one. Its methods are widely applied in industry, logistics, transportation, manufacturing, health care, defense, and finance. It allows decision-makers to analyze complex problems, identify optimal or near-optimal solutions, and improve the overall performance of systems. Whether it is minimizing costs, maximizing profits, improving scheduling, or optimizing supply chains, Operations Research provides structured and scientifically grounded tools to approach such problems.

This Course Handout has been designed for second-year undergraduate students (L2) enrolled in the Operations Research course. It has been specially designed to make the subject more accessible by presenting it in a simplified and progressive manner. The focus of the Course Handout is placed on two fundamental and highly applicable areas:

- Linear Programming – including problem formulation, graphical solution, and the Simplex algorithm.
- Transportation Problems – including methods for obtaining initial feasible solutions and optimization techniques.

These topics have been selected due to their wide range of applications and their foundational role in more advanced areas of OR.

I. Purpose of the Course Handout

Despite the importance of Operations Research, many students find it challenging due to its mathematical nature and abstract formulations. This Course Handout aims to:

- Break down complex concepts into manageable parts.
- Use clear academic language suitable for students with limited mathematical background.
- Provide step-by-step solved examples to enhance understanding.
- Include practical exercises and real-world inspired problems to build analytical skills.
- Bridge the gap between theoretical knowledge and practical applications.

II. Educational Objectives

Upon completing this Course Handout, students will be able to:

1. Understand the scope and significance of Operations Research in business and economics.
2. Formulate real-life decision problems into mathematical models.
3. Solve linear programming problems using graphical and algebraic methods (Simplex).
4. Analyze transportation problems and determine optimal allocation strategies.
5. Interpret mathematical solutions within the context of real-world managerial decisions.
6. Develop logical and systematic approaches to problem-solving.

III. Prerequisite Knowledge

To successfully engage with the material presented in this Course Handout, students are expected to possess a basic level of mathematical and logical understanding. Specifically, they should be familiar with:

- Basic Algebra: Ability to manipulate linear equations and inequalities.
- Graphing Skills: Understanding how to plot lines and identify regions on a two-dimensional graph.
- Mathematical Reasoning: Comfort with symbolic representation and logical steps.
- Introductory Economics or Management Concepts (recommended): Familiarity with ideas such as resource constraints, cost minimization, and production efficiency can support better contextual understanding.

IV. Structure of the Course Handout

The Course Handout is organized into seven progressive chapters, each designed to include:

- Clearly stated learning objectives.
- A concise theoretical presentation of the chapter content.
- Solved examples to illustrate key concepts.
- Exercises with solutions for practice and self-assessment.

This structure ensures a balanced integration of theory and application, allowing students to develop both conceptual understanding and technical skills.

Chapter 1: Introduction to Operations Research

Learning Objectives

After completing this chapter, students should be able to:

1. Define operations research (OR) and identify its main characteristics.
2. Explain the historical development and evolution of operations research.
3. Define a model and explain how models are used in operations research.
4. Identify and describe the phases of an operations research study.
5. Discuss the major application areas of operations research.

Chapter contents

- 1-1 Historical overview
- 1-2 Definition of operations research
- 1-3 Phases of operations research study
- 1-4 Scope of operations research
- 1-5 Characteristics of Operations Research
- 1-6 Types of Operations Research (OR) Models
- 1-7 Questions and exercises
- 1-8 Solutions & Answers

1-1- Historical overview

It is difficult to determine the exact date when operations research methods began, as many early pioneers in various sciences conducted studies and carried out work that falls under what is today called operations research.

The roots of Operations Research (OR) go back many years, when people first began trying to use scientific methods to improve how organizations were managed. However, most writers agree that OR truly began during the early years of World War II. At that time, the military needed to use its limited resources in the most effective way possible. To do this, the British military, and later the U.S. military, asked groups of scientists to study their strategic and tactical problems and suggest scientific solutions. In other words, these scientists were asked to “research military operations,” which is how the field got its name.¹

These groups of scientists became the first OR teams. Their work helped improve the use of radar, which played an important role in the Allied victory in the Air Battle of Britain. They also developed better ways to manage naval convoys and fight submarines, which contributed greatly to winning the Battle of the North Atlantic. Similar research supported the success of the Island Campaign in the Pacific.

By the early 1950s, operations research had been widely adopted in Britain and subsequently in the United States. Since then, the field has expanded rapidly and has become an essential tool for decision-making in business, industry, government, and many other sectors. Its remarkable growth has been driven by two major factors:

- ✓ The rapid advancement of operations research theories and methodologies.
- ✓ The remarkable progress in computer technology, which has enabled the efficient solution of increasingly complex problems.

¹ Hillier, F.S. & Lieberman, G.J., *Introduction to Operations Research*, 11th ed., McGraw-Hill Education, New York, 2024, p. 1

1-2- Definition of operations research

In general, operations research is considered the scientific approach to making decisions related to the organization's operations, as it is applied to cases that concern the management and coordination of operations and activities in the organization.

Specifically, Operations Research can be defined as the use of quantitative methods to assist analysts and decision-makers in designing, analyzing, and improving the performance or operation of systems. The systems being studied may include financial, scientific, engineering, or industrial systems. Regardless of the context, however, practically all such systems lend themselves to analysis within the systematic framework of the scientific method.¹

Operations research aims to identify the optimal solution to decision-making problems involving limited resources. The term "operations research" is often associated with the use of mathematical methods to model and analyze decision problems.

Operations research follows the scientific method of solving problems, which begins with careful observation and defining the problem, then building a scientific model that represents, as much as possible, all the real dimensions of the problem in practical reality to reach the optimal solution from among all feasible alternatives.

Operations research should also be viewed as a science and an art. The science aspect is providing mathematical and computational methods to solve appropriate decision problems. Operations research is also considered an art because the success of all aspects that precede or follow the solution of the mathematical model depends entirely on the creativity and personal ability of the analysts making the decisions.

¹ Carter, M.W., Price, C.C. & Rabadi, G., *Operations Research: A Practical Introduction*, 2nd ed., CRC Press, Boca Raton, FL, 2018, p. 1

1-3- Phases of Operations Research Study

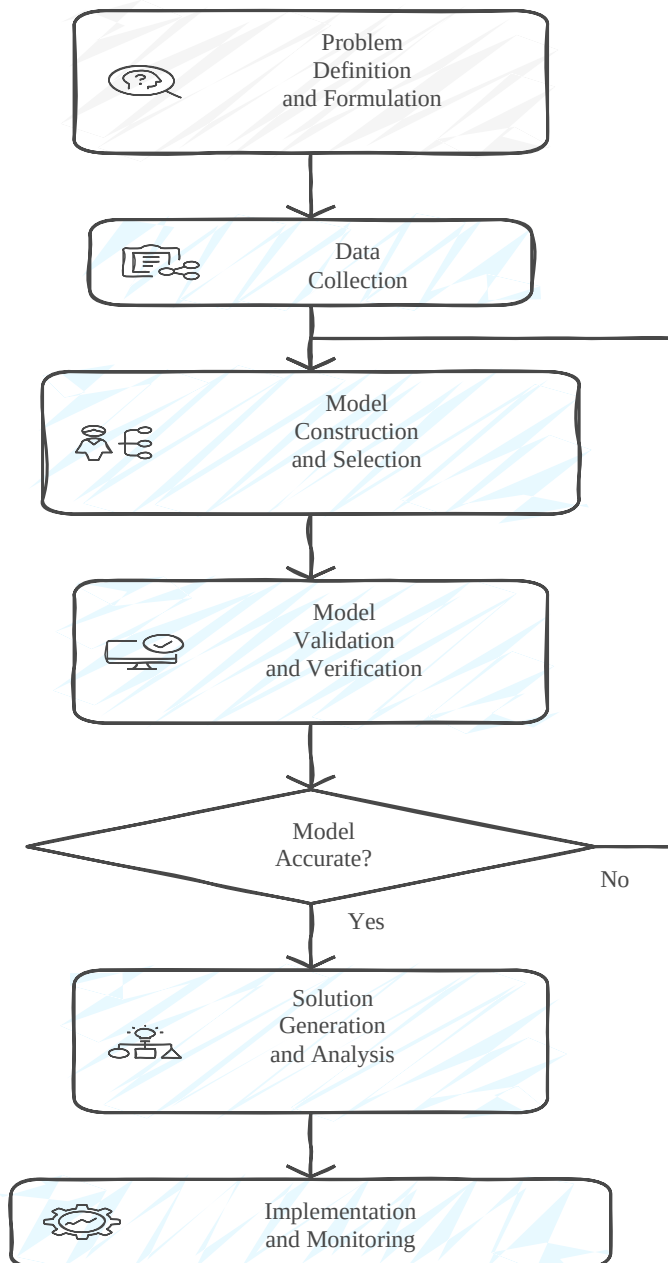
An O.R. study typically comprises six phases:¹

- a) Problem Definition and Formulation:** In this phase, the problem is identified and defined, and the objectives of the study are established. The scope of the study, the constraints, and assumptions are also defined in this phase. The primary output of this phase is a clear statement of the problem, which helps guide subsequent phases.
- b) Data Collection:** In this phase, the necessary data for the study are collected. Data can be collected from various sources, including databases, surveys, and historical records. Data quality is crucial at this stage, and it is essential to ensure that the data collected are complete, accurate, and relevant.
- c) Model Construction and Selection:** In this phase, a mathematical or simulation model is developed to represent the problem. The model should be simple enough to be solved using available techniques, but complex enough to capture the essential characteristics of the problem. Different modeling techniques can be used, including optimization, queuing theory, and simulation.
- d) Model Validation and Verification:** In this phase, the model is verified and validated to ensure that it accurately represents the problem. The model's assumptions are tested, and the output is compared to real-world data to verify its accuracy. If the model is found to be inaccurate, it is refined or reconstructed in the previous phase.
- e) Solution Generation and Analysis:** In this phase, the model is solved using appropriate techniques to obtain the optimal solution. The solution is analyzed to determine whether it meets the objectives and constraints of the problem. Sensitivity analysis is also performed to determine how changes in the input parameters affect the output.

¹ Topcu, Y. İ., *Operations Research: Lecture Notes* [Course material], Istanbul Technical University, p. 3 available at: <http://www.ilker.topcu.net> (accessed 1/2 /2025).

f) Implementation and Monitoring: In this phase, the solution is implemented, and the outcomes are monitored to ensure that they meet the objectives. This phase also involves developing an implementation plan, providing support to the implementation team, and addressing any challenges that arise.

Figure 1-1 : Operations Research Study Phases



1-4- Scope Of Operations Research

Scope of O.R. is very wide in today's world as it provides better solution to various decision-making problems with great speed and efficiency. Areas where methods/models developed in Operations Research can be applied are given here under:¹

a) In Agriculture: With the explosion of population and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from different resources. Problems of agriculture production under various restrictions can be solved by applications of Operations Research techniques.

b) In Defence Operations: Since the Second World War, operations research has been used for defence operations with the aim of obtaining maximum gains with minimum efforts.

c) In Finance: In these modern times, government of every country or every organisation wants to introduce such type of planning/policies regarding their finance and accounting which optimize capital investment, determine optimal replacement strategies, apply cash flow analysis for long range capital investments, formulate credit policies, credit risk. Techniques developed in O.R. can be applied for attaining above said things.

d) In Marketing: A Marketing Administrator has to face many problems like production selection, formulation of competitive strategies, distribution strategies, selection of advertising media with respect to cost and time, finding the optimal number of salesmen, finding optimum time to launch a product. All such problems can be overcome using Operations Research Techniques.

e) In Personnel Management: OR helps organizations recruit, assign, and schedule employees while minimizing labor costs and maximizing workforce efficiency. It needs to find the best combination of workers in different categories with respect to costs,

¹ Monika, M., *Operations Research Techniques (20MAT22C5)*, Directorate of Distance Education, Maharshi Dayanand University, Rohtak, India, 2021, p. 3

skills, age and nature of jobs. It also needs to frame recruitment policies, assign jobs to machines or workers.

f) In the Insurance Industry: Operations Research Techniques can be fruitfully applied in LIC offices as it enables the policy makers to decide the premium rates for various modes of policies.

g) In Research and Development: Operations research assists managers in identifying priority research areas, selecting the most promising projects, and allocating research resources efficiently. It also supports decision-making by evaluating alternative projects and optimizing research and development investments.

1-5- Characteristics of Operations Research

The characteristics of OR are as follows:¹

OR uses an interdisciplinary team approach to find optimal solutions:

OR study is performed by a team of scientists rather than an individual. It has been recognised that scientists from different disciplines can produce better results, than could be expected from the team of same number of scientists from the same discipline. Undoubtedly the scientists of different disciplines in a team will tackle the problem from different angles while the scientists of the same discipline cannot do so.

OR emphasizes the overall systems approach:

The main characteristic of Operations Research (OR) is that it considers all aspects of the problem being studied. In many cases, a decision that appears optimal from one point of view may be far from optimal from another. For example, in an industry, the marketing department may prefer to keep a large inventory, while the finance department may argue for keeping inventory as low as possible. Therefore, the final decision must be made by taking all aspects of the problem into account

OR tries to optimize the total output by maximizing the profit and minimizing the cost:

¹ Gupta, R.K. & Hira, D.S., *Operations Research*, 7th ed., Krishna Prakashan Media (P) Ltd., Meerut, India, 2015, p. 6

The management of an industry will apply *OR* methods to maximize the profit and minimize the costs. Thus, *OR* tries to optimize the total output.

 ***OR* uses scientific methods to arrive at an optimum solution:**

Scientific methods are used in *OR* to solve a problem and obtain an optimal solution.

1-6-Types of Operations Research Models

Operations research (*OR*) models can be classified into several categories based on different criteria. The most important classifications are outlined below: ¹

(a) Classification by Structure

Iconic Models: Iconic models are physical or visual representations that resemble the actual objects they represent, usually on a reduced or enlarged scale. For example, a toy car serves as an iconic model of a real car.

Analog Models: An analog model represents a real system by using another system with similar characteristics rather than its physical appearance. Examples include road maps and weather maps, where colors and symbols are used to represent real-world features.

Symbolic or Mathematical Models: In these models, the components of a problem are represented using mathematical symbols, variables, equations, or inequalities. Mathematical models are the most widely used in operations research due to their precision and analytical capability.

(b) Classification by Utility

Descriptive Models: Descriptive models are used to explain or describe certain aspects of a system or problem. They help decision-makers understand the behavior and structure of the system under study.

¹ Skedbooks, *Types of Operation Research Models*, Operations Research | Books, skedbooks.com, 2021, available at: <https://skedbooks.com/books/operations-research/types-of-operation-research-models> (accessed 4/4 /2025).

Predictive Models: Predictive models utilize historical or collected data to forecast or estimate the likely outcomes of a given situation.

Prescriptive Models: Prescriptive models go beyond prediction by recommending specific courses of action. When predictive results are reliable, these models can be used to prescribe optimal decisions.

(c) Classification by Nature of the Environment

Deterministic Models: In deterministic models, all parameters, variables, and resource levels are assumed to be known with certainty and remain constant throughout the planning horizon.

Probabilistic (Stochastic) Models: In stochastic models, some or all variables are subject to uncertainty and randomness. These models incorporate the element of risk and use probability distributions to represent uncertain outcomes.

(d) Classification Based on the Behavior of Variables

Static Models: Static models assume that the values of decision variables remain unchanged during the planning period, regardless of changes in environmental or system conditions.

Dynamic Models: Dynamic models allow variables to change over time or in response to variations in environmental or system conditions.

(e) Classification by Solution Method

Analytical Models: Analytical models have a well-defined mathematical structure and can be solved using exact mathematical and optimization techniques.

Simulation Models: Simulation models involve imitation of real-world systems. Although they may have a mathematical structure, they cannot be solved analytically and instead rely on computational experiments to evaluate system performance.

1-7 Questions and Exercises

Part 1: Theoretical Questions

A. Short Questions

- ✓ What is the significance of operations research?
- ✓ How does operations research provide solutions to various business problems?
- ✓ How is operations research useful in production management?
- ✓ What does the implementation and monitoring phase include?
- ✓ What is the first step in OR study?

B. Multiple Choice Questions

1. A model is
 - a) Abstraction of reality
 - b) Idealized representation.
 - c) Simple explanation
 - d) A and B
2. A mathematical model usually contains
 - a) Variables
 - b) Constants
 - c) Assumptions
 - d) All of the above
3. One of the following is not an application area of Operations Research
 - a) Transportation
 - b) Military
 - c) Personnel Assignment Problem
 - d) Operations System
 - e) None
4. Two of the steps in the management science approach to problem solving are:
 - a) Model construction and model interpretation
 - b) Model construction and model interpretation & analysis

- c) Finding a solution and testing the model
- d) Interpret and analyze the model and construct solution.

C. True or False

Answer true or false:

- 1) Operations Research originated during World War II to solve military logistics and planning problems. **(T/F)**
- 2) After World War II, OR techniques expanded rapidly into industry and business applications. **(T/F)**
- 3) Operations Research is a scientific approach to decision-making that uses mathematical models. **(T/F)**
- 4) OR aims to find the best possible solution among several available alternatives. **(T/F)**
- 5) OR is concerned only with theoretical mathematics and not with practical applications. **(T/F)**
- 6) OR helps managers make better decisions under conditions of limited resources. **(T/F)**
- 7) The first step in an OR study is to define the problem clearly. **(T/F)**
- 8) Model validation is performed to check if the model accurately represents the real system. **(T/F)**
- 9) Data collection always comes after model formulation. **(T/F)**
- 10) Implementing the solution is a necessary phase in an OR study. **(T/F)**
- 11) OR can be applied in areas such as transportation, finance, production, and healthcare. **(T/F)**
- 12) One major scope of OR is helping organizations allocate their limited resources optimally. **(T/F)**
- 13) OR techniques are frequently used in supply chain and logistics management. **(T/F)**
- 14) OR uses a systems approach, analyzing the whole system rather than individual parts. **(T/F)**
- 15) OR models always include unpredictable human behavior. **(T/F)**

- 16) Interdisciplinary collaboration is a key characteristic of OR. (T/F)
- 17) Linear programming models are considered deterministic OR models. (T/F)

1-8 Solutions & Answers

Part A: Short Questions

1. What is the significance of operations research?

Operations Research (OR) is significant because it provides a scientific and systematic approach to decision-making. It helps organizations utilize limited resources efficiently, reduce costs, and improve productivity by applying mathematical models and analytical techniques to complex problems.

2. How does operations research provide solutions for various business problems?

Operations research provides solutions by developing mathematical models that represent real-world business problems. Through methods such as linear programming, simulation, and network analysis, OR helps managers identify the best possible decisions under given constraints and objectives.

3. How is operations research useful to production management?

In production management, OR helps in optimizing production schedules, minimizing costs, managing inventory, improving quality control, and allocating resources efficiently. Techniques like linear programming and queuing theory are widely used to enhance operational efficiency.

4. What does the implementation and monitoring phase include?

The implementation and monitoring phase involves applying the proposed solution, monitoring the outcomes, and ensuring that the results meet the objectives of the study. It also includes addressing any challenges that may arise during implementation.

5. What is the first step in OR study?

The first step in an OR study is to define and formulate the problem. This includes clearly defining the objectives, identifying constraints, determining decision variables, and understanding the environment in which the problem exists.

B. Multiple Choice Questions

1. Answer: (d). A and B (A model is both an abstraction of reality and an idealized representation.)
2. Answer: (d). All of the above
3. Answer: (d). Operations system
4. Answer: (a). Model construction and model interpretation.

C. True or False

- 1) True
- 2) True
- 3) True
- 4) True
- 5) False
- 6) True
- 7) True
- 8) True
- 9) False
- 10) True
- 11) True
- 12) True
- 13) True
- 14) True
- 15) False
- 16) True
- 17) True

Chapter 2: Linear Programming Problems

Learning Objectives

After completing this chapter, students will be able to:

1. Recognize the importance of Linear Programming Problems.
2. Explain the concept of Linear Programming.
3. Analyze the mathematical formulation of Linear Programming Problems.
4. Describe the general form of a Linear Programming Model.
5. Illustrate the mathematical formulation of Linear Programming through examples

Chapter contents

2-1 Definition and Scope of Linear Programming

2-2 Linear Programming Conditions

2-3 Linear Programming Assumptions

2-4 Advantages and Limitations of LPP

2-5 Guidelines on Linear Programming Model Formulation

2-6 The general formula for the Linear Programming Model

2-7 Questions and Exercises

2-8 Solutions & Answers

2-1- Definition and Scope of Linear Programming

Linear Programming (LP), also called Linear Optimization, is concerned with the allocation of limited resources among a set of competing activities. To achieve the optimal use of these resources, they must be allocated properly in a way that ensures the attainment of management objectives.

Linear Programming is a special class of mathematical programming models in which both the objective function and the constraints are expressed as linear functions of the decision variables. As in general mathematical programming models, the decision variables represent quantities that can be controlled within the system being studied. The objective function reflects the main goal of the system and is used to measure its performance, such as maximizing profit or productivity, or minimizing cost or resource consumption.¹

In practice, the system is always subject to limitations on available resources, including time, materials, machines, energy, or manpower. These limitations are represented by constraints, which take the form of linear equations or inequalities involving the decision variables. Solving a Linear Programming problem involves finding the values of the decision variables that optimize the objective function while satisfying all the given constraints

The term programming refers to the process of determining a sequence of steps leading to the optimal solution of the problem under study. The term linear indicates that the relationships between the variables are linear, meaning that a constant change in one variable results in a constant proportional change in the others. The scope of Linear Programming includes:²

✚ Resource Allocation: LP is used to allocate scarce resources, such as labour, materials, and equipment, to maximize efficiency.

¹ Carter, M. W., Price & Rabadi, *op. cit.*, p.23

² Alex Antonyraj, P., *Operation Research*, Manonmaniam Sundaranar University, Tamil Nadu, India, 2021, p. 25

Chapter 2: Linear Programming Problems

- ✚ Cost Minimization: LP helps minimize costs, such as production costs, transportation costs, and inventory costs.
- ✚ Profit Maximization: LP is used to maximize profits, revenue, or returns on investment.
- ✚ Optimization of Business Processes: LP can optimize business processes, such as supply chain management, logistics, and production planning.
- ✚ Portfolio Optimization: LP is used in finance to optimize investment portfolios, managing risk and return.
- ✚ Energy and Environment: LP can optimize energy production, consumption, and distribution, as well as environmental management.
- ✚ Transportation and Logistics: LP is used to optimize transportation routes, schedules, and inventory management.
- ✚ Manufacturing and Production: LP can optimize production planning, scheduling, and inventory control.

2-2- Linear Programming conditions

Certain conditions must be satisfied when applying Linear Programming, the most important of which are:

- Having a clear goal that can be translated into a mathematical equation and that expresses the maximum return or the lowest cost.
- There are a number of alternatives to choose from.
- The relationship between the variables and the objective function must be linear.
- The presence of constraints representing the scarcity of available resources.
- The values of the variables in the problem must be positive (non-negative condition).

- The ability to express the activities or variables that are the subject of programming in numerical form.

2-3- Linear Programming assumptions

LP is theoretically based on a set of assumptions that must be satisfied in order to solve the problem, summarized as follows:¹

a) Certainty: In LP Models, it is assumed that all its parameters such as: availability of resources, profit (or cost) contribution per unit of decision variable and consumption of resources per unit of decision variable must be known and constant.

b) Additivity: The value of the objective function and the total amount of each resource used (or supplied), must be equal to the sum of the respective individual contribution (profit or cost) of the decision variables. For example, the total profit earned from the sale of two products A and B must be equal to the sum of the profits earned separately from A and B. Similarly, the amount of a resource consumed for producing A and B must be equal to the total sum of resources used for A and B individually.

c) Linearity (or proportionality): The amount of each resource used (or supplied) and its contribution to the profit (or cost) in objective function must be proportional to the value of each decision variable. For example, if production of one unit of a product uses 5 hours of a particular resource, then making 3 units of that product uses $3 \times 5 = 15$ hours of that resource.

d) Divisibility (or continuity): The solution values of decision variables are allowed to assume continuous values. For instance, it is possible to collect 6.254 thousand litres of milk by a milk dairy and such variables are divisible. But, it is not desirable to produce 2.5 machines and such variables are not divisible and therefore must be assigned integer values. Hence, if any of the variable can assume only integer values or are limited to discrete number of values, LP Model is no longer applicable.

¹ Sharma, J.K., *Operations Research: Theory and Applications*, 6th ed., Laxmi Publications, New Delhi, 2017, p.27

2-4- Advantages and limitations of Linear Programming Problems

2-4-1 Advantages of Using Linear Programming:

Linear Programming offers several advantages that make it a valuable quantitative decision-making tool. The most important advantages include the following: ¹

- **Efficient Use of Resources:** Linear Programming helps decision-makers allocate limited productive resources in the most efficient manner possible.
- **Improved Quality of Decisions:** The technique enhances the quality of decision-making by promoting an objective and systematic approach, thereby reducing reliance on intuition or subjective judgment.
- **Identification of Optimal Solutions:** Linear Programming enables decision-makers to determine the optimal solution to a problem while explicitly considering resource constraints. For instance, the ability to produce a certain quantity of a product does not necessarily imply that all produced units can be sold.
- **Identification of Bottlenecks:** The Linear Programming approach highlights bottlenecks within production or service systems. For example, a bottleneck may occur when a machine is unable to produce sufficient output to meet demand, while other machines remain underutilized.

2-4-2 Limitations of Linear Programming:

Despite its wide applicability and numerous advantages, Linear Programming has several limitations, which are outlined below: ²

- **Assumption of Linearity:** Linear Programming assumes linear relationships among decision variables in both the objective function and the constraints. However, real-world relationships are often nonlinear.

¹ Sharma, *op. cit.*, p. 27

² Ibid, p.28

- **Non-Integer Solutions:** Solutions obtained from Linear Programming Models are not guaranteed to be integers. In practical situations—such as determining the number of workers or machines required—fractional solutions may be unrealistic. Simply rounding such solutions may lead to suboptimal or infeasible results.
- **Neglect of Time and Uncertainty:** Linear Programming Models do not explicitly account for time dynamics or uncertainty, even though many real-life decision problems are influenced by these factors.
- **Assumption of Constant Parameters:** The model assumes that all coefficients and parameters are known with certainty and remain constant, whereas in practice they are often subject to change and estimation errors.
- **Single Objective Focus:** Linear Programming deals with a single objective function, while real-world decision problems frequently involve multiple and potentially conflicting objectives.

2-5- Guidelines on Linear Programming Model Formulation

The effective use and application requires, as a first step, the mathematical formulation of an LP Model. Steps of LP Model formulation are summarized as follows:¹

Step 1: Identify the decision variables

- (a) Express each constraint in words. For this you should first see whether the constraint is of the form $\geq$ (greater than or equal to), of the form $\leq$ (less than or equal to) or of the form $=$ (exactly equal to).
- (b) Express verbally the objective function.
- (c) Verbally identify the decision variables with the help of Step (a) and (b). For this you need to ask yourself the question – What decisions must be made in order to optimize the objective function?

¹ Sharma, *op. cit.*, p. 30

Having followed Step1 (a) to (c) decide the symbolic notation for the decision variables and specify their units of measurement. Such specification of units of measurement would help in interpreting the final solution of the LP problem.

Step 2: Identify the problem data

To formulate an LP Model, identify the problem data in terms of constants, and parameters associated with decision variables. It may be noted that the decision maker can control values of the variables but cannot control values in the data set.

Step 3: Formulate the constraints

Convert the verbal expression of the constraints in terms of resource requirement and availability of each resource. Then express each of them as linear equality or inequality, in terms of the decision variables defined in Step1.

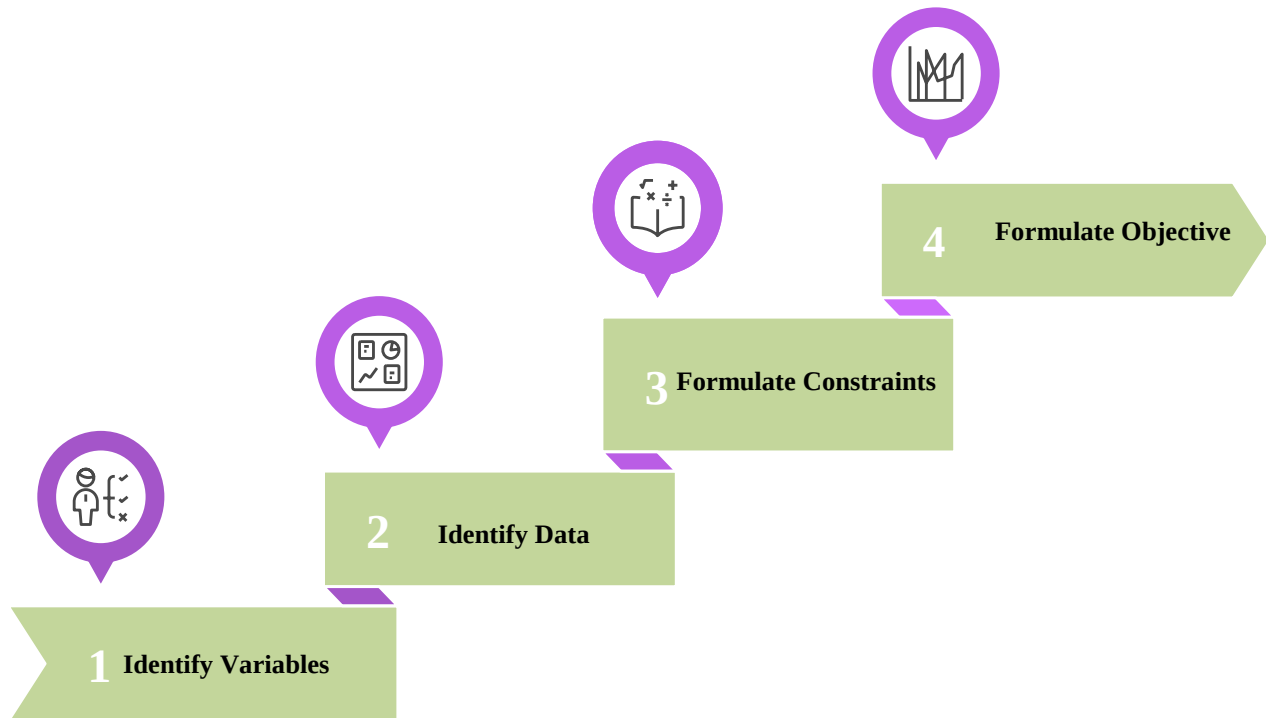
Values of these decision variables in the optimal LP problem solution must satisfy these constraints in order to constitute an acceptable (feasible) solution. Wrong formulation can either lead to a solution that is not feasible or to the exclusion of a solution that is actually feasible and possibly optimal.

Step 4: Formulate the objective function

Identify whether the objective function is to be maximized or minimized. Then express it in the form of linear mathematical expression in terms of decision variables along with profit (cost) contributions associated with them.

After gaining sufficient experience in model building, readers may omit the verbal description step. The following examples of LP Model Formulation are intended to strengthen the ability to translate real-life problems into mathematical models.

Figure 2-1: Steps to Formulate an LP Model



2-6- The General Form of the Linear Programming Model

We previously explained that the Linear Programming Model consists of two parts: the objective function and the constraints. The general formula of the mathematical model is as follows:

$$\text{Max or Min } Z = c_1X_1 + c_2X_2 + \dots + c_iX_i + \dots + c_nX_n$$

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1j}X_j + \dots + a_{1n}X_n \leq, =, \geq b_1$$

$$a_{21}X_1 + a_{22}X_2 + \dots + a_{2j}X_j + \dots + a_{2n}X_n \leq, =, \geq b_2$$

.....

$$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{ij}X_i + \dots + a_{in}X_n \leq, =, \geq b_i$$

.....

$$a_{m1}X_1 + a_{m2}X_2 + \dots + a_{mj}X_j + \dots + a_{mn}X_n \leq, =, \geq b_m$$

$$X_j \geq 0$$

$$\forall i = 1, 2, \dots, m$$

$$\forall j = 1, 2, \dots, n$$

Chapter 2: Linear Programming Problems

The general formula for the Linear Programming Model can be summarized as follows:

$$\begin{aligned} \text{Optimize (Max. or Min.)} \quad Z &= \sum_{j=1}^n C_j X_j \\ \sum_{j=1}^n a_{ij} X_j &\leq, =, \geq b_i \\ i &= 1, 2, 3 \dots m \\ j &= 1, 2, 3 \dots n \\ X_j &\geq 0 \end{aligned}$$

and $X_j \geq 0 ; j = 1, 2, \dots, n$ (Non-negativity conditions)

Where, the C_j are coefficients representing the per unit profit (or cost) of decision variable X_j to the value of objective function. The a_{ij} are referred as *technological coefficients (or input-output coefficients)*.¹

These represent the amount of resource, say i consumed per unit of variable (activity) X_j . These coefficients can be positive, negative or zero. The b_i represents the total availability of the i th resource. The term resource is used in a very general sense to include any numerical value associated with the right-hand side of a constraint. It is assumed that $b_i \geq 0$ for all i . However, if any $b_i < 0$, then both sides of the constraint is multiplied by -1 to make $b_i > 0$ and reverse the inequality of the constraint.

In the general LP problem, the expression ($\leq, =, \geq$) means that in any specific problem each constraint may take only one of the three possible forms:

- less than or equal to ($\leq$)
- equal to ($=$)
- greater than or equal to ($\geq$)

¹ Sharma, *op. cit.*, p. 29

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We notice that the objective function is a linear equation that takes either the maximization form (if the goal is to achieve the greatest profit) or the minimization form (if the goal is to achieve the lowest cost). The constraints may take the form of less-than-or-equal-to ($\leq$), greater-than-or-equal-to ($\geq$), or equality ($=$) relations. As for the last constraints, they are always in the form of inequalities greater than or equal to zero and are called non-negativity constraints because the decision variables have non-negative values.¹

Examples of lp Model Formulation:

The Andalus Woodworking Company produces two products: tables and chairs. The company aims to determine the best production plan for the next month in order to maximize profit. The main resources used in production are wood and metal rods, which are considered the critical supplies affecting the system's performance.

The following table shows the amount of each resource required to produce one unit of each product, as well as the available quantities of resources next month and the unit profit from each product:

Product	Metal Required (m ²)	Wood Required (m ²)	Unit Profit (\$)
Chair	1	2	5
Table	1/2	3	12
Available Next Month	1000	3000	

Market research indicates that the maximum demand for tables is 750 units, while all chairs produced can be sold without any limitation.

➤ The decision problem is to decide how many chairs and tables should be produced next month to maximize profit.

¹ *Ibid.*, p. 29

To prepare the mathematical model, we must first define the variables of the problem (The Decision Variables) and then express the objective and constraints in the form of mathematical functions (equations) for these variables.

a) The Decision Variables: For the furniture company problem, where the goal is to determine the number of chairs and tables to be produced next month, the decision variables can be defined as follows:

X_1 : The number of units of chairs to be produced next month.

X_2 : The number of units of tables to be produced next month.

b) Objective function: As one unit of chairs yields a profit of 5 \$ and one unit of tables yields a profit of 12 \$, the total profit equals the profit of one chair times the number of chairs produced and sold ($5X_1$) plus the profit of one table times the number of tables produced and sold ($12X_2$).

As the goal is to maximize total profit, the objective function can be written as:

$$\text{Maximize (total profit) } Z = 5X_1 + 12X_2$$

Where Z represents the total profit (chair sales are independent of table sales).

The goal is to determine the values X_1 and X_2 which will maximize the value.

The objective function is the criterion. The approved (scale) for comparison (evaluation and then ranking) between the various possible alternatives (various possible solutions).

c) Constraints: In the Andalus Company problem there are constraints on the use of raw materials and on demand. The constraint on the use of raw materials is expressed verbally as follows:

Uses of chairs and tables from raw material $\leq$ Maximum available raw material

can be translated into the following mathematical inequalities representing the constraints on raw materials :

➤ Wood constraint: $2X_1 + 3X_2 \leq 3000$

➤ Metal constraint: $X_1 + \frac{1}{2}X_2 \leq 1000$

Similarly, the market demand constraint on tables can be expressed verbally as follows:

The number of tables produced should not exceed the maximum market demand for tables:

$$\text{Number of tables produced} \leq \text{Market demand for tables}$$

Mathematically, this can be written as:

$$X_2 \leq 750$$

Remark: *There is no market demand constraint for chairs, as all chairs produced can be sold.*

There is an implicit restriction (logically understood) that it is not possible to produce both types in the negative (i.e., less than zero), i.e.

$$X_1 \geq 0$$

$$X_2 \geq 0$$

Accordingly, the specific values for each of the following are represented: X_1 and X_2 are possible solutions if all constraints imposed on the model are satisfied. Now the model for the furniture factory problem can be summarized as follows:

$$\text{Max } Z = 5X_1 + 12X_2$$

$$\text{subject to: } 2X_1 + 3X_2 \leq 3000$$

$$X_1 + \frac{1}{2}X_2 \leq 1000$$

$$X_2 \leq 750$$

$$X_1, X_2 \geq 0$$

while those accompanying the objective function in the form of minimization are in the form of less than or equal to inequalities or equations.

2-7 Questions and Exercises

Part 1: Theoretical Questions

A. Short Answer Questions

1. Define Linear Programming (LP) in your own words.
2. What are the three main components of an LP Model?
3. Explain the meaning of “decision variables” and give an example.
4. Why is the non-negativity condition important in LP?

B. Multiple Choice Questions

1. The main objective of Linear Programming is to:
 - a) Randomly allocate resources
 - b) Optimize resource allocation to maximize or minimize an objective
 - c) Eliminate constraints in decision-making
 - d) Increase the number of decision variables
2. Which of the following is NOT an assumption of LP?
 - a) Certainty
 - b) Divisibility
 - c) Non-linearity
 - d) Proportionality
3. In Linear Programming, the relationship between variables and the objective function must be:
 - a) Random
 - b) Linear
 - c) Quadratic
 - d) Exponential

C. True or False

1. In LP, decision variables can take negative values. (T/F)
2. The additivity assumption means that the total effect is the sum of the contributions of individual activities. (T/F)
3. Linear Programming deals with the optimization of a linear objective function subject to linear constraints. (T/F)
4. The objective function in an LPP can represent profit maximization or cost minimization. (T/F)
5. Divisibility requires decision variables to take only integer values. (T/ F)

Part 2: Practice Exercises

Exercise1:

A garment manufacturer has a production line making two styles of shirts. Style I requires 200 grams of cotton thread, 300 grams of dacron thread, and 300 grams of linen thread per unit . Style II requires 200 grams of cotton thread, 200 grams of dacron thread, and 100 grams of linen thread per unit. The manufacturer makes a net profit of \$19 on Style 1 and \$15 on Style II. He has in hand an inventory of 24 kg of cotton thread, 26 kg of dacron thread, and 22 kg of linen thread.

The manufacturer wishes to determine the number of shirts of each style to be produced using the available inventory in order to maximize total profit.

- ✓ Formulate a Linear Programming Model to determine an optimal production schedule.

Exercise 2:

Almarai Company specializes in manufacturing three (3) types of poultry feed: A, B, and C, using several types of inputs, the most important of which are soybeans and corn.

The production of each unit of A, B, and C requires, respectively, 2, 1, and 2 kg of soybeans and 1, 1, and 1/2 kg of corn.

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To meet its commitments for the next month, the company must produce at least 12000 units in total of the three products and exactly 5000 units of feed A.

The available quantities of raw materials for the next month are 26 tons of soybeans and 18 tons of corn. The profit per unit of feeds A, B, and C is \$250, \$210, and \$270, respectively.

- ✓ Formulate a Linear Programming Model to determine an optimal production schedule for the next month.

Exercise 3:

A farm owner has a flock of sheep that consumes at least 1,000 kg of a special feed per day. This feed is prepared by mixing two basic materials: barley grain (M1) and wheat bran (M2).

Each kilogram of these materials contains different amounts of three types of nutrients—proteins, carbohydrates, and mineral elements—as shown in the table below:

	Proteins (unit)	Carbohydrates (unit)	Mineral elements (unit)
M1 (1kg)	40	15	30
M2 (1kg)	20	30	15

In order for the nutrition to be balanced, the daily food of these livestock must include at least 9000 units of proteins, at least 6000 units of carbohydrates, and at least 8000 units of mineral elements. The cost of the two materials, M1 and M2, is \$10 and \$12 per kg, respectively.

- ✓ Formulate a Linear Programming Model in order to find the best combination of the two materials, M1 and M2 that minimizes the total feeding cost while meeting all nutritional requirements.

Exercise 4:

A carpentry company produces two types of school tables, Type A and Type B, using several inputs, including iron, wood, labor, and machines. For the next month, the company has the following available resources: 2,000 square meters of iron, 2,200 meters of wood, 900 labor hours, and 600 machine hours.

Producing one table of Type A requires 1 meter of iron, 2 square meters of wood, 1 hour of manual labor, and 15 minutes (0.25 hour) of machine time.

Producing one table of Type B requires 2 meters of iron, 1 square meter of wood, 30 minutes (0.5 hour) of manual labor, and 1 hour of machine time

The company is required to sell at least 600 tables of Type A and no more than 400 tables of Type B per month. The profit per unit is estimated at DA 1000 for each table of Type A and DA 1500 for each table of Type B.

- ✓ Formulate a Linear Programming Model to determine an optimal production schedule for the next month.

Exercise 5:

A commercial company plans to promote a new range of products available at all its branches. To reach potential customers in a residential area, the company intends to advertise through two media: local newspapers and radio broadcasts.

The advertising campaign is designed to reach at least 60000 men and 40000 women. Market research indicates that:

- One newspaper advertisement reaches approximately 6,000 men and 2,000 women.
- One radio advertisement reaches approximately 2,000 men and 4,000 women.

The cost of one newspaper advertisement is \$450, whereas the cost of one radio advertisement is \$260.

- ✓ Formulate a linear programming model to determine the optimal combination of newspaper and radio advertisements that minimizes the total advertising

Exercise 6:

To manufacture 100 units of a product in a factory, metal sheets must be cut into two shapes: A and B. Each finished product requires 3 units of shape A and 7 units of shape B.

The factory has four possible cutting methods for the metal sheets:

1. First method: Produces 4 pieces of type A and no pieces of type B, with a waste of 12 meters of metal.
 2. Second method: Produces 3 pieces of type A and 4 pieces of type B, with a waste of 5 meters of metal.
 3. Third method: Produces 1 piece of type A and 9 pieces of type B, with a waste of 3 meters of metal.
 4. Fourth method: Produces 12 pieces of type B and no pieces of type A, with no metal waste.
- ✓ Formulate a linear programming model to determine the optimal cutting plan that minimizes the total waste generated from the metal sheets.

2-8 Solutions & Answers

Part 1: Theoretical Questions

A. Short Answer Questions

1. Define Linear Programming (LP) in your own words.

Linear Programming (LP) is a mathematical method used to determine the best possible outcome—such as maximizing profit or minimizing cost—under a set of linear constraints and relationships.

2. What are the three main components of an LP Model?

The three main components are:

- a) Objective function – what we want to maximize or minimize.
- b) Decision variables – unknown quantities to be determined.
- c) Constraints – limitations or restrictions on the decision variables.

3. Explain the meaning of “decision variables” and give an example.

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Decision variables are the controllable quantities in an LP problem that represent choices to be made.

Example: In a factory, x_1 and x_2 could represent the number of units of product A and B to produce.

4. Why is the non-negativity condition important in LP?

Because negative quantities (like negative production or resources) have no practical meaning. The non-negativity condition ensures all decision variables are zero or positive.

B. Multiple Choice Questions

1. The main objective of Linear Programming is to:

- b) Optimize resource allocation to maximize or minimize an objective

2. Which of the following is NOT an assumption of LP?

- c) Non-linearity

3. In Linear Programming, the relationship between variables and the objective function must be:

- b) Linear

C. True or False

1. False
2. True
3. True
4. True
5. False

Part 2: Practice exercises

Exercise 1:

Step 1: Decision variables: These are the numbers of each style to be produced:

- Number of style I shirts for the next week, say X_1

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– Number of style II shirts, for the next week, say X_2

Step 2: Objective Function

Objective of the decision maker: Maximize total contribution, given that the contribution per unit

is \$19 for X_1 and \$15 for X_2

Hence the objective function (Maximize profit) is:

$$\text{Max } Z = 19X_1 + 15X_2$$

Step 3: Constraints

• Availability of cotton thread: Style I needs 200 g and style II needs 200 g. 24000 g is available. The corresponding constraint is:

$$200 X_1 + 200 X_2 \leq 24000$$

• Availability of dacron thread: Style I needs 300 g and style II needs 200 g. 26,000 g is available. The corresponding constraint is:

$$300 X_1 + 200 X_2 \leq 26000$$

• Availability of linen thread: Style I needs 300 g and style II needs 100 g. 22,000 g is available. The corresponding constraint is:

$$300 X_1 + 100 X_2 \leq 22000$$

Step 4: Non-negativity

Since the number of shirts cannot be negative, add the non-negativity restriction $X_1, X_2 \geq 0$ and

$$\text{Max } Z = 19X_1 + 15 X_2$$

$$\text{Sub to: } 200 X_1 + 200 X_2 \leq 24000 \text{ (cotton)}$$

$$300 X_1 + 200 X_2 \leq 26000 \text{ (dacron)}$$

$$300 X_1 + 100 X_2 \leq 22000 \text{ (linen)}$$

$$X_1, X_2 \geq 0 \text{ (non-negativity)}$$

Exercise 2:

Definition of the decision variables of the problem:

X_1 : The produced quantity of Type A poultry feed for the coming month.

X_2 : The produced quantity of Type B poultry feed for the coming month.

X_3 : The produced quantity of Type C poultry feed for the coming month.

$$\text{Max } Z = 250X_1 + 210X_2 + 270X_3$$

$$\text{Sub to: } 2X_1 + X_2 + 2X_3 \leq 26000$$

$$X_1 + X_2 + 1/2X_3 \leq 18000$$

$$X_1 + X_2 + X_3 \geq 12000$$

$$X_1 = 5000$$

$$X_2, X_3 \geq 0$$

Exercise 3:

Definition of the decision variables of the problem:

X_1 : Quantity of barley grain purchased (M1) in kg

X_2 : Quantity of wheat bran (M2)purchased (M2) in kg

$$\text{Min } Z = 10X_1 + 12X_2$$

$$\text{Sub to: } X_1 + X_2 \geq 1000$$

$$30X_1 + 15X_2 \geq 9000$$

$$15X_1 + 30X_2 \geq 6000$$

$$40X_1 + 20X_2 \geq 8000$$

$$X_1, X_2 \geq 0$$

Exercise 4:

Definition of the decision variables of the problem:

X_1 : Quantity produced of type A tables for the next month.

X_2 : Quantity produced of type B tables for the next month

$$\text{Max } Z = 1000X_1 + 1500X_2$$

$$\text{Sub to: } X_1 + 2X_2 \leq 2000$$

$$2X_1 + X_2 \leq 2200$$

$$X_1 + 1/2X_2 \leq 900$$

$$1/4X_1 + X_2 \leq 600$$

$$X_1 \geq 600$$

$$X_2 \leq 400$$

$$X_2 \geq 0$$

Exercise 5:

Definition of the decision variables of the problem:

X_1 : Number of advertisements placed in newspapers.

X_2 : Number of advertisements broadcast on the radio.

$$\text{Min } Z = 450X_1 + 260X_2$$

$$\text{Subj to: } 6000X_1 + 2000X_2 \geq 60000$$

$$2000X_1 + 4000X_2 \geq 40000$$

$$X_1, X_2 \geq 0$$

Exercise 6:

Definition of the decision variables of the problem:

X_1 : Number of metal sheets cut using method 1.

X_2 : Number of metal sheets cut using method 2.

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X_3 : Number of metal sheets cut using method 3.

X_4 : Number of metal sheets cut using method 4.

$$\text{Min } Z = 12X_1 + 5X_2 + 3X_3 + 0X_4$$

$$\text{Sub to: } X_1 + 3X_2 + X_3 \geq 300$$

$$4X_2 + 9X_3 + 12X_4 \geq 700$$

$$X_1, X_2, X_3, X_4 \geq 0$$

Chapter 3: Graphical Solution to LP Problem

Learning Objectives

After completing this chapter, students will be able to:

1. Define the graphical method for solving linear programming problems.
2. Identify the fundamental components of an LPP, including decision variables, the objective function, and constraints.
3. Graph linear inequalities in two-dimensional space to construct the feasible region.
4. Explain the concepts of feasible solutions and the feasible region.
5. Apply the corner-point method and the objective function line (trial-and-error) method to determine the optimal solution.
6. Distinguish graphically between maximization and minimization problems.
7. Assess the limitations of the graphical method in comparison with other solution techniques, such as the simplex method.

Chapter contents

3-1 Definition of the Graphical Method

3-2 Steps in the Graphical Solution

3-3 Special cases in the Graphical Method:

3-4 Advantages and Limitations of the Graphical Method

3-5 Questions and Exercises

3-6 Solutions & Answers

3-1-Definition of the Graphical Method

Once a problem is formulated as mathematical model, the next step is to solve the problem to get the optimal solution. A linear programming problem with only two variables presents a simple case, for which the solution can be obtained using the graphical (geometric) method. Though, in actual practice such small problems are rarely encountered, the graphical method provides a pictorial representation of the solution process and a great deal of insight into the basic concepts used in solving large L.P. problems.¹

The graphical method is characterized by simplicity in application, use and presentation, but it is only used to solve problems with two variables, while it is difficult to use if the number of variables is three (3) and impossible to use if the number of variables is four (4) or more.

According to this method, we use a graph with two axes (an axis for the variable X_1 and another for the variable X_2). Then we proceed to determine the region that satisfies all the inequalities at the same time, after converting all the constraints from inequalities to equations and drawing each one separately on the graph. After that, we determine the region for each constraint. This region is either below the equation line, above it, or on the line in the case of constraints that are in the form of equations. The intersection of all these regions gives us a region called the region of feasible solutions because it is the only region whose points satisfy all the constraints at the same time. Since the optimal solution always lies at a vertex or corner of this region, we substitute the coordinates of these vertices into the objective function and choose the vertex that gives us the best value for the objective function, which is the optimal solution.

¹. Gupta & Hira, op. cit., p. 102

3-2- Steps in the Graphical Solution

The procedure for solving an LPP by the graphical method is as follows:¹

- a) **Plot the constraints on a graph:** Consider each inequality constraints as an equation, and Plot each equation on the graph, as each equation will geometrically represent a straight line.
- b) **Shade the feasible region:** Every point on the line will satisfy the equation of the line. If the inequality constraints corresponding to that line are $\leq$, then the region below the line lying in the first quadrant (due to no negativity of variables) is shaded. For the inequality constraints with the $\geq$ sign, the region above the line in the first quadrant is shaded.
- c) **Identify the feasible region:** The feasible region is the intersection of all the shaded areas from the constraints. This region contains all possible solutions that satisfy all constraints.
- d) **Determine the corner points of the feasible region:** The optimal solution in an LP problem lies at one of the corner points (vertices) of the feasible region.
Evaluate the objective function at each corner point: Calculate the value of the objective function for each corner point.
- e) **Select the corner point with the optimal value:** Depending on whether the problem is maximization or minimization, choose the point with the highest or lowest objective function value.

Example 1:

Solve the following LPP graphically:

$$\text{Max} Z = 5X_1 + 12X_2$$

$$2X_1 + 3X_2 \leq 3000$$

$$X_1 + \frac{1}{2}X_2 \leq 1000$$

$$X_2 \leq 750$$

$$X_1, X_2 \geq 0$$

¹ Srinivasan, G., *Operations Research: Principles and Applications*, 2nd ed., PHI Learning Pvt. Ltd., New Delhi, 2010, p.22

Solution:

First, we need to draw the straight lines of the constraints. This means finding X_1 - intercept and X_2 - intercept.

For the constrain $2X_1 + 3X_2 \leq 3000$

Convert inequality into an equation: $2X_1 + 3X_2 = 3000$

X_1 - intercept (set $X_2 = 0$) is

$2X_1 + 0 = 3000$, then $X_1 = 1500$, the point (1500, 0)

X_2 - intercept (set $X_1 = 0$) is

$0 + 3X_2 = 3000$, then $X_2 = 1000$, the point is (0, 1000)

$$2X_1 + 3X_2 = 3000 \rightarrow (0,1000)(1500,0)$$

For the constrain $X_1 + \frac{1}{2}X_2 \leq 1000$

Convert inequality into an equation: $X_1 + \frac{1}{2}X_2 = 1000$

X_1 - intercept (set $X_2 = 0$) is

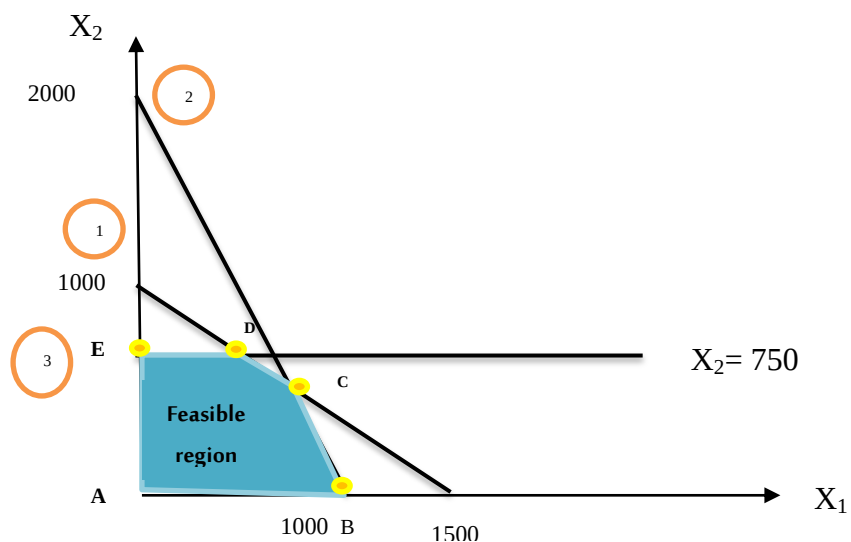
$X_1 + 0 = 1000$, then $X_1 = 1000$, the point (1000, 0)

X_2 - intercept (set $X_1 = 0$) is

$0 + \frac{1}{2}X_2 = 1000$, then $X_2 = 2000$, the point is (0, 2000)

$$X_1 + \frac{1}{2}X_2 = 1000 \rightarrow (0,2000)(1000,0)$$

For the constrain $X_2 = 750$



Any point inside or on the boundary of the feasible region is considered to satisfy all constraints and is therefore considered a possible solution.

In a next step, finding optimal solution among the possible solutions that lie on the corners of the possible solutions area, and this is done in one of two ways:

3-2-1-Corner-Point Evaluation Method (Evaluating the objective function at the corners of the feasible solution region):

Using this method, the values are compensated for X_j at each corner of the region of possible solutions in the objective function and choose the corner that achieves the best results (the largest value in the case of maximization and the smallest value in the case of minimization).

From the previous example, we have four corners of the region of possible solutions at the points:

at the point A:

$$A(0,0) \Rightarrow Z = 5(0) + 12(0) = 0$$

at the point B:

$$B(1000,0) \Rightarrow Z = 5(1000) + 12(0) = 5000$$

at the point C:

Point C is the point of intersection of the first and second constraints. To determine the coordinates of this point, we solve the following system of equations:

$$\begin{cases} 2X_1 + 3X_2 = 3000 & \dots(1) \\ X_1 + \frac{1}{2}X_2 = 1000 & \dots(2) \end{cases}$$

Multiply equation (2) by -2 and add it to equation (1) to obtain $X_2=500$. Substituting this value into either equation gives $X_1=750$. Therefore, point C (750, 500)

$$C(750,500) \Rightarrow Z = 5(750) + 12(500) = 9750$$

at the pointD:

Point D is the point of intersection of the first and third constraints. To determine the coordinates of this point, we solve the following system of equations:

$$\begin{cases} 2X_1 + 3X_2 = 3000 & \dots(1) \\ X_2 = 750 & \dots(3) \end{cases}$$

$$D(375,750) \Rightarrow Z = 5(375) + 12(750) = 10875$$

at the point E:

$$E(0,750) \Rightarrow Z = 5(0) + 12(750) = 9000$$

The maximum value of the objective function occurs **at** point D (375, 750). Therefore, the optimal production plan for the organization is to produce 375 chairs and 750 tables in the next month. This production plan yields a total profit of \$10875, which is the highest possible profit achievable under the given constraints.

3-2-2- Objective Function Line Method:

This method is based on the idea that the optimal solution can be determined by trying different values of the objective function and searching for the point that achieves the largest value, provided that this point is within the feasible region.

You represent the objective function graphically, then draw parallel lines to it, with the same slope, and the last corner of the feasible region through which a line parallel to the objective function passes is the optimal solution; this is for problems in which the objective function is in the form of Max. As for the problems in which the objective function is in the form Min. The first corner of the region of possible solutions through which a line parallel to the objective function passes is the optimal solution.

We start by assigning an arbitrary (default) value to the objective function. Let this value be:

$$Z = 5X_1 + 12X_2$$

$$Z = 0 \Rightarrow 5X_1 + 12X_2 = 0$$

$$\Rightarrow 5X_1 = -12X_2$$

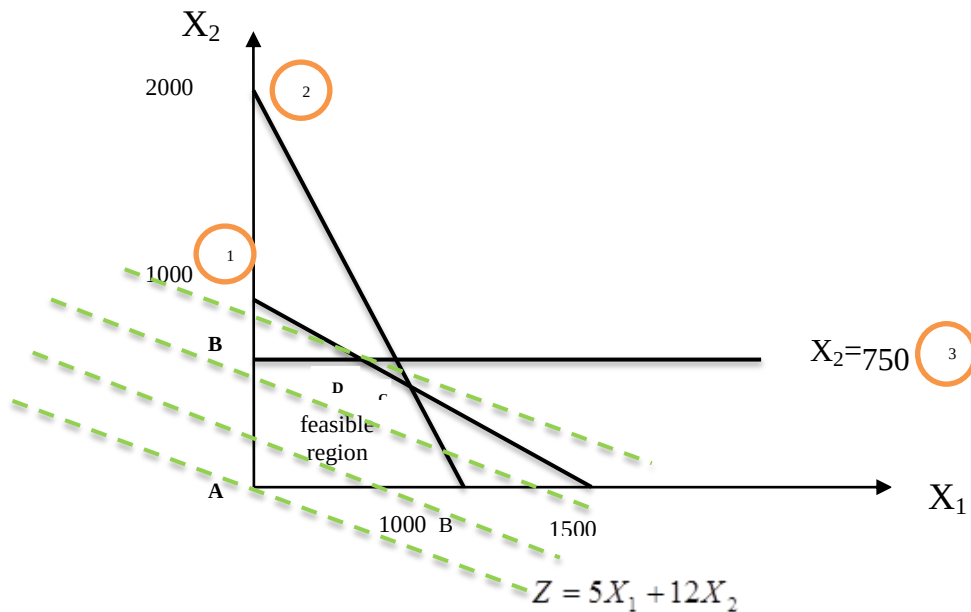
$$X_1 = 0 \Rightarrow X_2 = 0$$

$$(0,0)$$

$$X_2 = 500 \Rightarrow X_1 = \frac{-6000}{5} = -1200$$

$$(-1200,500)$$

To find the optimal solution (maximum profit according to the example) we must move the profit line to the right (parallel) to the furthest point touching the region of possible solutions.



From the figure it appears that the last corner of the region of possible solutions through which a line parallel to the objective function passes is located at point D

3-3- Special cases in the Graphical Method:

There are special cases that we may encounter when solving any problem using quantitative methods

3-3-1 Multiple Optimal solution: In a linear programming problem, a multiple optimal solution (also called an alternative optimal solution) occurs when the problem has more than one optimal solution. This situation arises when the objective function is parallel to a binding (non-redundant) constraint.

A critical (binding) constraint is a constraint that is satisfied as an equation at the optimal solution. When the objective function is parallel to such a constraint, it can attain the same optimal value at more than one point along the feasible region. As a result, several solution points yield the same optimal value of the objective function, leading to alternative optimal solutions.

Example 2:

$$\text{Max } Z = 2X_1 + 4X_2$$

$$X_1 + 2X_2 \leq 30$$

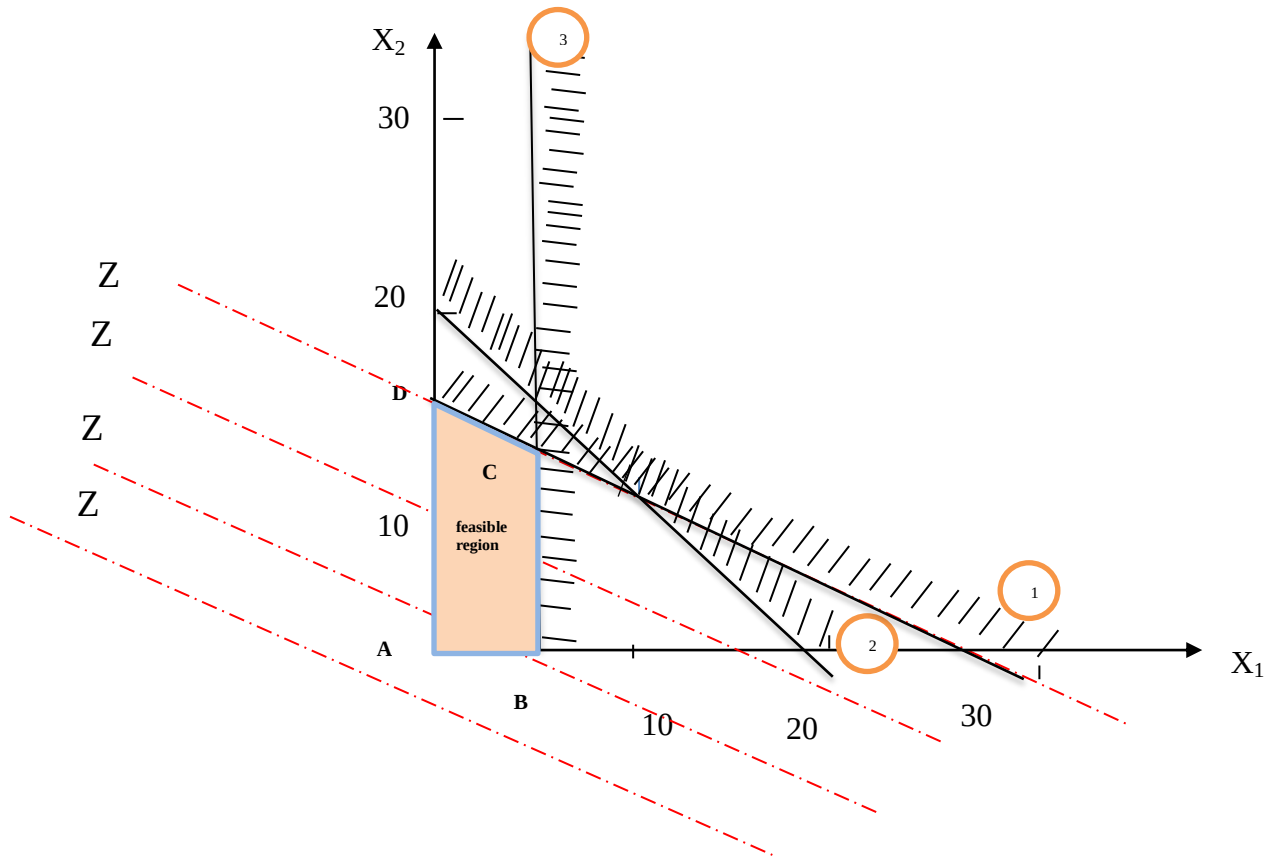
$$X_1 + X_2 \leq 20$$

$$X_1 \leq 5$$

$$X_1, X_2 \geq 0$$



$$\begin{array}{rcl} X_1 + 2X_2 & = & 30 \quad (0,15) \quad (30,0) \\ X_1 + X_2 & = & 20 \quad (0,20) \quad (20,0) \\ X_1 & = & 5 \end{array}$$



From the figure, we observe that the feasible solution region is represented by the trapezoid ABCD. We also observe that the slope of the constraint line (1) is equal to the slope of the objective function. Therefore, if the objective function is drawn, we will find that one of the parallel lines matches the constraint line (1). Therefore, the two points C and D represent an optimal solution. Moreover, all the points on the line CD give us alternative optimal solutions, as the values of X_1 and X_2 differ from one point to another on the straight line segment CD, while the value of the objective function Z remains constant. Therefore, we say that the problem has alternative optimal solutions. To confirm this, we calculate the coordinates of the two points C and D and substitute them into the objective function.

$$D(0,15) \Rightarrow Z = 2(0) + 4(15) = 60$$

$$C(5,12.5) \Rightarrow Z = 2(5) + 4(12.5) = 60$$

To make sure that the other point B is not an optimal solution, we substitute its coordinates into the objective function.

$$B(5, 0) \Rightarrow Z = 2(5) + 4(0) = 10$$

From the above discussion, and as illustrated in the example, we conclude that if the optimal solution is attained at two corner points of the feasible region, then there exists an infinite number of optimal solutions lying on the line segment connecting these two points.

3-3-2- Unbounded solution: Sometimes an LP problem may have an infinite solution. Such a solution is referred as an *unbounded solution*. It happens when value of certain decision variables and the value of the objective function (maximization case) are permitted to increase infinitely, without violating the feasibility condition.¹

It may be noted that there is a difference between unbounded *feasible region* and *unbounded solution* to a LP problem. It is possible that for a particular LP problem the feasible region may be unbounded but LP problem solution may not be unbounded, i.e. an unbounded feasible region may yield some definite value of the objective function. In general, an unbounded LP problem solution exists due to improper formulation of the real-life problem.²

Example 3:

$$\text{Min } Z = 10 X_1 + 25 X_2$$

$$3X_1 + 5 X_2 \geq 75$$

$$X_2 \leq 12$$

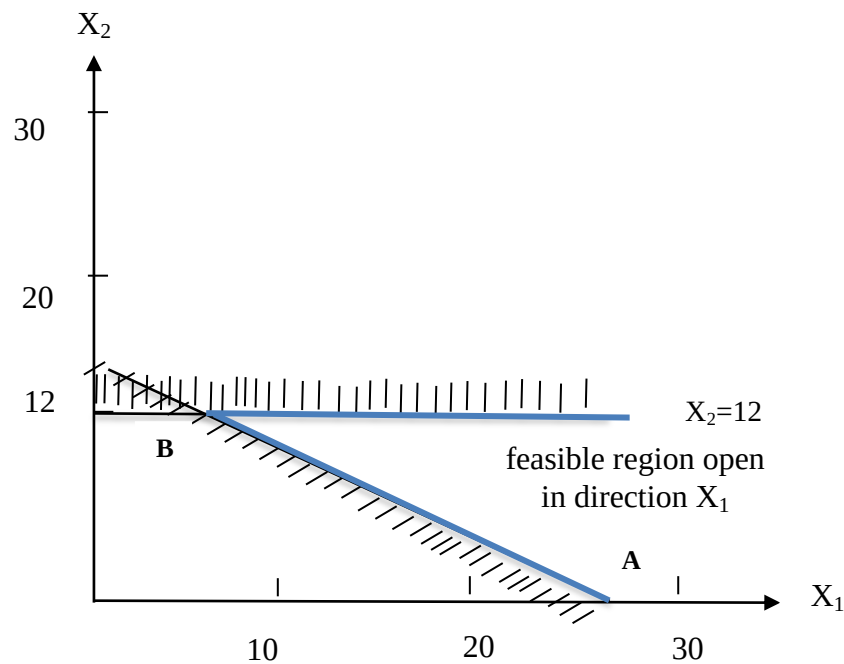
$$X_1, X_2 \geq 0$$



$$\begin{array}{l} 3X_1 + 5X_2 = 75 \quad (0,15) \quad (25,0) \\ X_2 = 12 \end{array}$$

¹ Sharma, *op. cit.*, p.88

² Ibid, p.88



From the figure above, we notice that the feasible region is unlimited and has only two corners, which are : A and B To determine the optimal solution point, we substitute the coordinates of these two points into the objective function.

$$B(5,12) \Rightarrow Z = 10(5) + 25(12) = 350$$

$$A(25,0) \Rightarrow Z = 10(25) + 25(0) = 250$$

Since the objective function is of the form Min and since point A gives us the lowest possible value of this function, the optimal solution is:

$$X_1 = 25 \text{ and } X_2 = 0$$

$$Z = 250$$

Remark: If the objective function is of the form Max It is not possible to obtain the optimal solution because any point that can be determined within this region can be obtained that gives us a greater value for the objective function. Therefore, we say that the program is unlimited. There is an error in the formulation of the issue.

3-3-3- Infeasible/ No solution: An infeasible solution to an LP problem arises when there is no solution that satisfies all the constraints simultaneously. This happens when there is no unique (single) feasible region. This situation arises when a LP

model that has conflicting constraints. Any point lying outside the feasible region violates one or more of the given constraints.¹

Example 4:

$$\text{Max } Z = 20X_1 + 15X_2$$

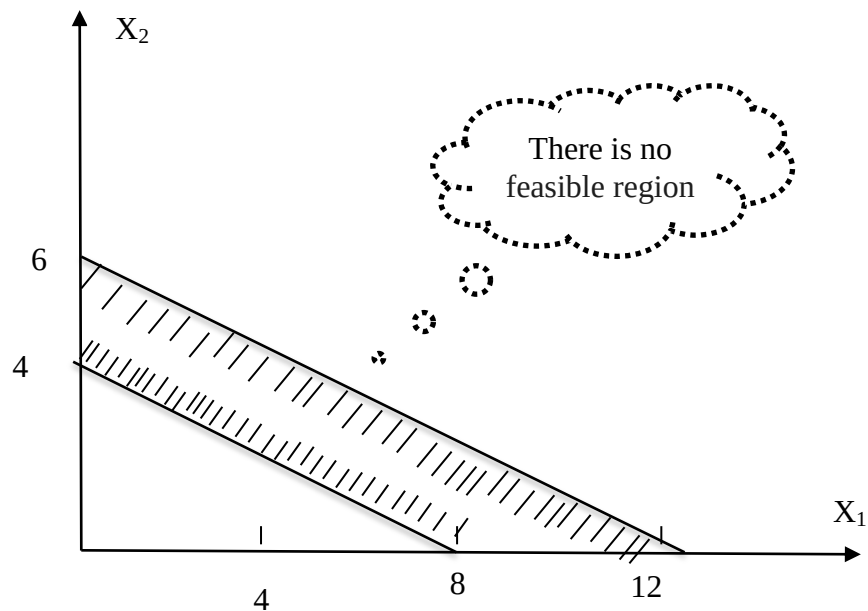
$$5X_1 + 10X_2 \leq 40$$

$$4X_1 + 8X_2 \geq 48$$

$$X_1, X_2 \geq 0$$



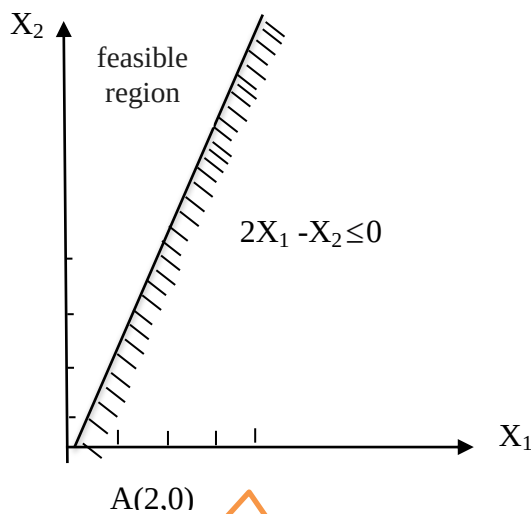
$$\begin{aligned} 5X_1 + 10X_2 &= 40 \quad (0,4) \quad (8,0) \\ 4X_1 + 8X_2 &= 48 \quad (0,6) \quad (12,0) \end{aligned}$$



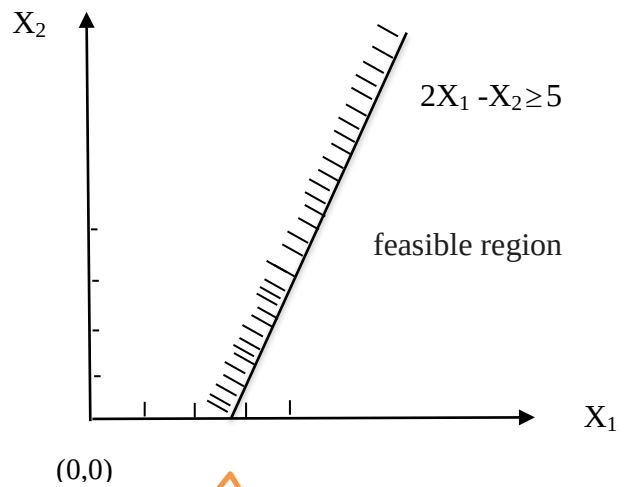
Remark: If it is not possible to determine the direction of the constraint directly, the origin point (0 0) can be tested or any point from the two spaces separated by the constraint can be chosen and whether this point satisfies the constraint or not

¹ Sharma, op. cit, p.90

Example 5:



If we test, for example, point A (2, 0) and find that it does not satisfy the constraint, then the region on the right-hand side of the constraint line represents the infeasible (rejected) region



If we test the origin point (0, 0) and find that it does not satisfy the constraint, then the left-hand side of the constraint line represents the rejected (infeasible) region region.

3-4- Advantages and Limitations of the Graphical Method

a) Advantages of the Graphical Method

- **Visual Insight:** It gives a clear visual representation of the feasible region, helping you understand the relationship between constraints and objective.
- **Simple for Two Variables:** The method is straightforward and easy to apply for problems with only two variables.
- **Intuitive:** It is an excellent tool for teaching and understanding the basics of linear programming.

b) Limitations of the Graphical Method

- **Limited to Two Variables:** The graphical method is only feasible for problems with two decision variables. For larger problems (three or more variables), other methods such as the **Simplex Method** are needed.
- **Less Practical for Large Problems:** As the number of constraints or decision variables increases, the graphical method becomes cumbersome and impractical.

3-5- Questions and Exercises

Part 1: Theoretical Questions

A. Short Questions

1. Define the graphical method for solving linear programming problems. When can it be used?
2. What is meant by the *feasible region*? How is it determined?
3. Why does the optimal solution always lie at a corner point of the feasible region?
4. Differentiate between the **corner-point method** and the **trial-and-error (objective function line) method**.
5. State the main limitations of the graphical method compared with the simplex method.

B. Multiple Choice Questions

1. In graphical method of solving LPP, the feasible region is:

- a) Always a triangle
- b) A region that satisfies only the objective function
- c) The region that satisfies all constraints including non-negativity
- d) A single point only

2. What is the first step in solving an LPP graphically?

- a) Maximizing the objective function
- b) Drawing the coordinate axes
- c) Plotting the constraints as lines
- d) Guessing the optimal solution

3. The corner points of the feasible region are used to:

- a) Estimate profit
- b) Solve for maximum or minimum values of the objective function
- c) Plot the objective function
- d) Eliminate unbounded solutions

4. In a graphical solution, if the feasible region is unbounded, the LPP may have:

- a) No feasible solution
- b) Infinite feasible solutions but no optimal solution
- c) An optimal solution only if the objective function is bounded
- d) unique solution only

C. True or False

1. The graphical method of solving LP problems can only be used when the problem has two decision variables. **(T/F)**
2. The feasible region represents all points that satisfy the problem's constraints. **(T/F)**
3. If a point satisfies all constraints, it lies inside or on the boundary of the feasible region. **(T/F)**
4. If the feasible region is empty, the LP problem has no feasible solution. **(T/F)**
5. Every LP problem must have a corner point. **(T/F)**
6. The feasible region for a linear programming problem is always convex. **(T/F)**
7. The objective function helps determine which direction to move along the feasible region to find the optimal point. **(T/F)**
8. The value of the objective function is constant along any line parallel to it. **(T/F)**
9. In maximization problems, the optimal solution is always at the farthest corner point from the origin. **(T/F)**
10. Both minimization and maximization LP problems can be solved using the graphical method if there are only two variables. **(T/F)**
11. The optimal solution of a graphical LP must always lie at a corner (extreme) point of the feasible region. **(T/F)**
12. A linear programming problem may have more than one optimal solution. **(T/F)**
13. If the objective function is parallel to a boundary of the feasible region, the LP problem may have infinitely many optimal solutions. **(T/F)**
14. If the LP problem is unbounded, a maximum value always exists. **(T/F)**
15. If the feasible region is unbounded, the LP problem may still have an optimal solution. **(T/F)**
16. If all constraint lines intersect at a single point, the LP problem cannot have alternative optimal solutions. **(T/F)**

Part 2: Practice Exercises

Exercise 1 – Maximization Problem

$$\begin{aligned}\text{Max } Z &= 3X_1 + 5X_2 \\ \text{S/to: } X_1 + 2X_2 &\leq 8 \\ 3X_1 + 2X_2 &\leq 12 \\ X_1, X_2 &\geq 0\end{aligned}$$

Tasks:

1. Plot the constraints graphically.
2. Identify the feasible region.
3. Calculate the value of the objective function at each corner point.
4. Find the optimal solution and the maximum value of Z .

Exercise 2 – Minimization Problem

$$\begin{aligned}\text{Min } Z &= 4X_1 + 6X_2 \\ \text{S/to: } 3X_1 + 2X_2 &\geq 12 \\ X_1 + 4X_2 &\geq 8 \\ X_1, X_2 &\geq 0\end{aligned}$$

Tasks:

1. Plot the constraints.
2. Determine the feasible region.
3. Using the graphical method (objective function line approach), find the optimal solution and the minimum value of Z .

Exercise 3 – Maximization Problem

$$\begin{aligned}\text{Max } Z &= 5X_1 + 6X_2 \\ \text{S/to: } X_1 + 2X_2 &\leq 30 \\ X_1 + 1/2 X_2 &\leq 10 \\ X_2 &= 5 \\ X_1, &\geq 0\end{aligned}$$

Tasks:

1. Plot the constraints.
2. Determine the feasible region.
3. Using the graphical method (objective function line approach), find the optimal solution and the maximum value of Z .

Exercise 4 – Real-Life Application

A company that works in the leather industry produces two types of school bags, A and B. The management wants to decide the best production plan for the next month. Each unit of bag A needs 2 square meters of leather, and each unit of bag B needs 3 square meters. Both products go through two departments: tailoring and assembly. The table below shows the time required for each product in each department.

Sections	The product		Available working hours (per month)
	A	B	
detail	1 hour	30 min	1500hour
Collection	1 hour	2 hours	2000hour

The company has 6,000 square meters of leather available for the next month and must produce at least 600 units of type A bags to meet its commercial obligations to customers.

- Find the best production plan if the net profit for each unit of type A and type B bags is \$150 and \$120, respectively.

Exercise 5– Real-Life Application:

A pharmacist wants to prepare a medical ointment for skin moisturizing using a mixture of two basic ingredients: M1 and M2. The ointment must contain at least 200 g of fatty elements, 150 g of antibiotics, and 90 g of anti-inflammatory agents.

Each kilogram of M1 provides 10 g of fatty elements, 10 g of antibiotics, and 3 g of anti-inflammatory agents.

Each kilogram of M2 provides 10 g of fatty elements, 5 g of antibiotics, and 9 g of anti-inflammatory agents.

The cost of purchasing M1 and M2 is \$200 and \$300 per kilogram, respectively.

- Determine the optimal mixture of M1 and M2 that minimizes the cost of preparing the medical ointment.

1-6 Solutions & Answers

Part 1: Theoretical Questions

A. Short Questions

1. Define the graphical method for solving linear programming problems. When can it be used?

The graphical method is a visual approach used to solve linear programming problems with two decision variables. It involves plotting all the constraints on a graph to identify the feasible region, then determining the values of the objective function (such as profit or cost) at each corner point of that region to find the optimal solution (maximum or minimum). It can be used only when there are two variables, because a problem with more than two variables cannot be represented accurately on a two-dimensional graph.

2. What is meant by the feasible region? How is it determined?

The feasible region is the set of all possible points that satisfy all the given constraints of a linear programming problem, including non-negativity conditions. It is determined by:

- Graphing each constraint as a straight line.
- Identifying the side of each line that satisfies the inequality.
- The common overlapping area of all these sides represents the feasible region.

3. Why does the optimal solution always lie at a corner point of the feasible region?

The optimal solution of a linear programming problem always occurs at a corner (vertex) point because:

- The objective function and constraints are linear.
- The maximum or minimum value of a linear function over a convex polygonal region occurs at one of its vertices.

Thus, evaluating the objective function at each corner point guarantees finding the best possible solution.

4. Differentiate between the corner-point method and the trial-and-error (objective function line) method.

Aspect	Corner-Point Method	Trial-and-Error (Objective Function Line) Method
Approach	Compute the coordinates of each corner point of the feasible region and evaluate the objective function at each.	Draw several lines representing different values of the objective function and move them parallelly until the last point still touching the feasible region is found.
Accuracy	Exact, because all corner points are tested.	Approximate, since it relies on visual estimation.
Ease of Use	More systematic and analytical.	More visual and intuitive.
When Used	Preferred for precise solutions.	Used mainly for graphical understanding or teaching.

5. State the main limitations of the graphical method compared with the simplex method.

The graphical method has the following limitations:

- It is limited to two-variable problems only (cannot handle higher dimensions).
- It becomes impractical and inaccurate when constraints or data are numerous.
- It provides visual or approximate solutions, not always precise for complex systems.

- It cannot easily handle integer, non-linear, or large-scale optimization problems.
- The simplex method, on the other hand, can solve problems with many variables and constraints systematically using algebraic computations.

B. Multiple Choice Questions

1. Answer: (c). The region that satisfies all constraints including non-negativity
2. Answer: (c). Plotting the constraints as lines
3. Answer: (b). Solve for maximum or minimum values of the objective function
4. Answer: (c). An optimal solution only if the objective function is bounded

C. True or False

1. True
2. True
3. True
4. True
5. False
6. True
7. True
8. True
9. False
10. True
11. True
12. True
13. True
14. False
15. True
16. True

Part 2: Practice Exercises

Exercise 1:

$$\text{Max } Z = 3X_1 + 5X_2$$

$$\text{S/to: } X_1 + 2X_2 \leq 8$$

$$3X_1 + 2X_2 \leq 12$$

$$X_1, X_2 \geq 0$$

1- Graphical Plot of the Constraints

Convert each constraint into an equation:

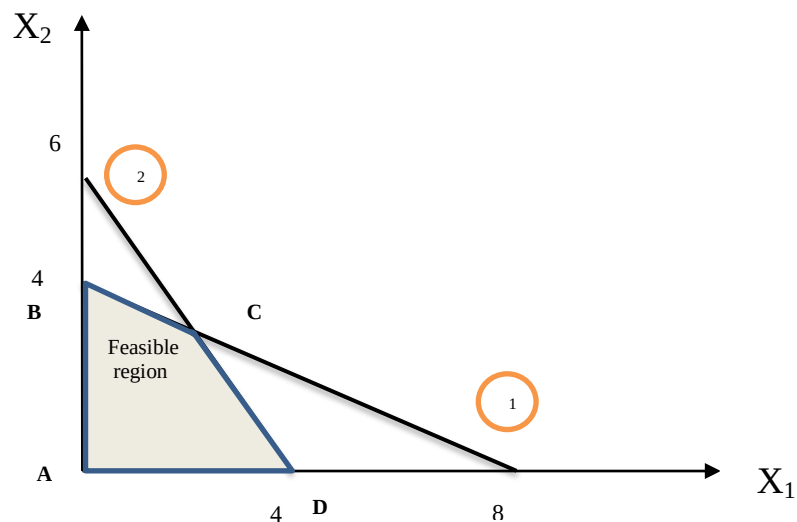
- $X_1 + 2X_2 = 8$
 - Intercepts:
 - $X_1 = 8$ when $X_2 = 0 \rightarrow (8, 0)$
 - $X_2 = 4$ when $X_1 = 0 \rightarrow (0, 4)$
- $3X_1 + 2X_2 = 12$
 - Intercepts:
 - $X_1 = 4$ when $X_2 = 0 \rightarrow (4, 0)$
 - $X_2 = 6$ when $X_1 = 0 \rightarrow (0, 6)$

The feasible region lies below both lines and in the first quadrant.

2- Feasible Region

The feasible region is the shaded area bounded by:

- The two constraint lines
- The X_1 -axis
- The X_2 -axis



3 - Corner Points and Objective Function Values

The corner (extreme) points of the feasible region are:

Corner Point	$Z = 3X_1 + 5X_2$
A(0, 0)	$3(0) + 5(0) = 0$
B(0, 4)	$3(0) + 5(4) = 20$
C(2, 3)	$3(2) + 5(3) = 21$
D(4, 0)	$3(4) + 5(0) = 12$

4- Optimal Solution

- Optimal point: $(X_1, X_2) = (2, 3)$
- Maximum value of the objective function: $Z_{\max} = 21$

Exercise 2:

$$\text{Min } Z = 4X_1 + 6X_2$$

$$\text{S/ to: } 3X_1 + 2X_2 \geq 12$$

$$X_1 + 4X_2 \geq 8$$

$$X_1, X_2 \geq 0$$

1- Graphical Plot of the Constraints

Convert each inequality into an equality:

Constraint 1:

$$\triangleright 3X_1 + 2X_2 = 12$$

Intercepts:

- **Constraint 1:**
- $X_1 = 4$ when $X_2 = 0 \rightarrow (4, 0)$
- $X_2 = 6$ when $X_1 = 0 \rightarrow (0, 6)$
- **Constraint 2:**

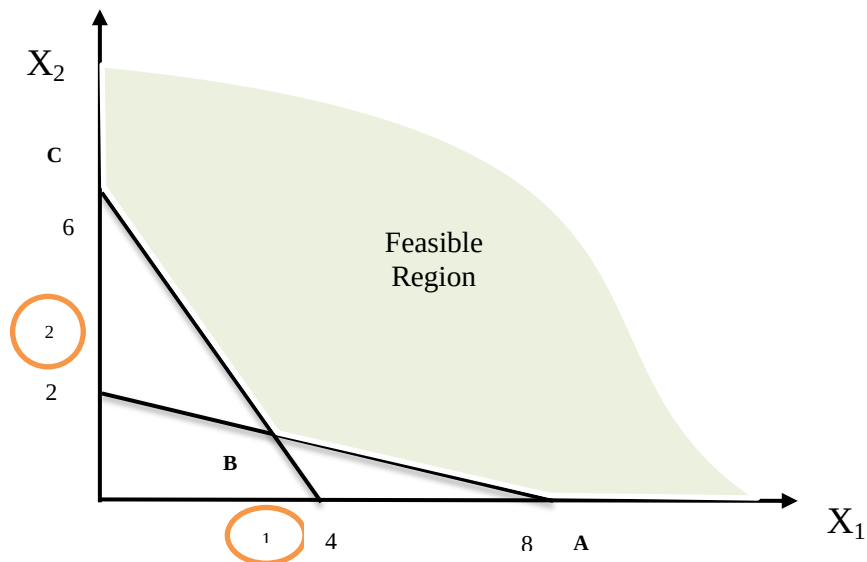
$$\triangleright X_1 + 4X_2 = 8$$

Intercepts:

- $X_1 = 8$ when $X_2 = 0 \rightarrow (8, 0)$
- $X_2 = 2$ when $X_1 = 0 \rightarrow (0, 2)$

2- Feasible Region

Since both constraints are $\geq$, the feasible region lies **above** each line.



3- Corner Points and Objective Function Values

For minimization problems, the optimal solution occurs at the corner point closest to the origin.

➤ Corner Points:

1. Intersection of the two constraints:

$$\begin{cases} 3X_1 + 2X_2 = 12 \\ X_1 + 4X_2 = 8 \end{cases}$$

Solving:

- $X_2=1.2$
- $X_1=3.2$

1. Point: **(3.2 , 1.2)**

2. Intersection with axes:

- (8, 0) satisfies both constraints
- (0, 6) satisfies both constraints

➤ Objective Function Values

Corner Point	$Z = 4X_1 + 6X_2$
A (8, 0)	$4(8)+6(0)=32$
B(3.2, 1.2)	$4(3.2)+6(1.2)=\mathbf{20}$
C (0, 6)	$4(0)+6(6)=36$

4- Optimal Solution :

- Optimal point: $(X_1, X_2) = (3.2, 1.2)$
- Minimum value of Z: $Z_{\min} = 20$

Exercise 3:

1- Graphical Plot of the Constraints

Max $Z = 5X_1 + 6X_2$

$X_1 + 2X_2 \leq 30$

$X_1 + 1/2 X_2 \leq 10$

$X_2 = 5$

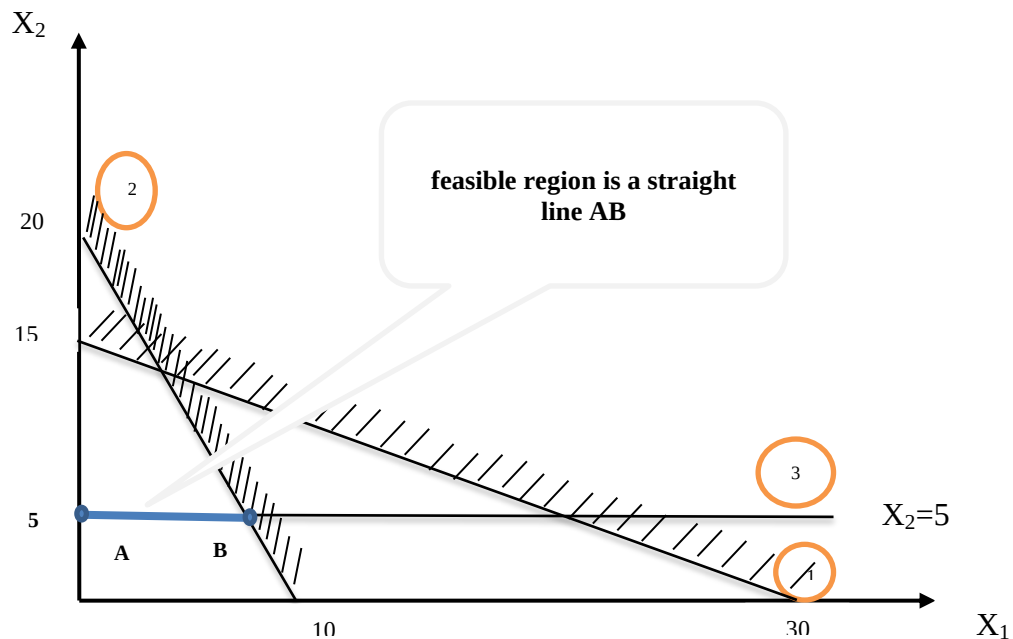
$X_1 \geq 0$

➔

$X_1 + 2X_2 = 30 \rightarrow (0, 15) (30, 0)$

$X_1 + 1/2 X_2 = 10 \rightarrow (0, 20) (10, 0)$

$X_2 = 5$



2- Feasible Region

feasible region is a straight line AB

3- Corner Points and Objective Function Values

At the point A:

$$B(0, 5) \Rightarrow Z = 5(0) + 6(5) = 30$$

At the pointB:

The point B it is the point of intersection of the second and third constraints. To find the coordinates of this point, we solve the set of equations:

$$\begin{cases} X_1 + \frac{1}{2}X_2 = 10 \\ X_2 = 5 \end{cases}$$

$$B(7.5,5) \Rightarrow Z = 5(7.5) + 6(5) = 67.5$$

4- Optimal Solution :

- Optimal point: $(X_1, X_2) = (7.5, 5)$
- Maximum value of the objective function: $Z_{\max} = 67.5$

Exercise 4:

Mathematical formulation of the problem:

Definition of the variables decision:

X_1 : Quantity of type A school bags produced for the next month.

X_2 : Quantity of type B school bags produced for the next month.

$$\text{Max } Z = 150X_1 + 120X_2$$

$$X_1 + \frac{1}{2}X_2 \leq 1500$$

$$X_1 + 2X_2 \leq 2000$$

$$2X_1 + 3X_2 \leq 6000$$

$$X_1 \geq 600$$

$$X_2 \geq 0$$

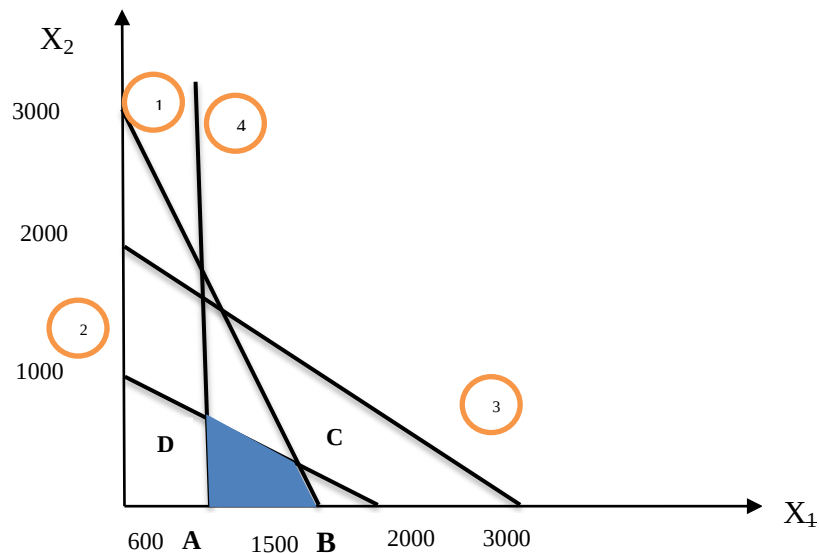
Solution by graphical method:

$$X_1 + \frac{1}{2}X_2 = 1500 \rightarrow (0, 3000)(1500, 0)$$

$$X_1 + 2X_2 = 2000 \rightarrow (0, 1000)(2000, 0)$$

$$2X_1 + 3X_2 = 6000 \rightarrow (0, 2000)(3000, 0)$$

$$X_1 = 600$$



A- Substitution method (evaluating the objective function at the corners of the feasible region): Using this method, the values of X_j are substituted at each corner of the possible solutions region in the objective function and the angle that achieves the best results is chosen (the largest value in the case of maximization, and the smallest value in the case of minimization).

The point A:

$$A(600,0) \Rightarrow Z = 150(600) = \$90000$$

The point B:

$$B(1500,0) \Rightarrow Z = 150(1500) = \$225000$$

The point C:

Point C is the point of intersection of the first and second constraints. To find the coordinates of this point, we solve the equations:

$$\begin{cases} X_1 + 1/2X_2 = 1500 & \dots(1) \\ X_1 + 2X_2 = 2000 & \dots(2) \end{cases}$$

$$\left(\frac{4000}{3}, \frac{1000}{3}\right) \Rightarrow Z = 150\left(\frac{4000}{3}\right) + 120\left(\frac{1000}{3}\right) = \$240000$$

The point D:

Point D is the point of intersection of the second and fourth constraints. To find the coordinates of this point, we solve the two equations:

$$\begin{cases} X_1 + 2X_2 = 2000 & \dots(2) \\ X_1 = 600 & \dots(4) \end{cases}$$

$$D(600,700) \Rightarrow Z = 150(600) + 120(700) = \$174000$$

The maximum value of the objective function is at point C, so the best production program is to produce 1333.33 units of type A bags and 333.33 units of type B bags to achieve a total profit of \$240000, which is the maximum possible profit given the constraints of the problem.

Exercise 5:

Mathematical formulation of the problem:

Definition of the variables decision:

X_1 : Quantity purchased of the material (M1) in kg.

X_2 : Quantity purchased of the material (M2) in kg.

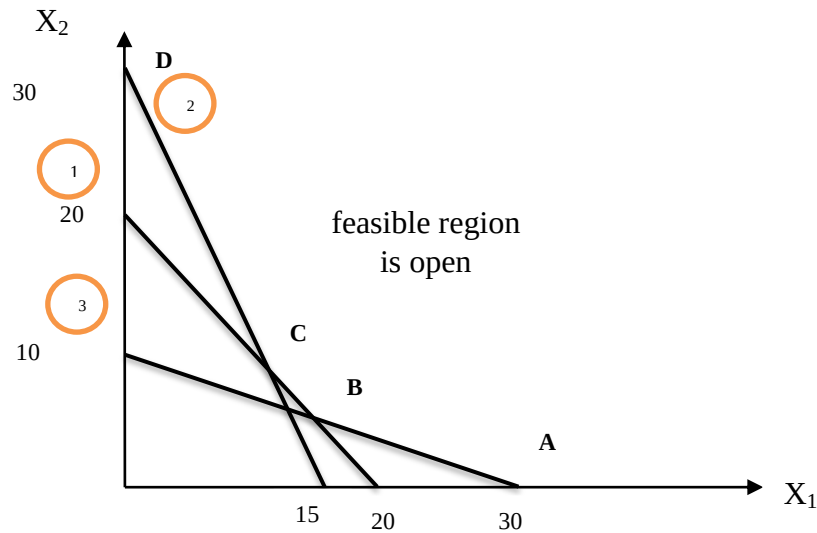
$$\begin{aligned} \text{Min } Z &= 200X_1 + 300X_2 \\ 10X_1 + 10X_2 &\geq 200 \\ 10X_1 + 5X_2 &\geq 150 \\ 3X_1 + 9X_2 &\geq 90 \\ X_1, X_2 &\geq 0 \end{aligned}$$

Solution by graphical method:

$$10X_1 + 10X_2 = 200 \rightarrow (0,20)(20,0)$$

$$10X_1 + 5X_2 = 150 \rightarrow (0,30)(15,0)$$

$$3X_1 + 9X_2 = 90 \rightarrow (0,10)(30,0)$$



The point A:

$$A(30,0) \Rightarrow Z = 30(200) = \$6000$$

The point B:

Point B is the point of intersection of the first and third constraints. To find the coordinates of this point, we solve the two equations:

$$\begin{cases} 10X_1 + 10X_2 = 200 & \dots(1) \\ 3X_1 + 9X_2 = 90 & \dots(3) \end{cases}$$

$$B(15,5) \Rightarrow Z = 200(15) + 300(5) = \$4500$$

The point C:

Point C is the point of intersection of the first and second constraints. To find the coordinates of this point, we solve the equations:

$$\begin{cases} 10X_1 + 10X_2 = 200 & \dots(1) \\ 10X_1 + 5X_2 = 150 & \dots(2) \end{cases}$$

$$C(10,10) \Rightarrow Z = 200(10) + 300(10) = \$5000$$

The point D:

$$D(0,30) \Rightarrow Z = 30(300) = \$9000$$

The minimum value of the objective function is at point B, so the best mixture to buy and use in the ointment of the two substances is 15 kg of substance M1 and 5 kg of substance M2 to achieve the lowest possible cost of preparing the medical ointment, which is \$4500, within the imposed constraints.

Chapter 4: The Simplex Method to LP Problems

Learning Objectives

After completing this chapter, students will be able to:

1. Understand the importance of the general linear programming problem.
2. Explain the canonical and standard forms (of linear programming problems).
3. Define the simplex method and its underlying concept.
4. Apply the simplex algorithm step by step to solve linear programming problems.
5. Interpret the final simplex tableau to obtain and analyze the optimal solution.

Chapter contents

4-1 Introduction

4-2 Canonical and standard forms of lpp

4-3 Simplex solution stages

4-4 Simplex method by introducing artificial variables: (two phase method)

4-4- Practical method of solution

4-5 Special cases in simplex method

4-6 Questions and Exercises

4-7 Solutions & Answers

4-1- Introduction

Graphic method of LPP is limited to two variables. We have to look to other procedure which offers an efficient means of solving more complex LPP. Although the graphical method of solving LPP is an invaluable aid to understand the basic structure, the method is of limited application in industrial

problems as the number of variables occurring there, is substantially large. So another method known as Simplex method is suitable for solving LPP with a large number of variables. The method though an iterative process, progressively approaches and ultimately reaches to the maximum or minimum value of the objective function. The method also helps the decision maker to identify an unbounded solution, multiple solution and infeasible solution.¹

4-2 Canonical and Standard Forms of Lpp

The general LPP can be put in either canonical or standard forms. In the standard form, irrespective of the objective function, namely maximize or minimize, all the constraints are expressed as equations. Moreover, RHS of each constraint and all variables are non-negative.²

4-2-1 Characteristics of the Canonical Form:

In the canonical form, if the objective function is of maximization, all the constraints other than non-negativity conditions are ' $\leq$ ' type. If the objective function is of minimization, all the constraints other than non-negative conditions are ' $\geq$ ' type.

- ✓ The objective function is of maximization type.
- ✓ All constraints are of ' $\leq$ ' type.
- ✓ All variables X_i are non-negative.

¹ Vineethan, T., *MCM2C10: Management science* [Study material], School of Distance Education, University of Calicut, , Kerala, India, 2019. p.42

² Alagappa University, *Operations Research (113 52)*, Directorate of Distance Education, Vikas Publishing House Pvt. Ltd., Noida, India, 2020, p.60

4-2-2 Characteristics of the Standard Form

The standard form of the LPP should have the following characteristics;

- ✓ The objective function is of maximization type.
- ✓ All constraints are expressed as equations.
- ✓ Right hand side of each constraint is non-negative.
- ✓ All variables are non-negative.

⇒ If the constraints of a general LPP be,

$$\sum_{j=1}^n a_{ij} X_j \leq b_i \quad (i = 1, 2, 3 \dots m)$$

then the non-negative variables S_i , which are introduced to convert the inequalities ($\leq$) to the equalities $\sum_{j=1}^n a_{ij} X_j + S_i = b_i \quad (i = 1, 2, 3 \dots m)$ are called slack variables.

Slack variables are also defined as the non-negative variables which are added in the LHS of the constraint to convert the inequality ' $\leq$ ' into an equation.

⇒ If the constraints of a general LPP be,

$$\sum_{j=1}^n a_{ij} X_j \geq b_i \quad (i = 1, 2, 3 \dots m)$$

then the non-negative variables S_i , which are introduced to convert the inequalities ($\geq$) to the equalities $\sum_{j=1}^n a_{ij} X_j - S_i = b_i \quad (i = 1, 2, 3 \dots m)$ are called surplus variables.

Surplus variables are defined as the non-negative variables which are removed from the LHS of the constraint to convert the inequality ' $\geq$ ' into an equation.

By entering these variables, the general standard form of the LPP is expressed as follows:

$$\text{Max} Z = C_1 X_1 + C_2 X_2 + \dots + C_n X_n + 0S_1 + 0S_2 + \dots + 0S_m$$

Subject to:

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1j}X_j + \dots + a_{1n}X_n + S_1 = b_1$$

$$a_{21}X_1 + a_{22}X_2 + \dots + a_{2j}X_j + \dots + a_{2n}X_n + S_2 = b_2$$

.....

$$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{ij}X_j + \dots + a_{in}X_n + S_i = b_i$$

.....

$$a_{m1}X_1 + a_{m2}X_2 + \dots + a_{mi}X_j + \dots + a_{mn}X_n + S_m = b_m$$

$$X_1, X_2, \dots, X_n, S_1, S_2, \dots, S_m \geq 0$$

With this final version of the program, we now have a new type of variable, which is: S_i ($S_1, S_2, \dots, S_m$). These variables are the basis of the first table and are equal to the values of the second side of the restrictions, i.e. the values of b_i ($b_1, b_2, \dots, b_m$). Respectively because we start the solution from the origin (0, 0), i.e. at $X_1, X_2, \dots, X_n = 0$.

4-3 Simplex solution stages (maximization LP)

The Simplex Method is an iterative algorithm that gradually improves the value of the objective function by moving from one corner point (basic feasible solution) to an adjacent one, until no further improvement in the objective function is possible. After expressing the linear programming problem in standard form, the Simplex Method can be summarized in the following steps:

Step 1: Preliminary checks

- Determine whether the objective function is a maximization or minimization problem.
- If the problem is a minimization problem, convert it into a maximization problem using: $\text{Min } Z = \text{Max } (-Z)$
- Check that all right-hand-side constants b_i ($i = 1, 2, \dots, m$) are positive.
- If any one of b_i is negative, then multiply the inequation of the constraint by -1 so as to get all b_i to be positive.

Note: Multiplying both sides of an inequality by -1 reverses the direction of the inequality.

Step 2: Convert the problem into standard form

Express the problem in the standard form by introducing slack/surplus variables to convert the inequality constraints into equations.

By introducing the slack variables S_1 and S_2 , the problem in the previous example can be converted into standard form as follows:

$$\begin{aligned}\text{Max} Z &= 5X_1 + 12X_2 \\ \text{Subject to, } 2X_1 + 3X_2 &\leq 3000 \\ X_1 + \frac{1}{2}X_2 &\leq 1000 \\ X_2 &\leq 750 \\ X_1, X_2 &\geq 0\end{aligned}$$



$$\begin{aligned}\text{Max} Z &= 5X_1 + 12X_2 + S_1 + S_2 + S_3 \\ 2X_1 + 3X_2 + S_1 &= 3000 \\ X_1 + \frac{1}{2}X_2 + S_2 &= 1000 \\ X_2 + S_3 &= 750 \\ X_1, X_2, S_1, S_2, S_3 &\geq 0\end{aligned}$$

S_1, S_2, S_3 : slack variable represent energy (resources).opportunities) unexploited

Step 3: Set up the Initial Solution (Initial Simplex Tableau)

The **Simplex tableau** is a tabular form used to perform the necessary row operations in the Simplex Method and to test whether a given solution is optimal.

The tableau contains:

- The coefficients of the decision variables and slack variables from the constraint equations,
- The coefficients of the objective function.

In the tableau, the top row indicates the variables represented by each column. The next rows correspond to the coefficients of the constraint equations, while the last row represents the coefficients of the objective function.

This initial tableau represents the **initial basic feasible solution**, from which the Simplex algorithm proceeds iteratively toward the optimal solution.

In order to set up the initial solution, we must first identify the basic and non-basic variables.

- In the initial simplex tableau, the slack variables are chosen as the basic variables, while the decision variables are treated as non-basic variables.
- If a linear programming problem has n variables and m constraints, then exactly m variables must be selected as basic variables, and the remaining $(n - m)$ variables are non-basic variables whose values are set equal to zero.

Basic variables are those whose values are determined from the basic solution, whereas non-basic variables are variables that are assigned a value of zero in the basic solution

In the previous example, we set the two decision variables equal to zero ($X_1=0$, $X_2=0$) Then we calculate the value of the other three variables. Since we start from the origin, setting $X_1=0$, $X_2=0$ and we get: $S_1=3000$, $S_2=1000$, $S_3=750$

Associated with this initial tableau, the non-basic variables are X_1 and X_2 and the basic variables are S_1, S_2 and S_3 . Therefore, the initial (or current) basic feasible solution is: $(X_1, X_2, S_1, S_2, S_3) = (0, 0, 3000, 1000, 750)$.

The value of the objective function for this solution is zero, which appears in the rightmost entry of the bottom row of the simplex tableau.

After converting the linear programming problem into standard form, the initial simplex tableau, denoted by T1, is constructed as follows:

Basic Variable	identity matrix					Basic Variable Value (b_i)
	X_1	X_2	S_1	S_2	S_3	
S_1	2	3	1	0	0	3000
S_2	1	1/2	0	1	0	1000
S_3	0	1	0	0	1	750
Z	-5	-12	0	0	0	0

Remark: From the tableau above, we observe that the variables appearing in the basis and also reappearing outside the basis within the same tableau form an identity matrix. This matrix does not affect the calculations when moving from one tableau to another. Therefore, it can be omitted, and we may work with condensed (reduced) tableaus, in which basic variables do not appear outside the basis. This approach simplifies the computations and reduces the time required to perform the Simplex Method. The resulting tableau is as follows:

Basic Variable	X_1	X_2	Basic Variable Value (b_i)
S_1	2	3	3000
S_2	1	1/2	1000
S_3	0	1	750
Z	-5	-12	0

Step 4: Check Optimality: In a maximization linear programming problem, the optimal solution is the set of values assigned to the decision variables that yields the maximum value of the objective function Z. Graphically, the optimal solution occurs at one of the corner points of the feasible region.

In the Simplex tableau, optimality is checked by examining the values in the last row (Z-row). If all the coefficients in the Z-row are greater than or equal to zero, the current solution is optimal.

If any coefficient in the Z-row is negative, this indicates that the corresponding variable can enter the basis to improve the value of the objective function, and therefore the current solution is not optimal. In such a case, the next step is to identify the pivot element by determining both the entering variable (the variable that will enter the basis) and the leaving variable (the variable that will leave the basis).

Step 5: Select the entering variable: The entering variable is determined by selecting the variable (which is initially a non-basic variable) that has the most negative coefficient in the objective function row (Z-row) of the simplex tableau. The column corresponding to this variable is then identified as the **pivot column**.

Since the problem is a maximization problem, the value of the objective function Z can be increased by allowing the variable with the most negative coefficient in the Z-row to enter the basis. In this case, the variable X_2 is selected as the entering variable because it provides the largest potential increase in the value of Z. If more than one variable has the same most negative coefficient in the Z-row, any one of them may be chosen arbitrarily as the entering variable.

Step 6: Identification of the Leaving Variable: The next step is to determine the leaving variable, that is, the basic variable which will be removed from the basis and become a non-basic variable.

To identify the leaving variable, we focus on the pivot column, which corresponds to the entering variable X_2 . Negative values and zeros in this column are ignored. The values in the right-hand side (RHS) column are then divided by the corresponding positive coefficients in the pivot column.

The leaving variable is the basic variable associated with the smallest positive ratio,

given by: $= \text{Min} \left\{ \frac{3000}{3}, \frac{1000}{1/2}, \frac{750}{1} \right\}$, which corresponds to S_3

Therefore, the leaving variable is the basic variable S_3 , and the corresponding row is called the **pivot row**.

Step 7: Selection of the Pivot Element: The pivot element is the entry located at the intersection of the pivot row and the pivot column. This element must be positive and non-zero, since it is used to perform the row operations required to obtain the next simplex tableau.

Basic Variable	X ₁	X ₂	Basic Variable Value (RHS)	Ratio
S ₁	2	3	3000	3000 / 3 = 1000
S ₂	1	1/2	1000	1000 / 1/2 = 2000
S ₃	0	1	750	750 / 1 = 750
Z	-5	-12	0	

↑

←

Step 8: Construction of the New Basic Solution (Next Simplex Tableau): After selecting the pivot element, the new basic solution is obtained by constructing the next simplex tableau as follows:

1. Replace the pivot element with its reciprocal.
2. Obtain the remaining elements in the pivot row by dividing each of the old elements in that row by the pivot element.
3. Obtain the remaining elements in the pivot column by dividing each of the old elements by the negative of the pivot element.
4. Calculate all other elements in the tableau using the following formula:

$$\text{New element} = \text{old element} - \left[\frac{\begin{array}{c} \text{Element corresponding} \\ \text{to it in the pivot line} \end{array} \times \begin{array}{c} \text{Element corresponding} \\ \text{to it in the pivot column} \end{array}}{\text{pivot element}} \right]$$

Basic Variable	X_1	S_3	Basic Variable Value (RHS)	Ratio
S_1	2	-3	750	$750 / 2 = 375$
S_2	1	-1/2	625	$625 / 1 = 625$
X_2	0	1	750	N
Z	-5	12	9000	

Step 9: Perform optimality test for second feasible solution: To test the optimality of the second feasible solution, we examine the Z-row of the simplex tableau:

- If there exists any negative coefficient in the Z-row, the current solution is not optimal, and we return to Step 5 to select a new entering variable.
- If all coefficients in the Z-row are non-negative, the current solution is optimal.

By examining the last table, we find that by applying the same conditions, we choose X_1 to be the entering variable and S_1 to be the leaving variable, which would result in a further improvement in the objective function.

Basic Variable	S_1	S_3	Basic Variable Value (RHS)	Ratio
X_1	1/2	-3/2	375	
S_2	-1/2	1	250	
X_2	0	1	750	
Z	5/2	9/2	10875	

Since all the values in the Z-row are either zero or positive, the current solution is optimal. Therefore, the optimal solution is

$$X_1 = 375, X_2 = 750 \text{ and Max. } Z = 5 \times 375 + 12 \times 750 = 10875$$

Hence, the maximum profit is 10875

Remarks:

- ✓ *The optimal solution corresponds to point D in the feasible region of the graph.*
- ✓ *The solution obtained satisfies the non-negativity constraints, indicating that no other combination of X_1 and X_2 yields a higher value of Z .*
- ✓ *The slack variable S_1 does not appear in the basis column, which means the first resource is fully utilized. In contrast, S_2 is in the basis column with a value of 250, indicating that 250 units of the second resource remain unused.*

4-4 Simplex Method Using Artificial Variables (Two-Phase Method)

The problem discussed above has all its constraints in the form of “less than or equal to” ($\leq$), which makes it straightforward to identify variables that can serve as basic variables in the initial basic solution.

However, if the original constraints are equalities ($=$) or “greater than or equal to” ($\geq$), it becomes difficult to determine a suitable set of basic variables for the first simplex table. In such cases, we introduce non-negative artificial variables on the left-hand side of each equation that lacks an initial basic variable. These artificial variables act like slack variables, providing a starting basic solution.

Artificial variables are used only to initiate the solution process. During the simplex procedure, they are removed from the basis and forced to become zero in the final solution, because they are imaginary variables with no real physical meaning.

This type of problem is solved in two phases as follows:

Phase I: After introducing the necessary artificial variables to obtain an initial basic solution, the focus is on moving into the feasible region (i.e., obtaining a feasible solution) by minimizing the sum of all artificial variables. The goal is to force all artificial variables to become zero, ensuring that the solution corresponds to a real feasible solution of the original problem.

$$\text{Min } W = a_1 + a_2 + \dots$$

If the optimal objective value $W > 0$: in this case, the original LP has no feasible solution.

Phase II: The optimal basic solution obtained in Stage 1 is used as the initial solution for the original problem. From this starting point, the search for the optimal solution of the original linear programming problem is carried out.

There is no guarantee that the auxiliary function w will attain a minimum value of zero, which means that proceeding to Phase II may not be possible. If this minimum value is nonzero, then the original problem is infeasible.¹

Example 2:

$$\text{Max } Z = 4X_1 + 2X_2$$

$$3X_1 + X_2 = 30$$

$$4X_1 + 3X_2 \geq 60$$

$$X_1 + 2X_2 \leq 40$$

$$X_1, X_2 \geq 0$$



$$\text{Max } Z = 4X_1 + 2X_2$$

$$3X_1 + 2X_2 + a_1 = 30$$

$$4X_1 + 3X_2 - S_2 + a_2 = 60$$

$$X_1 + 2X_2 + S_3 = 40$$

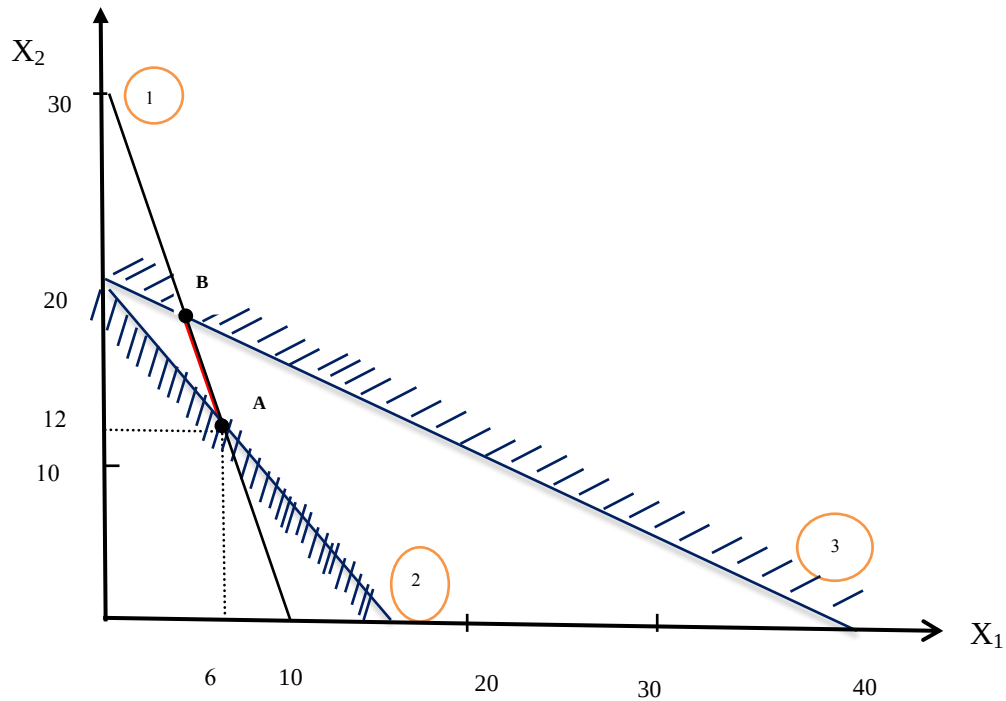
$$X_1, X_2, S_2, S_3 \geq 0 \quad a_1, a_2 \geq 0$$

Since the optimal solution of a linear programming problem is always located at a corner point of the feasible region, the first objective is to enter the feasible region. This is achieved by eliminating the artificial variables from the basis, because the search for the optimal solution cannot begin until a feasible solution is obtained.

To accomplish this, an artificial objective function, denoted W , is constructed and optimized in the first stage.

Referring to the graph, we observe that the solution corresponding to table T1 coincides with the origin point (0,0), since $X_1 = X_2 = 0$. However, this point lies outside the feasible region, which confirms the need to use artificial variables and proceed with Stage 1 to reach feasibility.

¹ Sultan, A., *Linear Programming: An Introduction with Applications*, Academic Press, New York, 1993, p. 92



$$\text{Min } W = a_1 + a_2 \Leftrightarrow \text{Max } -W = -a_1 - a_2$$

$$a_1 = 30 - 3X_1 - X_2$$

$$a_2 = 60 - 4X_1 - 3X_2 + S_2$$

$$\Rightarrow W = -7X_1 - 4X_2 + S_2 + 9$$

$$\Leftrightarrow -W - 7X_1 - 4X_2 + S_2 = -90$$

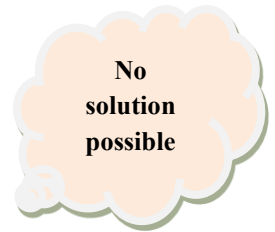
$$\Leftrightarrow -W = 7X_1 + 4X_2 - S_2 - 90$$

No
solution
possible

Thus the initial table becomes:

Basic Variable	X_1	X_2	S_2	Basic Variable Value (RHS)	Ratio
a_1	3	1	0	30	$30/3=10$
a_2	4	3	-1	60	$60/4=15$
S_3	1	2	0	40	$40/1=10$
$-W$	-7	-4	1	-90	

Basic Variable	a_1	X_2	S_2	Basic Variable Value (RHS)	Ratio
X_1	$1/3$	$1/3$	0	10	$10 / 1/3 = 30$
a_2	$-4/3$	$5/3$	-1	20	$20 / 5/3 = 12$
S_3	$-1/3$	$5/3$	0	30	$30 / 5/3 = 18$
$-W$	$7/3$	$-5/3$	1	-20	



To enter the feasible region, it is necessary to remove the artificial variables from the basis. This is achieved when the artificial variables satisfy: $a_1=0$, $a_2=0$

Once this condition is met, the original objective function Z can be reintroduced, and the search for the optimal solution of the original problem can begin.

Basic Variable	a_1	a_2	S_2	Basic Variable Value (RHS)	Ratio
(4) X_1	$3/5$	$-1/5$	$1/5$	6	$6 / 1/5 = 30$
(2) X_2	$-4/5$	$3/5$	$-3/5$	12	N
S_3	1	-1	1	10	$10/1 = 10$
Z	$4/5$	$2/5$	$-2/5$	48	

$(4) \times (3/5) + (2) \times (-4/5) = 4/5$

$(4) \times (-1/5) + (2) \times (3/5) = 2/5$

$(4) \times (1/5) + (2) \times (-3/5) = -2/5$

From tableau T3, it is evident that the artificial variables a_1 and a_2 have been completely eliminated from the basis. This indicates that a basic feasible solution has been successfully established and that the solution process has entered the feasible region of the original linear programming problem.

By referring to the graphical representation, we observe that tableau T3 corresponds to point A, where $X_1=6$ and $X_2=12$ which is a point lying within the feasible region.

At this stage, Phase I of the simplex method is complete. We now proceed to Phase II by reinstating the original objective function in the simplex tableau. The coefficients of the objective function row are then recalculated based on the current basic variables, and the simplex iterations continue until the optimal solution to the original linear programming problem is obtained.

The elements of the **Z-row** are calculated according to the following relationship:

$$\beta_{0j} = \sum (\alpha_{ij} \times C_{Bi}) - C_j^{NB}$$

where:

β_{0j} represents the element in the objective function **Z-row** in column j .

α_{ij} denotes the coefficient of variable X_j in row i of the constraint matrix, corresponding to a basic variable;

C_{Bi} is the coefficient of The coefficient of the variable basic variable X_j in the objective function;

C_j^{NB} is the coefficient of the non-basic variable X_j in the objective function.

According to the optimality condition of the simplex method for a maximization problem, the tableau is not optimal if there exists at least one negative element in the objective-function (Z) row. Since the above tableau contains negative values in the Z-row, the current solution is not optimal, and we therefore continue the search for the optimal solution by applying the usual simplex iteration steps.

Basic Variable	a_1	a_2	S_3	Basic Variable Value (RHS)	Ratio
X_1	2/5	0	-1/5	4	
X_2	-1/5	0	3/5	18	
S_2	1	-1	1	10	
Z	6/5	0	2/5	52	

The table 4 is an optimal table, so the optimal solution is:

$$X_1=4 \quad X_2=18 \quad S_2=10 \quad Z=52$$

Remark: If there are no negative elements remaining in the objective-function (Z) row, while one or more artificial variables still remain in the basis of the solution, then the linear programming problem is said to be infeasible. This indicates that the original constraints are inconsistent, meaning that no solution exists that satisfies all the constraints simultaneously.

4-5 Practical Method of Solution

To reduce the number of artificial variables introduced into a linear programming problem and thereby simplify the process of finding the optimal solution, a practical method was developed based on the same solution principles used in the two-phase simplex method. According to this method:

- Artificial variables are introduced only into constraints that are in the form of equations (=).
- For constraints in the form of $\geq$ the inequality is first converted to $\leq$ by multiplying both sides by -1 , and then a slack variable S is added.

This approach reduces the number of artificial variables, making the simplex iterations more efficient while maintaining the correctness of the solution process.

Solution Rules:

We apply the solution rules to the same example discussed previously

$$\text{Max } Z = 4X_1 + 2X_2$$

$$3X_1 + X_2 = 30$$

$$4X_1 + 3X_2 \leq 60$$

$$X_1 + 2X_2 \leq 40$$

$$X_1, X_2 \geq 0$$



$$\text{Max } Z = 4X_1 + 2X_2$$

$$3X_1 + X_2 + a_1 = 30$$

$$-4X_1 - 3X_2 + S_2 = -60$$

$$X_1 + 2X_2 + S_3 = 40$$

$$X_1, X_2 \geq 0 \quad S_2, S_3 \geq 0 \quad a_1 \geq 0$$

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Basic Variable	X_1	X_2	Basic Variable Value (RHS)	Ratio
$(-1) a_1$	3	1	30	$30 / 3 = 10$
$(+1) S_2$	-4	-3	-60	$-60 / -4 = 15$
S_3	1	2	40	$40 / 1 = 10$
-W	-7	-4	-90	

No solution possible

$$(-1) \times 3 + (+1) \times (-4) = -7$$

$$(-1) \times 1 + (+1) \times (-3) = -4$$

According to the first tableau, the artificial variable a_1 appears as a basic variable with a positive value, while the slack variable S_2 also appears as a basic variable but with a negative value. This indicates that the initial solution lies outside the feasible region. Therefore, the search for the optimal solution is postponed, and the first goal is to enter the feasible region by satisfying the following two conditions:

1. Remove the artificial variables from the basis.
2. Eliminate negative values on the right-hand side of the tableau.

To calculate the elements of the artificial objective function W in the first stage:

- Multiply all elements in rows containing a basic artificial variable by -1 .
- Multiply all elements in rows containing a negative basic variable by $+1$.
- Multiply all other elements by 0.

Finally, sum the resulting values vertically column by column to obtain the elements of the W -row in the first stage

No solution possible

Basic Variable	a_1	X_2	Basic Variable Value (RHS)	Ratio
X_1	$1/3$	$1/3$	10	$10 / 1/3 = 30$
S_2	$4/3$	$-5/3$	-20	$-20 / -5/3 = 12$
S_3	$-1/3$	$5/3$	30	$30 / 5/3 = 18$
-W	$7/3$	$-5/3$	-20	

Basic Variable	a_1	S_3	Basic Variable Value (RHS)	Ratio
(4) X_1	$2/5$	$-1/5$	4	
S_2	1	1	10	
(2) X_2	$-1/5$	$3/5$	18	
Z	$6/5$	$2/5$	52	

Possible and optimal solution

Remark: if we have a test between a positive and a negative pivot, we always choose the positive one.

The table 3 is an optimal table, so the optimal solution is:

$$X_1=4 \quad X_2=18 \quad S_2=10 \quad Z=52$$

Example 2:

$$\text{Max } Z = 30X_1 + 20X_2 + 15X_3$$

$$2X_1 + X_2 + 3X_3 \leq 1800$$

$$X_1 + 2X_2 + 2X_3 \leq 1500$$

$$X_1 + X_2 + X_3 \geq 800$$

$$X_3 = 300$$

$$X_1, X_2 \geq 0$$



$$\text{Max } Z = 30X_1 + 20X_2 + 15X_3$$

$$2X_1 + X_2 + 3X_3 + S_1 = 1800$$

$$X_1 + 2X_2 + 2X_3 + S_2 = 1500$$

$$-X_1 - X_2 - X_3 + S_3 = -800$$

$$X_3 + a_4 = 300$$

$$X_1, X_2 \geq 0, S_1, S_2, S_3 \geq 0, a_4 \geq 0$$

Basic Variable	X_1	X_2	X_3	Basic Variable Value (RHS)	Ratio
S_1	2	1	3	1800	$1800/3=600$
S_2	1	2	2	1500	$1500/2=750$
(+1) S_3	-1	-1	-1	-800	$-800/-1=800$
(-1) a_4	0	0	1	300	$300/1=300$
-W	-1	-1	-2	-1100	

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Basic Variable	X_1	X_2	a_4	Basic Variable Value (RHS)	Ratio
S_1	2	1	-3	900	$900/2=450$
S_2	1	2	-2	900	$900/1=900$
S_3	-1	-1	1	-500	$-500/-1=500$
X_3	0	0	1	300	N
-W	-1	-1	2	-900	

Basic Variable	S_1	X_2	a_4	Basic Variable Value (RHS)	Ratio
X_1	1/2	1/2	-3/2	450	$450 / 1/2=900$
S_2	-1/2	3/2	-1/2	450	$450 / 3/2=300$
S_3	1/2	-1/2	-1/2	-50	$-50/-1/2 =100$
X_3	0	0	1	300	N
-W	1/2	-1/2	1/2	-50	

Basic Variable	S_1	S_2	a_4	Basic Variable Value (RHS)	Ratio
(30) X_1	2/3	-1/3	-4/3	300	
(20) X_2	-1/3	2/3	-1/3	300	
S_3	1/3	1/3	-2/3	100	
(15) X_3	0	0	1	300	
Z	40/3	10/3	-95/3	19500	

This tableau represents an optimal solution. Although there is a negative value in the Z-row, it appears in a column corresponding to an artificial variable. According to the rules of the simplex method, once an artificial variable is removed from the basis, it **cannot re-enter** the solution. Therefore, the current solution is considered optimal with respect to the original problem.

Special Note on Choosing the Pivot Element in the Two-Stage Method

When applying the two-stage simplex method, the following situation may arise during the selection of the pivot element:¹

➤ **Case: The pivot column contains both positive and negative elements**

In this case, only rows with positive coefficients in the pivot column are considered for determining the leaving variable. Among these rows, the leaving variable is chosen by applying the minimum positive ratio test, that is, selecting the row with the smallest positive ratio of the right-hand-side value to the corresponding positive coefficient in the pivot column.

Accordingly, the pivot element must be positive, ensuring that the next basic feasible solution remains feasible.

Example 3:

Basic Variable	X_3	X_2	X_1	Basic Variable Value (RHS)
S_1	3	2	1	1800
S_2	1	1/2	1	800
a_3	1	1	0	400
S_4	-1	0	0	-300
-W	-2	-1	0	

We observe that although the fourth row yields the smallest positive ratio when dividing the right-hand-side values by the corresponding elements of the pivot column, the third row is preferred. This is because the pivot element in the third row is positive, which facilitates the subsequent mathematical operations required to construct the next simplex tableau and helps maintain feasibility.

¹ حاج صحراوي، حمودي، رياضيات المؤسسة كمدخل إلى التقنيات الكمية وفق البرنامج الوزاري المعتمد، دار جيطلي للنشر، برج بوعريش، الجزائر، 2014، ص 31.

➤ Case where all elements of the pivot column are negative:

In this situation, the pivot element will necessarily be negative. Therefore, when determining the leaving variable, we select the row that yields the largest positive value from the ratio of the right-hand-side elements to the corresponding elements of the pivot column. This choice ensures a feasible transition to the next tableau and allows the algorithm to proceed correctly.

Example 4:

Basic Variable	X_1	X_2	X_3	Basic Variable Value (RHS)
S_1	-1	-2	-1	-1800
S_2	-1	-1	-1	-800
S_4	-1	-1/2	0	-400
-W	-3	-7/2	-2	

Working with these instructions enables us to either reach the optimal solution in the least number of tables or facilitate the calculations that may be somewhat complicated due to the choice of a negative pivot element. However, it is worth noting that the solution remains correct in the event that we apply the general rule for choosing the pivot element and do not take these instructions into account.

4-6 Special cases in simplex method

4-6-1 - Alternative optimal solutions:

Alternative optimal solutions occur when the objective function attains the same optimal value at more than one feasible solution point. This situation arises when, in the final simplex tableau, there exists at least one zero coefficient in the objective function Z- row corresponding to a non-basic variable, indicating that more than one optimal solution is possible

Example 5:

$$\text{Max } Z = 2X_1 + 4X_2$$

$$X_1 + 2X_2 \leq 5$$

$$X_1 + X_2 \leq 4$$

$$X_1, X_2 \geq 0$$

Basic Variable	X_1	X_2	Basic Variable Value (RHS)	Ratio
S_1	1	2	5	$5/2=2,5$
S_2	1	1	4	$4/1=4$
Z	-2	-4	0	

Basic Variable	X_1	S_1	Basic Variable Value (RHS)	Ratio
X_2	$1/2$	$1/2$	$5/2$	$5/2 \cdot 1/2=5$
S_2	$1/2$	$-1/2$	$3/2$	$3/2 \cdot 1/2=3$
Z	0	2	10	

Basic Variable	S_2	S_1	Basic Variable Value (RHS)	Ratio
X_2	-1	1	1	
X_1	2	-1	3	
Z	0	2	10	

Possibility of X_1 entering the basis without changing the value of the objective function.

There is a practical advantage in identifying alternative optimal solutions, as they provide management with the flexibility to select the solution that best suits the prevailing conditions, without causing any reduction or change in the optimal value of the objective function.

4-6-2 - Degeneracy:

If the ratios used to determine the leaving variable are equal, then one or more of the basic variables will take a value of zero in the next simplex tableau. In this case, the resulting solution is called a degenerate solution. Degeneracy occurs because the linear programming model contains at least one redundant constraint.

Example 6:

$$\text{Max } Z = 3X_1 + 9X_2$$

$$X_1 + 4X_2 \leq 8$$

$$X_1 + 2X_2 \leq 4$$

$$X_1, X_2 \geq 0$$

Basic Variable	X_1	X_2	Basic Variable Value (RHS)	Ratio
S_1	1	4	8	$8/4=2$
S_2	1	2	4	$4/2=2$
Z	-3	-9	0	

Basic Variable	X_1	S_1	Basic Variable Value (RHS)	Ratio
X_2	$1/4$	$1/4$	2	$2 / 1/4=8$
S_2	$2/4$	$-2/4$	0	$0 / 2/4=0$
Z	$-3/4$	$9/4$	18	

Basic Variable	S_2	S_1	Basic Variable Value (RHS)	Replacement Ratio
X_2	$-1/2$	$1/2$	2	
X_1	2	-1	0	
Z	$3/2$	$3/2$	18	

4-6-3 - Unbounded Solutions:

If the coefficients of a non-basic variable in all constraint rows are non-positive, then no limiting constraint exists to restrict the increase of that variable. As a result, the feasible region extends infinitely in the direction of that variable, leading to an unbounded solution.

Example 7:

$$\begin{aligned}\text{Max } Z &= 10X_1 + 25X_2 \\ 3X_1 + 5X_2 &\geq 60 \\ X_2 &\leq 15 \\ X_1, X_2 &\geq 0\end{aligned}$$

Basic Variable	X_1	X_2	Basic Variable Value (RHS)	Ratio
(+1) S_1	-3	-5	-60	N
S_2	0	1	15	15/1=15
-W	-3	-5		

Basic Variable	(25) X_1	S_2	Basic Variable Value (RHS)	Ratio
S_1	-3	5	15	
(10) X_2	0	1	15	
Z	-25	10	20	

The possibility of increasing X_1 infinitely without exceeding either of the two constraints, and thus an infinite solution in the direction of X_1 .

4-6-4 - Infeasibility:

There will be no feasible solution to the model if all constraints cannot be satisfied simultaneously. In the simplex method, this situation is identified when one or more artificial variables remain with positive values in the final solution, or when a

negative value appears on the right-hand side of the final tableau, indicating that the model is infeasible..

Key Terms:¹

Basic Variable: A variable that belongs to a basic feasible solution and takes a non-negative value in that solution.

Non-Basic Variable: A variable that has a value equal to zero in a feasible solution.

Artificial Variable: A non-negative variable temporarily introduced into a linear programming model to obtain an initial basic feasible solution and to initiate the simplex method.

Slack Variable: A non-negative variable added to a constraint of the $\leq$ type in order to convert the inequality into an equation.

Surplus Variable: A non-negative variable added to a constraint of the $\geq$ type in order to convert the inequality into an equation.

Basic Solution: In a system of m equations and n variables, where $m < n$, a solution is called a basic solution if at least $n - m$ variables are set equal to zero.

Basic Feasible Solution: A basic solution in which the remaining m variables satisfy the non-negativity condition, while the other $n - m$ variables are equal to zero.

Optimal Solution: A solution that yields the best possible value of the objective function, either maximizing or minimizing it, while satisfying all model constraints.

¹ Anna University, *MBA-H2040: Quantitative Techniques for Managers* [Course handout], Department of Management Studies, Chennai, India, 2018 , p.58

4-7 Questions and Exercises

Part 1: Theoretical Questions

Part A: Short Questions

1. What is the purpose of the Simplex Method in linear programming?
2. Define a slack variable in your own words.
3. When is the Graphical Method not valid for solving LP problems?

Part B: Multiple Choice Questions

1. Who developed the Simplex Method?
 - a) Albert Einstein
 - b) George Dantzig
 - c) Isaac Newton
 - d) John von Neumann
2. In the Simplex tableau, the “pivot element” is:
 - a) The largest value in the table
 - b) The intersection of the pivot row and pivot column
 - c) Always equal to zero
 - d) The right-hand side value
3. The Simplex Method stops when:
 - a) There are still negative values in the Z-row
 - b) All values in the Z-row are non-negative
 - c) Slack variables are zero
 - d) The objective function equals zero

Part C: True or False

1. The Simplex Method can only solve problems with two decision variables. (T / F)
2. Adding slack variables helps to convert inequalities into equalities. (T / F)

3. A basic feasible solution always satisfies all the constraints. (T / F)

Part 2: Practice Exercises

Exercise1: Solve the following problem using the graphical method and then using the simplex method.

$$\begin{aligned}\text{Max } Z &= 5X_1 + 6X_2 \\ 2X_1 + X_2 &\leq 30 \\ \frac{1}{2}X_1 + X_2 &\leq 15 \\ X_1 &\leq 12 \\ X_1, X_2 &\geq 0\end{aligned}$$

Exercise2: Solve the following problems

$$\begin{aligned}\text{Max } Z &= 10X_1 + 15X_2 + 12X_3 \\ X_1 + X_2 + X_3 &\leq 100 \\ 3X_1 + X_2 + X_3 &\leq 150 \\ X_1 + 2X_2 + 2X_3 &\leq 200 \\ X_1, X_2, X_3 &\geq 0\end{aligned}$$

$$\begin{aligned}\text{Max } Z &= 110X_1 + 100X_2 + 120X_3 \\ 2X_1 + X_2 - X_3 &\leq 60 \\ X_1 + X_3 &\leq 80 \\ X_1 + X_2 + X_3 &\leq 120 \\ X_1, X_2, X_3 &\geq 0\end{aligned}$$

Exercise3: Solve the following problem using two phase method simplex.

$$\begin{aligned}\text{Max } Z &= 250X_1 + 210X_2 + 270X_3 \\ 2X_1 + X_2 + 2X_3 &\leq 26000 \\ X_1 + X_2 + \frac{1}{2}X_3 &\leq 18000 \\ X_1 + X_2 + X_3 &\geq 12000 \\ X_1 &= 5000 \\ X_2, X_3 &\geq 0\end{aligned}$$

$$\begin{aligned}\text{Max } Z &= 15X_1 + 20X_2 + 18X_3 \\ X_1 + \frac{1}{2}X_2 + X_3 &\leq 20000 \\ X_1 + X_2 + 2X_3 &\leq 30000 \\ X_2 + X_3 &\geq 10000 \\ X_3 &= 5000 \\ X_1, X_2 &\geq 0\end{aligned}$$

$$\begin{aligned}\text{Min } Z &= 2X_1 + X_2 + 3X_3 \\ X_1 + X_2 + X_3 &\geq 120 \\ 2X_1 + X_2 + 2X_3 &\geq 140 \\ X_1 + 2X_2 + 2X_3 &\leq 180 \\ X_1, X_2, X_3 &\geq 0\end{aligned}$$

Exercise4:

KYRO Company specializes in the production of three types of paint: A, B, and C. The manufacturing process requires two main raw materials: binder and paint powder. Producing one unit of paints A, B, and C requires $\frac{2}{3}$, 1, and 1 liter of binder, respectively, and 1, 2, and 1 kg of paint powder, respectively.

To fulfill its contractual obligations for the coming month, the company must produce at least 5000 units in total of paints A and C, and exactly 2000 units of paint B.

The available resources for the next month are 10,000 liters of binder and 15 tons (15000 kg) of paint powder. The profit earned per unit of paints A, B, and C is €100, €110, and €115, respectively.

Required:

- Formulate the above problem as a linear programming model.
- Determine the optimal product mix that maximizes the company's total profit.

4-8 Solutions & Answers

Part A: Short Questions

1. What is the purpose of the Simplex Method in linear programming?

- Answer: The Simplex Method is used to find the optimal solution of a linear programming problem with multiple constraints.

2. Define a slack variable in your own words.

- Answer: A slack variable is an extra variable added to a $\leq$ constraint to convert it into an equality by measuring unused capacity.

3. When is the Graphical Method not valid for solving LP problems?

- Answer: The graphical method is not valid when the LP problem has more than two decision variables.

Part B: Multiple Choice Questions

1. Who developed the Simplex Method?

- Answer: b) George Dantzig

2. In the Simplex tableau, the “pivot element” is:

- Answer: b) The intersection of the pivot row and pivot column

3. The Simplex Method stops when:

- Answer: b) All values in the Z-row are non-negative

Part C: True or False

1. False
2. True
3. True

Part 2: Practice exercises

Exercise1:

graphical method:

$$\text{Max } Z = 5X_1 + 6X_2$$

$$2X_1 + X_2 \leq 30$$

$$1/2 X_1 + X_2 \leq 15$$

$$X_1 \leq 12$$

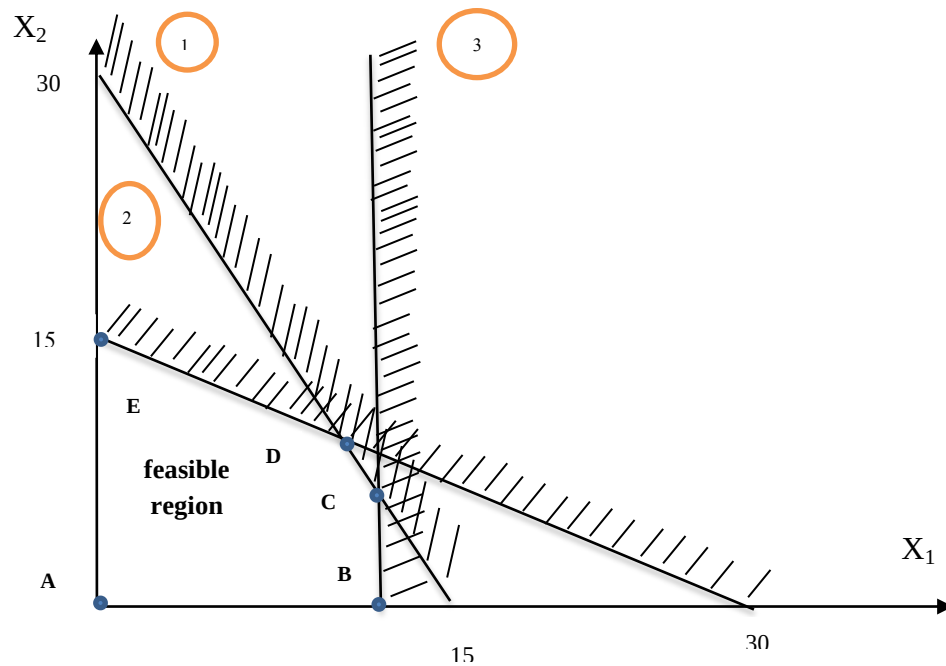
$$X_1, X_2 \geq 0$$



$$2X_1 + X_2 = 30 \quad (0,30) \quad (15,0)$$

$$1/2 X_1 + X_2 = 15 \quad (0,15) \quad (30,0)$$

$$X_1 = 12$$



at the point A: $A(0,0) \Rightarrow Z = 5(0) + 6(0) = 0$

at the point B: $B(12,0) \Rightarrow Z = 5(12) + 6(0) = 60$

at the point C:

The point C is the point of intersection of the first and third constraints. To find the coordinates of this point, we solve the set of equations:

$$\begin{cases} 2X_1 + X_2 = 30 \\ X_1 = 12 \end{cases}$$

$$C(12,6) \Rightarrow Z = 5(12) + 6(6) = 96$$

at the point D:

The point D It is the point of intersection of the first and second constraints. To find the coordinates of this point, we need to solve the system of equations:

$$\begin{cases} 2X_1 + X_2 = 30 \\ \frac{1}{2}X_1 + X_2 = 15 \end{cases}$$

$$D(10,10) \Rightarrow Z = 5(10) + 6(10) = 110$$

at the point E: $E(0,15) \Rightarrow Z = 5(0) + 6(15) = 90$

The maximum value of the objective function is at the point D (10,10)

simplex method:

standard form:

$$\text{Max } Z = 5X_1 + 6X_2 + 0S_1 + 0S_2 + 0S_3$$

$$2X_1 + X_2 + S_1 = 30$$

$$\frac{1}{2}X_1 + X_2 + S_2 = 15$$

$$X_1 + S_3 = 12$$

$$X_1, X_2 \geq 0 \quad S_1, S_2, S_3 \geq 0$$

Basic Variable	X_1	X_2	Basic Variable Value (RHS)	Ratio
S_1	2	1	30	$30/1=30$
S_2	1/2	1	15	$15/1=15$
S_3	1	0	12	N
Z	-5	-6	0	

The current simplex tableau corresponds to point A in the feasible region shown in the graph

Basic Variable	X_1	S_2	Basic Variable Value (RHS)	Ratio
S_1	3/2	-1	15	$15/3/2=10$
X_2	1/2	1	15	$15/1/2=30$
S_3	1	0	12	$12/1=12$
Z	-2	6	90	

The current simplex tableau corresponds to point E in the feasible region shown in the graph

Basic Variable	S_1	S_2	Basic Variable Value (RHS)	Ratio
X_1	2/3	-2/3	10	
X_2	-1/3	4/3	10	
S_3	-2/3	2/3	2	
Z	4/3	14/3	110	

The current simplex tableau corresponds to point D in the feasible region shown in the graph.

optimal solution:

$$X_1 = 10 \quad X_2 = 10 \quad Z = 110$$

Exercise2:

$$\text{Max } Z = 10X_1 + 15X_2 + 12X_3$$

$$X_1 + 1/2X_2 + X_3 \leq 100$$

$$3X_1 + X_2 + X_3 \leq 150$$

$$X_1 + 2X_2 + 2X_3 \leq 200$$

$$X_1, X_2, X_3 \geq 0$$

$$\text{Max } Z = 10X_1 + 15X_2 + 12X_3 + 0S_1 + 0S_2 + 0S_3$$

$$X_1 + 1/2X_2 + X_3 + S_1 = 100$$

$$3X_1 + X_2 + X_3 + S_2 = 150$$

$$X_1 + 2X_2 + 2X_3 + S_3 = 200$$

$$X_1, X_2, X_3 \geq 0 \quad S_1, S_2, S_3 \geq 0$$

standard form

Basic Variable	X_1	X_2	X_3	Basic Variable Value (RHS)	Ratio
S_1	1	1/2	1	100	100 / 1/2 = 200
S_2	3	1	1	150	150 / 1 = 150
S_3	1	2	2	200	200 / 2 = 100
Z	-10	-15	-12	0	

Basic Variable	X_1	S_3	X_3	Basic Variable Value (RHS)	Ratio
S_1	3/4	-1/4	1/2	50	50 / 3/4 = 66,67
S_2	5/2	-1/2	0	50	50 / 5/2 = 20
X_2	1/2	1/2	1	100	100 / 1/2 = 200
Z	-5/2	15/2	3	1500	

Basic Variable	S_2	S_3	X_3	Basic Variable Value (RHS)	Ratio
S_1	-3/10	-1/10	1/2	35	
X_1	2/5	-1/5	0	20	
X_2	-1/5	3/5	1	90	
Z	1	7	3	1550	

optimal solution:

$$X_1 = 20 \quad X_2 = 90 \quad X_3 = 0 \quad Z = 1550$$

Chapter 4: The Simplex Method to LP Problems

$$\text{Max } Z = 110X_1 + 100X_2 + 120X_3$$

$$2X_1 + X_2 - X_3 \leq 60$$

$$X_1 + X_3 \leq 80$$

$$X_1 + X_2 + X_3 \leq 120$$

$$X_1, X_2, X_3 \geq 0$$

$$\text{Max } Z = 110X_1 + 100X_2 + 120X_3$$

$$2X_1 + \frac{1}{2}X_2 - X_3 + S_1 = 60$$

$$X_1 + X_3 + S_2 = 80$$

$$X_1 + X_2 + X_3 + S_3 = 120$$

$$X_1, X_2, X_3 \geq 0, S_1, S_2, S_3 \geq 0$$

standard form

Basic Variable	X_1	X_2	X_3	Basic Variable Value (RHS)	Ratio
S_1	2	1/2	-1	60	N
S_2	1	0	1	80	80/ 1= 80
S_3	1	1	1	120	120/ 1=120
Z	-110	-100	-120	0	

Basic Variable	X_1	X_2	S_2	Basic Variable Value (RHS)	Ratio
S_1	3	1/2	1	140	140/ 1/2=280
X_3	1	0	1	80	N
S_3	0	1	-1	40	40/ 1= 40
Z	10	-100	120	9600	

Basic Variable	X_1	S_3	S_2	Basic Variable Value (RHS)	Ratio
S_1	3	-1/2	3/2	120	
X_3	1	0	1	80	
X_2	0	1	-1	40	
Z	10	100	20	13600	

optimal solution:

$$X_1 = 0 \quad X_2 = 40 \quad X_3 = 80 \quad Z = 13600$$

Exercise3:

$$\text{Max } Z = 250X_1 + 210X_2 + 270X_3$$

$$2X_1 + X_2 + 2X_3 \leq 26000$$

$$X_1 + X_2 + 1/2X_3 \leq 18000$$

$$X_1 + X_2 + X_3 \geq 12000$$

$$X_1 = 5000$$

$$X_2, X_3 \geq 0$$

standard form

$$\text{Max } Z = 250X_1 + 210X_2 + 270X_3$$

$$2X_1 + X_2 + 2X_3 + S_1 = 26000$$

$$X_1 + X_2 + 1/2X_3 + S_2 = 18000$$

$$-X_1 - X_2 - X_3 + S_3 = -12000$$

$$X_1 + a_4 = 5000$$

$$X_2, X_3 \geq 0 \quad S_1, S_2, S_3 \geq 0 \quad a_4 \geq 0$$

Basic Variable	X_1	X_2	X_3	Basic Variable Value (RHS)	Ratio
S_1	2	1	2	26000	$26000/2=13000$
S_2	1	1	1/2	18000	$18000/1=18000$
$(+1)S_3$	-1	-1	-1	-12000	N
$(-1)a_4$	1	0	0	5000	$5000/1=5000$
-W	-2	-1	-1	-17000	

Basic Variable	a_4	X_2	X_3	Basic Variable Value (RHS)	Ratio
S_1	-2	1	2	16000	$16000/1=16000$
S_2	-1	1	1/2	13000	$13000/1=13000$
S_3	1	-1	-1	-7000	N
X_1	1	0	0	5000	N
-W	2	-1	-1	-7000	

Chapter 4: The Simplex Method to LP Problems

Basic Variable	a_4	S_2	(270) X_3	Basic Variable Value (RHS)	Ratio
S_1	-1	-1	3/2	3000	3000/ 3/2=2000
(210) X_2	-1	1	1/2	13000	13000/ 1/2=26000
S_3	0	1	-1/2	6000	N
(250) X_1	1	0	0	5000	N
Z	40	210	-165	3980000	

Basic Variable	a_4	S_2	S_1	Basic Variable Value (RHS)	Ratio
X_3	-2/3	-2/3	2/3	2000	
X_2	-2/3	4/3	-1/3	12000	
S_3	-1/3	2/3	1/3	7000	
X_1	1	0	0	5000	
Z	-70	100	110	4310000	

this is the optimal solution. Although the Z-row contains a negative value, it is associated with an artificial variable. Since an artificial variable cannot re-enter the basis once it has been eliminated, the current solution is optimal

optimal solution:

$$X_1 = 5000 \quad X_2 = 12000 \quad X_3 = 2000 \quad Z = 4310000$$

Chapter 4: The Simplex Method to LP Problems

$$\text{Max } Z = 15X_1 + 20X_2 + 18X_3$$

$$X_1 + 1/2X_2 + X_3 \leq 2000$$

$$X_1 + X_2 + 2X_3 \leq 30000$$

$$X_2 + X_3 \geq 10000$$

$$X_3 = 5000$$

$$X_1, X_2 \geq 0$$

standard form

$$\text{Max } Z = 15X_1 + 20X_2 + 18X_3$$

$$X_1 + 1/2X_2 + X_3 + S_1 = 20000$$

$$X_1 + X_2 + 2X_3 + S_2 = 30000$$

$$-X_2 - X_3 + S_3 = -10000$$

$$X_3 + a_4 = 5000$$

$$X_1, X_2 \geq 0 \quad S_1, S_2, S_3 \geq 0 \quad a_4 \geq 0$$

Basic Variable	X_1	X_2	X_3	Basic Variable Value (RHS)	Ratio
S_1	1	1/2	1	20000	20000/ 1=20000
S_2	1	1	2	30000	30000/ 2=15000
$(+1)S_3$	0	-1	-1	-10000	N
$(-1)a_4$	0	0	1	5000	5000/ 1=5000
-W	0	-1	-2	-15000	

Basic Variable	X_1	X_2	a_4	Basic Variable Value (RHS)	Ratio
S_1	1	1/2	-1	15000	15000/ 1/2=30000
S_2	1	1	-2	20000	20000/ 1=20000
S_3	0	-1	1	-5000	N
X_3	0	0	1	5000	N
-W	0	-1	2	-5000	

Chapter 4: The Simplex Method to LP Problems

Basic Variable	(15) X_1	S_2	a_4	Basic Variable Value (RHS)	Ratio
S_1	1/2	-1/2	0	5000	
(20) X_2	1	1	-2	20000	
S_3	0	1	-1	15000	
(18) X_3	0	0	1	5000	
Z	5	20	-22	490000	

optimal solution:

$$X_1 = 0 \quad X_2 = 20000 \quad X_3 = 5000 \quad Z = 490000$$

$$\text{Min } Z = 2X_1 + X_2 + 3X_3$$

$$X_1 + X_2 + X_3 \geq 120$$

$$2X_1 + X_2 + 2X_3 \geq 140$$

$$X_1 + 2X_2 + 2X_3 \leq 180$$

$$X_1, X_2, X_3 \geq 0$$

standard form

$$\text{Max } -Z = -2X_1 - X_2 - 3X_3$$

$$-X_1 - X_2 - X_3 + S_1 = -120$$

$$-2X_1 - X_2 - 2X_3 + S_2 = -140$$

$$X_1 + 2X_2 + 2X_3 + S_3 = 180$$

$$X_1, X_2, X_3 \geq 0 \quad S_1, S_2, S_3 \geq 0$$

Basic Variable	X_1	X_2	X_3	Basic Variable Value (RHS)	Ratio
(+1) S_1	-1	-1	-1	-120	N
(+1) S_2	-2	-1	-2	-140	N
S_3	1	2	2	180	180/2= 90
-W	-3	-2	-3	-260	

A positive pivot element is selected even if it does not satisfy the minimum ratio test.

Chapter 4: The Simplex Method to LP Problems

Basic Variable	S_3	(-1) X_2	(-3) X_3	Basic Variable Value (RHS)	Ratio
S_1	1	1	1	60	$60/1=60$
S_2	2	3	2	220	$220/3=73,33$
$(-2)X_1$	1	2	2	180	$180/2=90$
$-Z$	-2	-3	-1	-360	

Basic Variable	S_3	S_1	X_3	Basic Variable Value (RHS)	Ratio
X_2	1	1	1	60	
S_2	-1	-3	-1	40	
X_1	-1	-2	0	60	
$-Z$	1	3	2	-540	

optimal solution:

$$X_1 = 60 \quad X_2 = 60 \quad X_3 = 0 \quad Z = 540$$

$$\text{Max } Z = 100X_1 + 110X_2 + 115X_3$$

$$2/3X_1 + X_2 + X_3 \leq 10000$$

$$X_1 + 2X_2 + X_3 \leq 15000$$

$$X_1 + X_3 \geq 5000$$

$$X_2 = 2000$$

$$X_1, X_3 \geq 0$$

$$\text{Max } Z = 100X_1 + 110X_2 + 115X_3$$

$$2/3X_1 + X_2 + X_3 + S_1 = 10000$$

$$X_1 + 2X_2 + X_3 + S_2 = 15000$$

$$-X_1 - X_3 + S_3 = 5000$$

$$X_2 + a_4 = 2000$$

$$X_1, X_3 \geq 0 \quad S_1, S_2, S_3 \geq 0 \quad a_4 \geq 0$$

standard form

Chapter 4: The Simplex Method to LP Problems

Basic Variable	X_1	X_3	X_3	Basic Variable Value (RHS)	Ratio
S_1	2/3	1	1	10000	10000/ 1=10000
S_2	1	2	1	15000	15000/ 2=7500
(+1) S_3	-1	0	-1	-5000	N
(-1) a_4	0	1	0	20000	2000/ 1= 2000
-W	-1	-1	-1	-25000	

Basic Variable	X_1	a_4	X_3	Basic Variable Value (RHS)	Ratio
S_1	2/3	-1	1	8000	8000/ 2/3=12000
S_2	1	-2	1	11000	11000/ 1= 11000
S_3	-1	0	-1	-5000	N
X_2	0	1	0	2000	N
-W	-1	1	-1	-5000	

Basic Variable	S_2	a_4	(115) X_3	Basic Variable Value (RHS)	Ratio
S_1	-2/3	1/3	1/3	2000/3	2000/3/ 2/3= 1000
(100) X_1	1	-2	1	11000	11000/ 1=11000
S_3	1	-2	0	6000	N
(110) X_2	0	1	0	2000	N
Z	100	-90	-15	1320000	

Basic Variable	S_2	a_4	S_1	Basic Variable Value (RHS)	Ratio
(115) X_3	-2	1	3	2000	
(100) X_1	3	-3	-3	9000	
S_3	1	-2	0	6000	
(110) X_2	0	1	0	2000	
Z	70	-75	45	1350000	

The best product mix for the next month is to produce 9000 units of type A paint, 2000 units of type B, and 2000 units of type C to achieve the maximum possible profit of €1350 000.

Chapter 5: Linear Programming Duality

Learning Objectives

After completing this chapter, students will be able to:

1. Understand the concept of duality in linear programming.
2. Explain the general relationship between primal and dual problems.
3. Analyze the formulation of a dual problem from a given primal problem.
4. Illustrate the primal–dual pair using matrix notation.
5. Explain and interpret the concept of shadow prices.

Chapter contents

5-1 Concept of primal and dual

5-2 Formulation of the Dual problem

5-3 Methods of solving the dual problem

5-3 Economic interpretation of dual variables

5-4 Questions and Exercises

5-5 Solutions & Answers

$$\begin{aligned} \text{Max} Z &= \sum_{j=1}^n c_j X_j \\ \sum_{j=1}^n a_{ij} X_j &\leq b_i \\ i &= 1, 2, 3, \dots, m \end{aligned}$$

The standard form of this primal model is written as follows:

$$\text{P.MaxZ} = C_1 X_1 + C_2 X_2 + \dots C_j X_j + \dots C_n X_n$$

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1j}X_j + \dots + a_{1n}X_n + S_1 = b_1$$

$$a_{21}X_1 + a_{22}X_2 + \dots + a_{2j}X_j + \dots + a_{2n}X_n + S_2 = b_2$$

$$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{ij}X_j + \dots + a_{in}X_n + S_i = b_i$$

$$a_{m1}X_1 + a_{m2}X_2 + \dots + a_{mi}X_i + \dots + a_{mn}X_n + S_m = b_m$$

$$X_1, X_2, \dots, X_n \geq 0 \quad , \quad S_1, S_2, \dots, S_m \geq 0$$

The transformation of a given primal problem into a dual problem involves the following considerations:¹

1. If the objective of the primal is maximization, the objective of the dual is minimization.
2. The primal has m -constraints while its dual has m -unknowns.
3. The primal has n -unknowns while its dual has n -constraints
4. The n -coefficients of the objective function of primal (C_j) become the n constant terms (b_i) of its dual.
5. The m -constant terms of the primal (b_i) become the m -constant terms of the objective function (C_j) of its dual.
6. The coefficients of the variables of the primal are transformed in their position in the dual. This means that the first column of the coefficients in the primal becomes the first row in the dual, the second column becomes the second row and so on.

¹Alagappa University, *op.cit*, pp. 95-96

7. The n -variables (X_n) of the primal are replaced by the m new variables (y_m) of its dual. This change affects the system of restrictions as well as the objective function.
8. The sign of the inequalities in the set of restrictions of the primal ($\leq$) is reversed in the set of restrictions in its dual ($\geq$). In other words, if the inequalities in the primal are of the type $\leq$, then, they are of $\geq$ type in the dual.
9. For writing the dual of the given maximization problem, we should first ensure that all the constraint inequalities are of the $\leq$ type and for writing the dual of the given minimization problem, the constraint inequalities should be of the $\geq$ type.

The above steps can be presented as in the following table:

Primal	Dual
Objective is minimization	Objective is maximization
Objective is maximization	Objective is minimization
More than or equal $\geq$ (Type constraints)	Less than or equal $\leq$ (Type constraints)
Number of columns	Number of rows
Number of rows	Number of columns
Number of decision variables	Number of constraints
Number of constraints	Number of decision variables
Coefficient of objective function	RHS value
RHS value	Coefficient of objective function
Unrestricted in sign	Equality Constraint

المصدر: حسين محمود الجنابي، الأحدث في بحوث العمليات، دار الحامد للنشر والتوزيع، الطبعة الأولى، عمان، الأردن، 201، ص 143

If we assume that the variables of the dual problem are y_i and the objective function is W , then the dual problem (for the previous original problem) is as follows:

$$D: \text{Min } W = b_1 y_1 + b_2 y_2 + \dots b_i y_i + \dots b_m y_m$$

$$a_{11}y_1 + a_{21}y_2 + \dots a_{i1}y_3 + \dots + a_{m1}y_m \geq C_1$$

$$a_{12}y_1 + a_{22}y_2 + \dots a_{i2}y_3 + \dots + a_{m2}y_m \geq C_2$$

$$\dots\dots\dots$$

$$a_{1j}y_1 + a_{2j}y_2 + \dots a_{ij}y_3 + \dots + a_{mj}y_m \geq C_j$$

$$\dots\dots\dots$$

$$a_{1n}y_1 + a_{2n}y_2 + \dots a_{in}y_3 + \dots + a_{mn}y_m \geq C_n$$

$$y_1, y_2, \dots y_m \geq 0$$

$$\begin{aligned} \text{Min } W &= \sum_{i=1}^m b_i y_i \\ \sum_{i=1}^m a_{ij} y_i &\geq c_j \\ j &= 1, 2, 3, \dots, n \end{aligned}$$

The standard form of a primal problem is:

$$D: \text{Min } W = b_1 y_1 + b_2 y_2 + \dots b_i y_i + \dots b_m y_m$$

$$a_{11}y_1 + a_{21}y_2 + \dots a_{i1}y_3 + \dots + a_{m1}y_m - t_1 = C_1$$

$$a_{12}y_1 + a_{22}y_2 + \dots a_{i2}y_3 + \dots + a_{m2}y_m - t_2 = C_2$$

$$\dots\dots\dots$$

$$a_{1j}y_1 + a_{2j}y_2 + \dots a_{ij}y_3 + \dots + a_{mj}y_m - t_j = C_j$$

$$\dots\dots\dots$$

$$a_{1n}y_1 + a_{2n}y_2 + \dots a_{in}y_3 + \dots + a_{mn}y_m - t_n = C_n$$

$$y_1, y_2, \dots y_m \geq 0 \quad t_1, t_2, \dots t_n \geq 0$$

This is assuming that the differences variables in the primal problem are t_j

Example1:

$\begin{aligned} \text{P:Min } Z &= 4X_1 + 5X_2 + 3X_3 \\ 2X_1 + 3X_2 + 4X_3 &\geq 2500 \\ X_1 + X_2 + X_3 &\geq 750 \\ X_1 &\geq 250 \\ X_2 &\geq 300 \\ X_1, X_2, X_3 &\geq 0 \end{aligned}$		$\begin{aligned} \text{D:Max } W &= 2500y_1 + 750y_2 + 250y_3 + 300y_4 \\ 2y_1 + y_2 + y_3 &\leq 4 \\ 3y_1 + y_2 + y_4 &\leq 5 \\ 4y_1 + y_2 &\leq 3 \\ y_1, y_2, y_3, y_4 &\geq 0 \end{aligned}$
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$\begin{aligned} \text{P:Max } Z &= 5X_1 + 2X_2 + 4X_3 \\ X_1 + 2X_2 + X_3 &\leq 10 \\ 2X_1 - X_2 + 3X_3 &\leq 8 \\ X_1, X_2, X_3 &\geq 0 \end{aligned}$		$\begin{aligned} \text{D:Min } W &= 10y_1 + 8y_2 \\ y_1 + 2y_2 &\geq 5 \\ 2y_1 - y_2 &\geq 2 \\ y_1 + 3y_2 &\geq 4 \\ y_1, y_2 &\geq 0 \end{aligned}$
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Remarks :

- If there are constraints of the form greater than or equal to ($\geq$) in a primal problem whose goal is to maximize, they must be transformed into ($\leq$) by multiplying both sides of the inequality by (-1) before formulating the corresponding dual problem.
- If there are constraints of the form less than or equal to ($\leq$) in a primal problem whose goal is to minimize, they must be transformed into ($\geq$) by multiplying both sides of the inequality by (-1) before formulating the corresponding dual problem.
- Constraints in the form of equations are not transformed. Since the constraint in the primal problem corresponds to a variable in the dual problem, we expect the variables corresponding to these constraints to be unrestricted, meaning they can be positive or negative.

Example2:

$\begin{aligned} \text{P:Max } Z &= 45X_1 + 50X_2 + 60X_3 \\ X_1 + 1/2X_2 + X_3 &\leq 50000 \\ 2X_1 + X_2 + 1/2X_3 &\leq 90000 \\ X_1 + X_2 + X_3 &\geq 40000 \\ X_1 &= 5000 \\ X_2, X_3 &\geq 0 \end{aligned}$		$\begin{aligned} \text{P:Max } Z &= 45X_1 + 50X_2 + 60X_3 \\ X_1 + 1/2X_2 + X_3 &\leq 50000 \\ 2X_1 + X_2 + 1/2X_3 &\leq 90000 \\ -X_1 - X_2 - X_3 &\leq -40000 \\ X_1 &= 5000 \\ X_2, X_3 &\geq 0 \end{aligned}$
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$$D: \text{Min } W = 50000y_1 + 90000y_2 - 40000y_3 + 5000y_4$$

$$y_1 + 2y_2 - y_3 + y_4 \geq 45$$

$$1/2 y_1 + y_2 - y_3 \geq 50$$

$$y_1 + 1/2 y_2 - y_3 \geq 60$$

$$y_1, y_2, y_3 \geq 0 \quad y_4 \neq 0$$

5-3 Methods of solving the dual problem

5-3-1- The Complementary Slackness Method between the primal variables and the dual variables:

Through the relationships that link the primal problem to the dual problem, they can be exploited in the process of finding the optimal solution to the dual problem based on the optimal solution to the original problem.

Dual problem

$$\text{Max } Z = \sum_{j=1}^n C_j X_j$$

$$\sum_{j=1}^n a_{ij} X_j \leq b_i$$

$$\sum_{j=1}^n a_{ij} X_j + S_i = b_i$$

primal problem

$$\text{Min } W = \sum_{i=1}^m b_i y_i$$

$$\sum_{i=1}^m a_{ij} y_i \geq C_j$$

$$\sum_{i=1}^m a_{ij} y_i - t_j = C_j$$

At our optimal solution:

$$Z = W \Leftrightarrow \sum C_j X_j = \sum b_i y_i$$

We compensate with my value C_j and b_i we find:

$$\sum (\sum a_{ij} y_i - t_j) X_j = \sum (\sum a_{ij} X_j + S_i) y_i$$

$$\Rightarrow \sum \sum a_{ij} y_i X_j - \sum t_j X_j = \sum \sum a_{ij} X_j y_i + \sum S_i y_i$$

$$\Rightarrow -\sum t_j X_j = \sum S_i y_i$$

$$\Rightarrow \sum t_j X_j + \sum S_i y_i = 0$$

Since both: $t_j, X_j, S_i, y_i \geq 0$

$$\sum t_j X_j + \sum S_i y_i = 0 \Rightarrow \sum t_j X_j = 0 \quad \text{and} \quad \sum S_i y_i = 0$$

$$\sum X_j t_j = 0 \Rightarrow \begin{cases} X_j > 0 \quad \text{and} \quad t_j = 0 \\ \text{Or} \\ X_j = 0 \quad \text{and} \quad t_j > 0 \end{cases}$$

$$\sum S_i y_i = 0 \Rightarrow \begin{cases} y_i = 0 \quad \text{and} \quad S_i > 0 \\ \text{Or} \\ y_i > 0 \quad \text{and} \quad S_i = 0 \end{cases}$$

Example3: Let the following question be:

$$\text{Max } Z = 10X_1 + 15X_2 + 12X_3$$

$$X_1 + 1/2X_2 + X_3 \leq 100$$

$$3X_1 + X_2 + X_3 \leq 150$$

$$X_1 + 2X_2 + 2X_3 \leq 200$$

$$X_1, X_2, X_3 \geq 0$$

The optimal solution to this primal problem is:

$$X_1 = 20, X_2 = 90, X_3 = 0$$

$$S_1 = 35, S_2 = 0, S_3 = 0, Z = 1550$$

Formulating the dual problem:

$$\text{D.Min } W = 100 y_1 + 150 y_2 + 200 y_3$$

$$y_1 + 3y_2 + y_3 \geq 10$$

$$1/2y_1 + y_2 + 2y_3 \geq 15$$

$$y_1 + y_2 + 2y_3 \geq 12$$

$$y_1, y_2, y_3 \geq 0$$

Standard form:

$$\text{D.Min } W = 100 y_1 + 150 y_2 + 200 y_3$$

$$y_1 + 3y_2 + y_3 - t_1 = 10$$

$$1/2 y_1 + y_2 + 2 y_3 - t_2 = 15$$

$$y_1 + y_2 + 2y_3 - t_3 = 12$$

$$y_1, y_2, t_1, t_2, t_3 \geq 0$$

$$\sum X_j t_j = 0 \Rightarrow \begin{cases} X_1 = 20 \Rightarrow t_1 = 0 \\ X_2 = 90 \Rightarrow t_2 = 0 \\ X_3 = 0 \Rightarrow t_3 > 0 \end{cases}$$

$$\sum S_i y_i = 0 \Rightarrow \begin{cases} S_1 = 35 \Rightarrow y_1 = 0 \\ S_2 = 0 \Rightarrow y_2 > 0 \\ S_3 = 0 \Rightarrow y_3 > 0 \end{cases}$$

$$W = 1550$$

By substitution, we get the following set of equations:

$$\begin{cases} 3y_2 + y_3 = 10 & \dots(1) \\ y_2 + 2 y_3 = 15 & \dots(2) \\ y_2 + 2y_3 - t_3 = 12 & \dots(3) \end{cases}$$

Solving the two equations (1) and (2) we get: $y_2 = 1$ and $y_3 = 7$

And by compensation in (3) we get: $t_3 = 3$

To verify the results, we substitute the values of y_i into the objective function W and we get: $W = 100(0) + 150(1) + 200(7) = 1550$

Since $Z = W$, then the solution obtained for the dual problem is actually the optimal solution.

This method is simple and fast at the same time, but it is not without some drawbacks, including:

- It relies entirely on the optimal solution to the original problem. This method requires solving the original problem first, which can sometimes take a long time.
- This method cannot be used, as is, if the primal problem is characterized by a degeneracy case.

5-3-2- Derive the optimal solution to the dual problem from the optimal tableau of the original problem:

From the above example, we can draw an important conclusion regarding the solution of the primal problem. By looking at the optimal tableau of the original and exploiting the symmetry relations between the variables of the primal problem and the primal problem, we can obtain the optimal solution of the dual problem.

Example4: Let the optimal tableau for the primal problem

Basic Variable	S_2	S_3	X_3	Basic Variable Value (RHS)
S_1	-3/10	-1/10	1/2	35
X_1	2/5	-1/5	0	20
X_2	-1/5	3/5	1	90
Z	1	7	3	1550

We note that when $X_3 = 0$, we find the value of its complementary variable, t_1 , in the same column and in the Z row: $t_3 = 3$.

We also note that when $S_2 = 0$, we find the value of its complementary variable, y_2 , in the same column and in the Z row: $y_2 = 1$.

Likewise, we also note that when $S_3 = 0$, we find the value of its complementary variable, y_3 , in the same column and in the Z row: $y_3 = 7$.

Since the variables X_1 , X_2 , and S_1 are originally positive, their complementary variables, t_1 , t_2 , and y_1 , respectively, have zero values.

$$y_1 = 0, t_1 = 0, t_2 = 0$$

This is the same solution that we find in the optimal tableau for the primal problem in the previous example.

Basic Variable	y_1	t_1	t_2	Basic Variable Value (RHS)
y_2	3/10	-2/5	1/5	1
y_3	1/10	1/5	-3/5	7
t_3	-1/2	0	-1	3
-W	35	20	90	-1550

5-4 Economic interpretation of dual variables

The importance of the dual problem lies primarily in the economic meaning of its variables, as these variables represent important indicators that help the organization's management in making appropriate decisions at the appropriate time.

5-4-1- The Economic Meaning of y_i :

We saw earlier that at the optimal solution $Z = W$ Since $W = \sum b_i y_i$ It can be concluded

$$Z = \sum b_i y_i = b_1 y_1 + b_2 y_2 + b_3 y_3 + \dots + b_m y_m$$

We also know from the primal problem that b_i represents the value of the second side of the constraint, the quantity of inputs in constraint i (for example), so if the quantity of resources in one of the constraints, let it be constraint i , changes by an amount of Δb_i , while the other data remain the same, that is, the second side of constraint i becomes:

$$b'_i = b_i + \Delta b_i$$

This means that instead of Z The objective function becomes Z' where:

$$\begin{aligned}
 Z' &= b_1 y_1 + b_2 y_2 + b_3 y_3 + \dots + \mathbf{b'_i y_i} + \dots b_m y_m \\
 &= b_1 y_1 + b_2 y_2 + b_3 y_3 + \dots + (b_i + \Delta b_i) y_i + \dots b_m y_m \\
 &= b_1 y_1 + b_2 y_2 + b_3 y_3 + \dots + b_i y_i + \Delta b_i y_i + \dots b_m y_m \\
 Z' &= Z + \Delta b_i y_i
 \end{aligned}$$

$$\Rightarrow Z' - Z = \Delta Z = \Delta b_i y_i$$

$$\Rightarrow y_i = \Delta Z / \Delta b_i$$

If there is a change in the second side of the constraint i by one unit (i.e. $\Delta b_i = 1$), then $\Delta Z = y_i$, which means that:

Therefore, y_i represents the marginal increase in the maximum profit resulting from the addition of one extra unit of resource i . It can thus be interpreted as the manufacturer's implicit valuation of resource i , commonly referred to -following Paul Samuelson- as the shadow price. The shadow price y_i indicates the maximum price the manufacturer is willing to pay for an additional unit of resource i .

If the market price of resource i is below its shadow price y_i , the manufacturer should consider purchasing additional units of that resource in order to expand production. Conversely, if the market price exceeds y_i , it would be economically rational to reduce the production scale or sell the excess resource.

Moreover, when the optimal solution X^* satisfies the i -th primal constraint with slack, this indicates that resource i is not fully utilized. In such a case, acquiring additional units of the resource is unnecessary, regardless of how low its market price may be. Accordingly, the shadow price y_i becomes zero..¹

For instance, consider a production problem in which the objective is to maximize profit. If, at the optimal solution, the slack variable S_i associated with the raw material constraint equals zero, this indicates that the available quantity of this resource is fully utilized. Furthermore, if the corresponding dual variable y_i takes a value of 4 in the optimal solution of the dual problem, this implies that each additional unit of this input would increase the firm's profit by \$4. In other words, the shadow price of this raw material is \$4, reflecting its marginal contribution to profit under the current constraints.

Since the entire available quantity of raw materials in the establishment must be fully utilized, the company manager should consider purchasing additional units of

¹ Pan, P.-Q., *Linear Programming Computation*, 2nd ed., Springer, New York, 2014, p. 137

these inputs from the market. If the market price p is lower than the shadow price y_i ($y_i > p$), the manager should consider buying more, because each additional unit would generate an extra net profit of $(y_i - p)$.

Conversely, if the market price is higher than the shadow price ($p > y_i$), the manager may consider selling part of the available quantity directly (without processing it), as each unit sold would yield an additional net profit of $(p - y_i)$.

This is why the variable y_i is called the *shadow price*: it represents a value that exists within the organization and can differ between companies, even for the same type of input.

5-4-2- The Economic Meaning of t_j

The variable t_j has a direct relationship with X_j in the optimal solution:

- If $X_j=0$ at the optimal solution, then t_j is positive in the optimal dual solution. For example, if $t_j=6$, this indicates that the optimal solution of the original problem requires producing no units of type j . However, if the firm decides to produce this type for some reason, it should expect a decrease in profit of \$6 for each unit produced.
- If $X_j>0$, then $t_j=0$. This means that the optimal solution requires producing a positive quantity of type j , so producing the recommended amount does not reduce profit.

This is why the variable t_j is called the **opportunity cost**: it represents the potential loss in profit associated with producing a unit of a product that is not required in the optimal solution.

Remark: It is possible to have $S_i = 0$ and $y_i = 0$ or $X_j = 0$ with $t_j = 0$. This is a rare case, but it can occur if there are alternative optimal solutions or if there is Degeneracy case.

5-5 Questions and Exercises

Part 1: Theoretical Questions

A. Short Questions

1. Explain the duality in linear programming.
2. Describe the formulation of the dual problem.
3. Discuss some important results in duality.
4. Explain the economic interpretation of dual variables
5. Explain the primal-dual relationship.
6. Briefly discuss ‘duality’ in linear programming.
7. What is the principle of duality in linear programming? Explain its advantages.

B. Multiple Choice Questions

1. The dual of the primal maximization LP problem having m constraints and n non-negative variables should
 - (a) have n constraints and m non-negative variables
 - (b) be a minimization LP problem
 - (c) both (a) and (b)
 - (d) none of the above
2. For any primal problem and its dual,
 - (a) optimal value of objective functions is same
 - (b) primal will have an optimal solution if and only if dual does too
 - (c) both primal and dual cannot be infeasible
 - (d) all of the above
3. The right-hand side constant of a constraint in a primal problem appears in the corresponding dual as
 - (a) a coefficient in the objective function
 - (b) a right-hand side constant of a constraint
 - (c) an input-out coefficient
 - (d) none of the above

4. Shadow price indicates how much one unit change in the resource value will change the

- (a) optimality range of an objective function
- (b) optimal value of the objective function
- (c) value of the basic variable in the optimal solution
- (d) none of the above

C. True or False

- 1. If the objective function of the primal LP problem is maximized. (T / F)
then the objective function of the dual is to be minimized. (T / F)
- 2. A dual variable is not defined for each constraint in the primal LP problem. (T / F)
- 3. The dual of dual is a primal problem. (T / F)
- 4. The shadow price indicates how much increase in a right hand side coefficient will change the optimality range of an objective function coefficient. (T / F)
- 5. The right hand side coefficient of some constraint in a primal LP problem appears in the corresponding dual as a coefficient in the objective function. (T / F)
- 6. The change in the objective function value due to per unit decrease in the right hand side of a constraint is given by the negative of the shadow price for that constraint. (T / F)

Part 2: Practice Exercises

Exercise 1: consider the following primal problem:

$$\begin{aligned}\text{Max } Z &= 150X_1 + 200X_2 + 180X_3 \\ 2X_1 + 2X_2 + X_3 &\leq 2200 \\ X_1 + X_2 + X_3 &\leq 1200 \\ X_1 + X_2 + 1/2X_3 &\leq 1000 \\ X_1, X_2, X_3 &\geq 0\end{aligned}$$

the optimal tableau for this problem is:

Basic Variable	X_1	S_3	S_2	Basic Variable Value (RHS)
S_1	0	-2	0	200
X_3	0	-2	2	400
X_2	1	2	-1	800
Z	50	40	160	232000

1. Write the dual problem corresponding to the primal problem above, and derive the optimal solution. solution for it.

Exercise 2:

A company specializing in chemical industries produces 3 types of combustible fuel using 2 types of crude oil. The mathematical formulation of this company's production problem for the next period was as follows:

$$\begin{aligned}
 \text{Max } Z &= 30X_1 + 20X_2 + 16X_3 \\
 X_1 + X_2 + X_3 &\leq 8000 && \text{Light crude oil (barrel)} \\
 1/2X_1 + X_2 + 2X_3 &\leq 5000 && \text{Heavy oil (barrel)} \\
 X_2 + X_3 &= 2500 && \text{The quantity produced of the second and third types} \\
 X_3 &\geq 1000 && \text{The quantity produced of the third type} \\
 X_1, X_2 &\geq 0
 \end{aligned}$$

The optimal tableau for the above question is:

Basic Variable	S_2	S_4	a_3	Basic Variable Value (RHS)
S_1	-2	-2	1	2500
X_1	2	2	-2	3000
X_3	0	-1	0	1000
X_2	0	1	1	1500
Z	60	64	-40	136,000

- 1- Formulate the dual problem corresponding to the primal problem above. And derive the optimal solution for it.
- 2- Explain the economic meaning of the variables in the dual problem.

3- If you know that the market price of heavy oil is \$55, what advice would you give to the company's managers?

Exercise 3:

A company specializing in textile industries produces three types of cotton clothing for children using two basic types of inputs: machines and cotton fabric. The mathematical formulation of the production problem of this company for the next period was as follows:

$$\begin{aligned} \text{Max } Z &= 550X_1 + 700X_2 + 650X_3 \\ X_1 + X_2 + 1/2X_3 &\leq 12000 && \text{Fabric (m}^2\text{)} \\ X_1 + X_2 + X_3 &\leq 15000 && \text{Work on machines (SA)} \\ X_1 + X_2 &\geq 5000 && \text{Minimum combined production requirement} \\ X_3 &= 2000 && \text{The quantity produced of the third type} \\ X_1, X_2 &\geq 0 \end{aligned}$$

1- Solve the dual problem and explain the economic meaning of its variables, given that the optimal solution to the original problem is: $X_1 = 0$, $X_2 = 11,000$,

$$X_3 = 2,000$$

2- If you know that the market price per square meter of fabric is €800, what advice would you give to the company's managers?

5-6 Solutions & Answers

Part 1: Theoretical Questions

A. Short Questions

1. Explain the duality in linear programming.

Duality in linear programming refers to the relationship between two optimization problems: the primal and its associated dual. Every linear programming problem has a dual problem whose structure is closely linked to the original. The optimal solution of one provides valuable information about the optimal solution of the other. Duality helps analyze resource values, feasibility, boundedness, and economic meaning in LP models.

2. Describe the formulation of the dual problem.

To formulate the dual of a primal LP problem:

- A primal with **n variables** and **m constraints** becomes a dual with **m variables** and **n constraints**.
- A **maximization** primal becomes a **minimization** dual (and vice versa).
- The **coefficients of the constraints matrix** become transposed in the dual.
- **Right-hand side constants** of primal constraints become coefficients in the dual objective function.
- **Objective function coefficients** in the primal become right-hand side constants in the dual constraints.
- **Signs** of inequalities flip according to the direction ($\leq$ becomes $\geq$).

3. Discuss some important results in duality.

Important results include:

1. **Weak Duality Theorem:** The value of the objective function in any feasible solution of the primal is always $\leq$ the value in any feasible solution of the dual.
2. **Strong Duality Theorem:** If both the primal and dual have optimal solutions, their optimal objective function values are equal.
3. **Complementary Slackness:** At optimality, a constraint is binding if and only if the corresponding dual variable is positive. This provides a condition to test optimality or construct solutions.
4. **Dual Feasibility and Primal Feasibility:** If one problem is unbounded, the other is infeasible.

4. Explain the economic interpretation of dual variables.

Dual variables represent **shadow prices** or **marginal values** of the resources in the primal problem. Economically:

- A dual variable shows how much the **objective function will improve** if one unit of a resource is increased.
- A positive dual variable means the resource is **scarce** and valuable.
- A zero-value dual variable means the resource is **not binding** (not fully used), and increasing it adds no benefit.

This interpretation is widely used in production planning, cost minimization, and resource allocation.

5. Explain the primal–dual relationship.

The primal and dual problems are mathematically linked:

- Their constraint matrices are transposes of each other.
- One's objective function coefficients appear on the RHS of the other.
- Their optimal values are equal under strong duality.
- Feasibility of one ensures certain properties of the other (boundedness, optimality).

This relationship allows solving one to infer information about the other, and is the basis for sensitivity analysis and shadow pricing.

6. Briefly discuss ‘duality’ in linear programming.

Duality provides an alternative representation of every LP model. It allows decision-makers to look at the same optimization problem from the **resource viewpoint (dual)** instead of the **activity viewpoint (primal)**.

It plays a key role in:

- Understanding resource values
- Sensitivity analysis
- Interpreting constraints and slack variables
- Ensuring accuracy and optimality checks

Thus, duality is fundamental to the theory and application of LP.

7. What is the principle of duality in linear programming? Explain its advantages.

Principle:

Every linear programming problem has a corresponding dual problem with interrelated structure and equal optimal value (when solutions exist). Solving either provides insights about the other.

Advantages:

- **Economic interpretation** of resources through shadow prices
- **Sensitivity analysis** becomes easier
- **Verification of optimality** using complementary slackness
- **Alternative solutions:** sometimes the dual is easier to solve
- **Infeasibility detection:** unboundedness in one implies infeasibility in the other

Duality enhances both the theoretical understanding and practical usefulness of LP models

B. Multiple Choice Questions

1. Correct answer: (c)
2. Correct answer: (d)
3. Correct answer: (a)
4. Correct answer: (b)

C. True or False

1. True
2. False
3. True
4. False
5. True

6. True

Part 2: Solve the Practice Exercises

Exercise 1:

Formulation of the dual problem:

$$\text{Min } W = 2200 y_1 + 1200 y_2 + 1000 y_3$$

$$2y_1 + y_2 + y_3 \geq 150$$

$$2y_1 + y_2 + y_3 \geq 200$$

$$y_1 + y_2 + \frac{1}{2}y_3 \geq 180$$

$$y_1, y_2, y_3 \geq 0$$

Deduce the optimal tableau for the dual problem:

Basic Variable	y_1	t_3	t_2	RHS
t_1	0	2	-1	50
y_3	-2	2	-2	40
y_2	0	-2	1	160
$-W$	200	400	800	-232000

Exercise 2:

$$\text{Max } Z = 30X_1 + 20X_2 + 16X_3$$

$$X_1 + X_2 + X_3 \leq 8000$$

$$\frac{1}{2}X_1 + X_2 + 2X_3 \leq 5000$$

$$X_2 + X_3 = 2500$$

$$X_3 \geq 1000$$

$$X_1, X_2 \geq 0$$



$$\text{Max } Z = 30X_1 + 20X_2 + 16X_3$$

$$X_1 + X_2 + X_3 \leq 8000$$

$$\frac{1}{2}X_1 + X_2 + 2X_3 \leq 5000$$

$$X_2 + X_3 = 2500$$

$$-X_3 \leq -1000$$

$$X_1, X_2 \geq 0$$

$$\text{D.Min } W = 8000 y_1 + 5000 y_2 + 2500 y_3 - 1000 y_4$$

$$y_1 + \frac{1}{2}y_2 \geq 30$$

$$y_1 + y_2 + y_3 \geq 20$$

$$y_1 + 2y_2 + y_3 - y_4 \geq 16$$

$$y_1, y_2, y_4 \geq 0, y_3 \neq 0$$

Basic Variable	y_1	t_1	t_3	t_2	RHS
y_2	2	-2	0	0	60
y_4	2	-2	1	-1	64
y_3	-1	2	0	-1	-40
-W	2500	3000	1000	800	-136000

Explanation of the economic meaning:

$t_1, t_2, t_3 = 0$. This means that the optimal solution requires producing specific quantities of the three types of fuel. Therefore, the company does not incur any decrease in the value of the objective function (the opportunity cost) if it produces these three types in optimal quantities.

$y_1 = 0$. This means that the optimal solution requires not exploiting all the available quantity of light crude oil. Therefore, any additional quantity of this type of oil will not lead to any change in the value of the objective function.

$y_2 = 60$. This means that the optimal solution requires exploiting all the available quantity of heavy crude oil. Each additional barrel of this type of oil will result in an additional profit of \$60.

$y_3 = -40$. This means that the optimal solution requires producing exactly 2500 units of the second and third types of fuel in total. Changing this limit, by one unit, will lead to a decrease in the value of the objective function by \$40

$y_4 = 64$, meaning that the optimal solution requires producing the lowest possible quantity (the minimum required) of the third type of fuel, and every additional unit produced that exceeds this limit will result in a decrease in the value of the objective function by \$64 (meaning that the relationship is inverse between the right side of the constraint and the value of Z in the case of constraints of type $\leq$).

$y_2 = 60 > p = 55$ the company's management should consider purchasing an additional quantity of heavy oil because each barrel purchased results in an additional net profit of \$5.

Exercise 3:

$$\text{Max } Z = 550X_1 + 700X_2 + 650X_3$$

$$X_1 + X_2 + 1/2X_3 \leq 12000$$

$$X_1 + X_2 + X_3 \leq 15000$$

$$X_1 + X_2 \geq 5000$$

$$X_3 = 2000$$

$$X_1, X_2 \geq 0$$



$$\text{Max } Z = 550X_1 + 700X_2 + 650X_3$$

$$X_1 + X_2 + 1/2X_3 \leq 12000$$

$$X_1 + X_2 + X_3 \leq 15000$$

$$-X_1 - X_2 \leq -5000$$

$$X_3 = 2000$$

$$X_1, X_2 \geq 0$$

$$\text{Min } W = 12000 y_1 + 15000 y_2 - 5000 y_3 + 2000 y_4$$

$$y_1 + y_2 - y_3 \geq 550$$

$$y_1 + y_2 - y_3 \geq 700$$

$$1/2y_1 + y_2 + y_4 \geq 650$$

$$y_1, y_2, y_3, \geq 0 \quad y_4 \neq 0$$

Standard

$$\text{Min } W = 12000 y_1 + 15000 y_2 - 5000 y_3 + 2000 y_4$$

$$y_1 + y_2 - y_3 - t_1 = 550$$

$$y_1 + y_2 - y_3 - t_2 = 700$$

$$1/2y_1 + y_2 + y_4 - t_3 = 650$$

$$y_1, y_2, y_3, \geq 0 \quad y_4 \neq 0$$

$$t_1, t_2, t_3 \geq 0$$

$$\sum X_j t_j = 0 \Rightarrow \begin{cases} X_1 = 0 \Rightarrow t_1 > 0 \\ X_2 = 11000 \Rightarrow t_2 = 0 \\ X_3 = 2000 \Rightarrow t_3 = 0 \end{cases}$$

$$\sum S_i y_i = 0 \Rightarrow \begin{cases} S_1 = 0 \Rightarrow y_1 > 0 \\ S_2 = 2000 \Rightarrow y_2 = 0 \\ S_3 = 6000 \Rightarrow y_3 = 0 \\ a_4 = 0 \Rightarrow y_4 > 0 \end{cases}$$

By substitution, we obtain the following set of equations:

$$y_1 - t_1 = 550$$

$$y_1 = 700$$

$$1/2y_1 + y_4 = 650$$

By solving the set of equations, we get: $y_1=700$ and $y_4=300$ and $t_1=150$

To verify the solution, we find: $W=Z=9000000$

Explanation of the economic meaning:

$t_2, t_3 = 0$. This means that the optimal solution requires producing certain quantities of the second and third types of clothing. Therefore, the company does not incur any decrease in the value of the objective function (opportunity cost) if it produces these two types in the appropriate quantities.

$t_1 = 150$. This means that the optimal solution requires not producing the first type of clothing. If the company decides to produce it, it must incur a decrease in the value of the objective function of \$150 for each unit produced.

$y_1 = 700$. This means that the optimal solution requires utilizing the entire available quantity of fabric, and each additional square meter will result in an additional profit of \$700.

$y_2 = 0$, meaning that the optimal solution requires not utilizing all the available energy from the machines. Therefore, any additional energy from the machines will not result in any change in the value of the objective function.

$y_3 = 0$, meaning that the optimal solution requires exceeding the minimum required, estimated at 5000 units of the first and second types in total (since $(X_1 + X_2 = 11,000)$). Therefore, producing quantities of the first and second types, in total, that exceed this minimum will not affect the value of the objective function by decreasing it.

$y_4 = 300$, meaning that the optimal solution requires producing exactly 2000 units of the third type. Changing this limit, by one unit, will lead to an increase in the value of the objective function by \$300 (i.e., it is in the company's interest to produce quantities greater than 2000 units, and the restriction is not in the company's favor).

$800 = p > y_1 = 700$ the company's management should consider converting part of the available quantity of fabric to sell it as is (i.e., without converting it), because every square meter of fabric sold results in an additional net profit of \$100.

Chapter 6: Integer Programming Problems

Learning Objectives

After completing this chapter, students will be able to:

1. Understand the importance of Integer Programming (IP).
2. Apply the principles of integer programming to real-life decision-making problems.
3. Solve integer programming problems using the cutting plane method.

Chapter contents

6.1 Introduction

6-2 Importance and applications of Integer Programming

6-3 Formulating Integer Programming Problems

6-4 Gomory's Cutting Plane Method of Integer Programming Problem

6-5 Questions and Exercises

6-6 Solutions and Answers

6-1- Introduction

An integer programming (IP) problem is a mathematical (linear or nonlinear) programming problem in which some or all of the variables are restricted to assume only integer or discrete values. If all variables take integer values, then the problem is called a pure IP. On the other hand, if both integer and continuous variables coexist, the problem is called a mixed integer program (MIP). In addition, if the objective function and all constraints are linear functions of all variables, such a problem is referred to as a pure integer linear programming (ILP) or mixed integer linear programming (MILP) model. The term linear may be omitted unless it is necessary to contrast these models with nonlinear programming problems.¹

The general form of an integer programming problem is expressed as follows,

$$\text{Max } Z = CX$$

Subject to constraints,

$$AX \leq B$$

$$X \geq 0 \text{ Where some or all variables are integers}$$

The simplex algorithm studied in Chapter 2 is based on the fact that the feasible region (the set of feasible solutions) of an LP problem is convex. This property plays a key role in the solution of linear models. In fact, the number of extreme points of a convex set of solutions is finite, and we have shown that the optimal solution is obtained in an extreme point. Therefore, even though the number of solutions is reduced when variables are restricted to be integer, IP problems are usually much more difficult to solve than LP problems because the set of feasible solutions is no longer convex.²

¹ Ravindran, A. R. (ed.). *Operations Research and Management Science*, Handbook. CRC Press, Boca Raton, FL, 2016, p.80

² University of the Basque Country (UPV/EHU). *Operations Research: Linear Programming*. penCourseWare, Bilbao, Spain, 2018, p. 205

According to the nature of the variables, we can distinguish three types of IP models:¹

- **Pure integer programming problems** are cases in which all variables are required to have integer values.
- **Mixed integer programming problems** are cases in which some, but not all, of the decision variables are required to have integer values.
- **Zero-one integer programming problems** are special cases in which all the decision variables must have integer solution values of 0 or 1.

6-2 Importance and applications of Integer Programming

Unlike linear programming, where decision variables may take any non-negative real values, integer programming requires some or all decision variables to assume integer values. This restriction is essential in many real-world situations where fractional solutions are impractical or meaningless.

Integer programming is particularly important in business and industrial applications that involve indivisible units. For example, production is often planned in terms of batches, lots, or manufacturing runs, while the transportation of goods requires a whole number of trucks, aircraft, or containers. Similarly, decisions involving the number of employees, machines, or facilities must be expressed as integers rather than fractions.

Integer programming has numerous applications in operations research, including production planning, scheduling, capital budgeting, facility location, logistics, and resource allocation. It is also widely used in binary decision problems, such as assignment, project selection, and the traveling salesman problem, where decision variables are restricted to either 0 or 1.

For binary decision variables,

$X_j = 1$, if the j -th activity is selected or performed.

¹ Render, B., Stair, R.M. & Hanna, M.E., *Quantitative Analysis for Management*, 11th ed., Pearson Education, New York, 2012, p.416

$X_j = 0$, if the j -th activity is not selected or performed.

Such binary variables are commonly used to model yes-or-no decisions. Examples include determining whether to undertake a project, assign a worker to a task, open a facility, or select a particular investment. Consequently, many practical decision-making problems can be naturally formulated as binary integer programming models.

6-3- Formulating Integer Programming Problems

Similar to LP, there are three basic steps in formulating an IP model: (1) identifying and defining all integer and continuous decision variables; (2) identifying all restrictions and formulating all corresponding constraints in terms of linear equations or inequalities; and (3) identifying and formulating the objective as a linear function of the decision variables to be optimized (either minimized or maximized).¹

Example 1:

Iris is evaluating four potential investment projects. Investment 1 yields a net present value (NPV) of \$16,000, Investment 2 yields an NPV of \$22,000, Investment 3 yields an NPV of \$12,000, and Investment 4 yields an NPV of \$8,000.

Each investment requires an initial cash outlay: \$5,000 for Investment 1, \$7,000 for Investment 2, \$4,000 for Investment 3, and \$3,000 for Investment 4. The total amount of capital currently available for investment is \$14,000.

Required: Formulate an integer programming (IP) model that determines which investments Iris should undertake in order to maximize the total net present value, subject to the available budget.

Solution: as in LP formulations, we begin by defining a variable for each decision that Iris must make. This leads us to define a 0–1 variable:

¹ Ravindran, A. R. (ed.), *op. cit.*, p. 82

$$X_j(1,2,3,4) = \begin{cases} 1 & \text{if investment } j \text{ is made} \\ 0 & \text{otherwise} \end{cases}$$

For example, $X_2 = 1$ if investment 2 is made, and $X_2 = 0$ if investment 2 is not made.

The NPV obtained by Iris (in thousands of dollars) is:

$$\text{Total NPV obtained by Iris} = 16x_1 + 22x_2 + 12x_3 + 8x_4 \dots \quad (1)$$

To see this, note that if $X_j = 1$, then (1) includes the NPV of investment j , and if $X_j = 0$, (1) does not include the NPV of investment j

This formulation implies that, for any selected combination of investments, equation (1) yields the corresponding total net present value (NPV). For instance, if Iris chooses to undertake Investments 1 and 4, the resulting NPV is:

$$16,000 + 8,000 = \$24,000$$

This combination of investments corresponds to $X_1 = X_4 = 1$, $X_2 = X_3 = 0$, Substituting these values into equation (1) gives:

$$16(1) + 22(0) + 12(0) + 8(1) = \$24 \text{ (thousand).}$$

This reasoning leads to the following objective function for Iris's investment problem:

$$\text{Max } Z = 16X_1 + 22X_2 + 12X_3 + 8X_4 \dots \quad (2)$$

Iris faces the constraint that at most \$14,000 can be invested. By the same reasoning used to develop (1), we can show that Total amount invested (in thousands of dollars) = $5X_1 + 7X_2 + 4X_3 + 3X_4 \dots \quad (3)$

For example, if $X_1 = 0$, $X_2 = X_3 = X_4 = 1$, then Iris makes investments 2, 3, and 4. In this case, Iris must invest $7 + 4 + 3 = \$14$ (thousand).

Equation (3) yields a total amount invested of $5(0) + 7(1) + 4(1) + 3(1) = \14 (thousand). Because at most \$14,000 can be invested, X_1 , X_2 , X_3 , and X_4 must satisfy:

$$5X_1 + 7X_2 + 4X_3 + 3X_4 \leq 14 \quad \dots (4)$$

Combining (2) and (4) with the constraints $X_j = 0$ or 1 ($j = 1, 2, 3, 4$) yields the following 0–1 IP:

$$\text{Max } Z = 16X_1 + 22X_2 + 12X_3 + 8X_4$$

$$\text{s.t. } 5X_1 + 7X_2 + 4X_3 + 3X_4 \leq 14 \quad \dots (5)$$

$$X_j = 0 \text{ or } 1 \quad (j = 1, 2, 3, 4)$$

Example 2:¹

The number of employees needed in a post office varies depending on the day of the week, as shown in the table:

Day	Employees
Monday	15
Tuesday	13
Wednesday	15
Thursday	18
Friday	14
Saturday	16
Sunday	10

Each employee works five consecutive days, followed by two consecutive days off. It becomes necessary to organize groups of employees to work in different shifts, so that the number of employees required is satisfied every day of the week. The objective is to employ the minimum number of workers.

We define the following decision variables:

X_j : number of employees whose 5-day work period starts on day j , $j = 1, \dots, 7$.

We can make sure that the number of employees required every day of the week is satisfied by introducing a constraint in the model. Each constraint ensures

¹ University of the Basque Country (UPV/EHU), *op. cit.*, pp. 206–208.

that the required number of employees is available each day. We obtain the following IP model:

$$\text{Min } Z = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7$$

subject to:

$$X_1 + X_4 + X_5 + X_6 + X_7 \geq 15$$

$$X_1 + X_2 + X_5 + X_6 + X_7 \geq 13$$

$$X_1 + X_2 + X_3 + X_6 + X_7 \geq 15$$

$$X_1 + X_2 + X_3 + X_4 + X_7 \geq 18$$

$$X_1 + X_2 + X_3 + X_4 + X_5 \geq 14$$

$$X_2 + X_3 + X_4 + X_5 + X_6 \geq 16$$

$$X_3 + X_4 + X_5 + X_6 + X_7 \geq 10$$

$$X_1, X_2, X_3, X_4, X_5, X_6, X_7 \geq 0 \text{ and integer}$$

Any IP may be viewed as the LP relaxation plus additional constraints (the constraints that state which variables must be integers or be 0 or 1). Hence, the LP relaxation is a less constrained, or more relaxed, version of the IP. This means that the feasible region for any IP must be contained in the feasible region for the corresponding LP relaxation. For any IP that is a max problem, this implies that optimal z-value for LP relaxation $\geq$ optimal z-value for IP. This result plays a key role when we discuss the solution of IPs.¹

6-4- Gomory's Cutting Plane Method of Integer Programming Problem

A systematic procedure for solving pure IPP was first developed by R.E. Gomory in 1956, which he later used to deal with the more complicated cases of mixed integer programming problems. This method consists of first solving the IPP as an ordinary LPP by ignoring the restriction of integer values and then introducing a new constraint to the problem such that the new set of feasible solution includes all the original feasible integer solutions, but does not include the optimum non-integer solution initially found. This new constraint is called 'Fractional Cut' or 'Gomorian

¹ Winston, W. L., *Operations research: Applications and algorithms*, 4th ed., Thomson books/Cole, 2004, P. 476

constraint'. Then the revised problem is solved using the simplex method, till an optimum integer solution is obtained.¹

6-4-1 Gomory's Cutting Plane Method steps

Step 1: Convert the minimization IPP into an equivalent maximization LPP. Ignore the integrality condition.

Step 2: Introduce slack and/or surplus variables if necessary to convert the given LPP in its standard form and obtain the optimum solution of the given LPP by using simplex method.

Step 3: Test the integrality of the optimum solution.

(i) If all $X_{Bi} \geq 0$ and are integers, then an optimum integer solution is obtained.

(ii) If all $X_{Bi} \geq 0$ and at least one X_{Bi} is not an integer, then go to the next step.

Step 4: Rewrite each X_{Bi} as $X_{Bi} = [X_{Bi}] + f_i$, where X_{Bi} is the integral part of X_{Bi} and f_i is the positive fractional part of X_{Bi} , $0 < f_i < 1$.

Choose the largest fraction of X_{Bi} i.e., choose $\text{Max}(f_i)$. If there is a tie, then select arbitrarily. Let $\text{Max}(f_i) = f_k$, corresponding to X_{Bk} (the k th row is called the 'source row').

Step 5: Express each negative fraction, if any, in the source row of the optimum simplex table as the sum of a negative integer and a non-negative fraction.

Step 6: Find the Gomory cut constraint or Gomorian constraint.

From the source row,

$$\sum_{j=1}^n a_{kj} X_j = X_{Bk}$$

$$\text{i.e.,} \quad \sum_{j=1}^n ([a_{kj}] + f_{kj} X_j) = [X_{Bk}] + f_k$$

The equation is in the form,

$$\sum_{j=1}^n f_{kj} X_j \geq f_k \Rightarrow - \sum_{j=1}^n f_{kj} X_j \leq -f_k$$

¹ Alagappa University, *op cit*, p. 123

Or,

$$\Rightarrow - \sum_{j=1}^n f_{kj} X_j + G_1 = -f_k$$

Where, G_1 is the Gomorian slack

Step 7: Add the Gomory cut constraint obtained in Step (6) at the bottom of the simplex table obtained in Step (2). Find the new feasible optimum solution using dual simplex method.

Step 8: Go to Step (3) and repeat the procedure until an optimum integer solution is obtained.

Example 3 ¹: Solve the following L.P.P. by Gomory technique.

$$\text{Maximize, } Z = X_1 + X_2$$

subject to:

$$3X_1 + 2X_2 \leq 5$$

$$X_2 \leq 2$$

$$X_1, X_2 \geq 0 \text{ and are integers}$$

Solution : We shall solve this example stepwise, so that the students may understand the procedure.

Step 1: Obtain the optimal solution to the LP problem ignoring the integer value restriction by the simplex method.

Basic Variable	S_1	S_2	Basic Variable Value (RHS)
X_1	$1/3$	$-2/3$	$1/3$
X_2	0	1	2
Z	$1/3$	$1/3$	$7/3$

since all bottom row elements ≥ 0 , the optimal solution of LP problem is:

$$X_1 = 1/3, X_2 = 2 \text{ and Max } Z = 7/2.$$

¹ Gupta, R.K., op. cit, p .462

Step 2: In the current optimal solution, shown in Table 1 all basic variables in the basis (X_B -column) did not assume integer value. Thus solution is not desirable. To obtain an optimal solution satisfying integer value requirement, go to step 3.

Step 3: Since X_1 is the only basic variable whose value is a non-negative fractional value, therefore consider first row (X_1 -row) as source row in Table1 to generate Gomory cut as follows:

$$\frac{1}{3} = X_1 + \frac{1}{3}S_1 - \frac{2}{3}S_2$$

The factoring of numbers (integer plus fractional) in the X_1 -source row gives:

$$\left(0 + \frac{1}{3}\right) = (1 + 0)x_1 + \left(0 + \frac{1}{3}\right)S_1 + \left(-1 + \frac{1}{3}\right)S_2$$

Each of the non-integer coefficients is factored into integer and fractional parts in such a manner that the fractional part is strictly positive.

Rearranging all of the integer coefficients on the left-hand side, we get:

$$\frac{1}{3} + (S_2 - x_1) = \frac{1}{3}S_1 + \frac{1}{3}S_2$$

Since value of variables X_1 and S_2 is assumed to be non-negative integer, left-hand side must satisfy

$$\frac{1}{3} \leq \frac{1}{3}S_1 + \frac{1}{3}S_2$$

$$\frac{1}{3} + S_{g1} = \frac{1}{3}S_1 + \frac{1}{3}S_2$$

or

$$S_{g1} - \frac{1}{3}S_1 - \frac{1}{3}S_2 = -\frac{1}{3} \quad (\text{Cut I})$$

where S_{g1} is the new non-negative (integer) slack variable.

Adding this equation (also called Gomory cut) at the bottom of Table 1, the new values so obtained is shown in Table 2.

Basic Variable	S_1	S_2	Basic Variable Value (RHS)
X_1	$1/3$	$-2/3$	$1/3$
X_2	0	1	2
S_{g1}	$-1/3$	$-1/3$	$-1/3$
Z	$1/3$	$1/3$	$7/2$

Step 4: Since the solution shown in Table 2 is infeasible, apply the dual simplex method to find a feasible as well as an optimal solution. The key row and key column are marked in Table 2. The new solution is obtained by applying the following row operations.

$$R_3(\text{new}) \rightarrow R_3(\text{old}) \times -3; R_1(\text{new}) \rightarrow R_1(\text{old}) - (1/3) R_3(\text{new})$$

The new solution is shown in Table 3.

Basic Variable	S_{g1}	S_2	Basic Variable Value (RHS)
X_1	1	-1	0
X_2	0	1	2
S_1	-3	1	1
Z	1	0	2

since all bottom row elements ≥ 0 and value of basic variables shown in X_B -column of Table 3 is integer, the solution:

$X_1 = 0$, $X_2 = 2$, $S_{g1} = 1$ and $\text{Max } Z = 2$, is an optimal basic feasible solution of the given ILP problem.

6-5-Questions and Exercises

Part 1: Theoretical Questions

A. Short Questions

1. Define integer programming.
2. Explain the importance of integer programming problems.
3. State the applications of integer programming.
4. Analyse the Gomory's cutting method.
5. Define the mixed integer programming problem.
6. Write first four steps of dual simplex algorithm.

B. Multiple Choice Questions

1. The LP relaxation of an integer program is obtained by:

- (a) Removing the objective function
- (b) Replacing integer constraints with non-negativity constraints
- (c) Replacing integrality constraints with continuous constraints
- (d) Adding cutting planes

2. The LP relaxation solution provides what in a maximization IP?

- (a) Lower bound
- (b) Upper bound
- (c) Exact optimal value
- (d) No relationship

3. The feasible region of an integer programming problem is:

- (a) Always convex
- (b) Always non-convex
- (c) Always empty
- (d) Always a polyhedron

4. A binary variable can take values:

- (a) Any real number
- (b) 0 or 1

(c) 0 to 10

(d) Negative values

5. Which statement about integer programming is correct?

(a) Integer programming is always easy to solve

(b) LP solvers can always solve IPs directly

(c) IPs are generally NP-hard

(d) Every IP has an optimal solution

C. True or False

1. Every integer programming problem can be solved by directly applying the simplex method to its LP relaxation. **(T/F)**

2. The optimal objective value of an IP is always at least as good (for maximization problems) as the optimal value of its LP relaxation. **(T/F)**

3. If the constraint matrix is totally unimodular and the RHS vector is integer, then every basic feasible solution of the LP relaxation is integral. **(T/F)**

4. Gomory cutting planes generate cuts that are always facets of the integer hull. **(T/F)**

5. The feasibility of the LP relaxation implies feasibility of the corresponding IP. **(T/F)**

6. In a minimization IP, a dual bound from the LP relaxation is a lower bound on the IP optimal value. **(T/F)**

Part 2: Practice Exercises

Exercise 1: There are four possible projects, which each run for three years and have the following characteristics:

Project	Return	Capital requirements		
		Year1	Year2	Year3
1	0.2	0.5	0.3	0.2
2	0.3	1	0.5	0.2
3	0.5	1.5	1.5	0.3
4	0.1	0.1	0.4	0.1
Available capital		3.1	2.5	0.4

✓ Which projects would you choose in order to maximize the total return?

Exercise 2: There are 6 cities in a district, and there is a project to communicate them by the construction of train stations in some of the cities. A decision has to be made about where to build the train stations. The final solution has to ensure that citizens of all cities can reach a train station in 30 minutes at most.

The objective is to build the minimum number of train stations. The table below shows how long it takes to go from one city to another:

	1	2	3	4	5	6
1	0	35	20	40	30	60
2	35	0	45	35	20	70
3	20	45	0	15	55	20
4	40	35	15	0	65	35
5	30	20	55	65	0	40
6	60	70	20	35	40	0

✓ where to build the train stations?

Exercise3: Find an optimum integer solution to the following IPP. Integer Programming

$$\text{Max } Z = X_1 + 2X_2$$

Subject to,

$$2X_2 \leq 7$$

$$X_1 + X_2 \leq 7$$

$$2X_1 \leq 11$$

$X_1, X_2 \geq 0$ and are integers.

6-6 Solutions and Answers

Part A: Short Questions

1. A linear programming problem in which all or some of the decision variables are constrained to assume non-negative integer values is called an Integer Programming Problem (IPP).

In a Linear Programming Problem (LPP) if all variables are required to take integer values then it is called the Pure (all) Integer Programming Problem (Pure IPP).

2. Integer programming is important because many real-world decision-making problems involve variables that must take integer values rather than fractional values. It enables organizations to model situations involving indivisible units, such as the number of employees, machines, vehicles, or projects. As a result, integer programming provides realistic and practical solutions in areas such as production planning, scheduling, transportation, capital budgeting, facility location, and resource allocation. By ensuring that decision variables satisfy integrality requirements, it helps decision-makers obtain feasible and implementable solutions.

3. Integer programming is applied in business and industry. Assignment problems are special cases of binary integer programming. Classical transportation problems are linear programming models that naturally yield integer solutions when supplies and demands are integers.

Other examples include capital budgeting and production scheduling problems. In fact, any situation involving decisions of the type ‘either to do a job or not’ can be viewed as an IPP. In all such situations,

$X_{ij} = 1$, if the j -th activity is performed.

$X_{ij} = 0$, if the j -th activity is not performed.

In addition, allocation problems involving the allocation of personnel or machines give rise to IPP, since such commodities can be assigned only in integers and not in fractions.

4. A systematic procedure for solving pure IPP was first developed by R.E.

Gomory in 1956, which he later used to deal with the more complicated cases of mixed integer programming problems. The Gomory Cutting Plane Method begins by solving the integer programming problem as a linear programming problem while temporarily ignoring the integrality restrictions. If the optimal solution obtained contains fractional values, a new constraint, known as a Gomory cut, is generated and added to the model. This cut eliminates the current fractional solution without excluding any feasible integer solutions. The revised problem is then solved again, typically using the dual simplex method. The process is repeated until an optimal solution in which all decision variables satisfy the required integer conditions is obtained.

5. In a mixed integer programming problem (MIP), some decision variables are required to take integer values, while the remaining variables are allowed to take continuous (real) values.

Mixed integer cutting plane procedure: The iterative procedure for the solution of mixed integer programming problem is as follows.

6. The dual simplex method is very similar to the regular simplex method, the only difference lies in the criterion used for selecting a variable to enter the basis and to leave the basis. In the dual simplex method, we first select the variable to leave the basis and then the variable to enter the basis. This method yields an optimal solution to the given LPP in a finite number of steps, provided no basis is repeated.

The dual simplex method is used to solve problems that start as dual feasible (i.e., whose primal is optimal but infeasible). In this method, the solution starts as optimum, but infeasible, and remains infeasible until the true optimum is reached, at which point the solution becomes feasible.

Multiple Choice Questions

1. Correct answer: (c)
2. Correct answer: (b)
3. Correct answer: (b)
4. Correct answer: (b)

5. Correct answer: (c)

C. True or False

1. False

2. False

3. True

4. False

5. False

6. True

Exercise 1:

We will use a 0-1 variable X_j for each project:

X_j is 1 if we decide to do project j ;

X_j is 0 otherwise (i.e. not do project j).

This leads to the 0-1 programming problem:

$$\begin{aligned}
 \text{Max} \quad & 0.2 X_1 + 0.3 X_2 + 0.5 X_3 + 0.1 X_4 \\
 \text{s.t.} \quad & 0.5 X_1 + 1 X_2 + 1.5 X_3 + 0.1 X_4 \leq 3.1 \\
 & 0.3 X_1 + 0.5 X_2 + 1.5 X_3 + 0.4 X_4 \leq 2.5 \\
 & 0.2 X_1 + 0.2 X_2 + 0.3 X_3 + 0.1 X_4 \leq 0.4 \\
 & X_j = 0 \text{ or } 1 \quad j = 1, \dots, 4
 \end{aligned}$$

Exercise 2:

We define a binary variable for each city j , $j = 1, \dots, 6$

$$x_j(1,2,3,4,5,6) = \begin{cases} 1 & \text{if a train station is built in the city} \\ 0 & \text{other wise} \end{cases}$$

The 0-1 IP model is:

$$\begin{aligned}
 \text{Min } Z = & X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \\
 \text{subject to} \quad & X_1 + X_3 + X_5 \geq 1 \\
 & X_2 + X_5 \geq 1 \\
 & X_1 + X_3 + X_4 + X_6 \geq 1 \\
 & X_3 + X_4 \geq 1 \\
 & X_1 + X_2 + X_5 \geq 1 \\
 & X_3 + X_6 \geq 1 \\
 & X_1, X_2, X_3, X_4, X_5, X_6 = 0 \text{ or } 1
 \end{aligned}$$

Each constraint refers to a city, and ensures that at least one train station will be located no further than a 30 minute drive from each city.

Exercise3:

Find an optimum integer solution to the following IPP. Integer Programming

$$\begin{aligned}
 &\text{Max } Z = X_1 + 2X_2 \\
 &\text{Subject to,} \quad 2X_2 \leq 7 \\
 &\quad \quad \quad X_1 + X_2 \leq 7 \\
 &\quad \quad \quad 2X_1 \leq 11 \\
 &\quad \quad \quad X_1, X_2 \geq 0 \text{ and are integers.}
 \end{aligned}$$

Solution: Introducing slack variables $S_1, S_2, S_3 \geq 0$, we get,

$$\begin{aligned}
 &\text{Max } Z = X_1 + 2X_2 + 0S_1 + 0S_2 + 0S_3 \\
 &\text{Subject to,} \quad 2X_2 + S_1 = 7 \\
 &\quad \quad \quad X_1 + X_2 + S_2 = 7 \\
 &\quad \quad \quad 2X_1 + S_3 = 11 \\
 &\quad \quad \quad X_1, X_2 \geq 0 \text{ and are integers. } S_1, S_2, S_3 \geq 0
 \end{aligned}$$

Ignoring the integrality condition, we get the optimum solution of the given IPP with initial basic feasible solution obtained by putting: $X_1 = 0$ and $X_2 = 0$ as $S_1 = 7$, $S_2 = 7$ and $S_3 = 11$.

Basic Variable	X_1	X_2	Basic Variable Value (RHS)	Replacement Ratio
S_1	0	2	7	$7/2=3.5$ ←
S_2	1	1	7	$7/1=7$
S_3	2	0	11	-
Z	-1	-2	0	0



Basic Variable	X_1	S_1	Basic Variable Value (RHS)	Replacement Ratio
X_2	0	$1/2$	$7/2$	-
S_2	1	$-1/2$	$7/2$	$7/2/1=3.5$ ←
S_3	2	0	11	$11/2=5.5$
Z	-1	1	7	



-1 /2	S ₂	S ₁	Basic Variable Value (RHS)	Replacement Ratio
X ₂	0	1 /2	7 /2	
X ₁	1	-1 /2	7 /2	
S ₃	-2	1	4	
Z	1	1 /2	21 /2	

Since the optimum solution obtained above is not an integer, we now select a constraint corresponding to,

$$\text{Max } (f_i) = \text{Max } (f_1, f_2, f_3)$$

$$X_1 = 7/2 = 3 + 1/2$$

$$X_2 = 7/2 = 3 + 1/2$$

$$S_3 = 4 = 4 + 0$$

$$\text{Max } (f_i) = \text{Max } \left(\frac{1}{2}, \frac{1}{2}, 0 \right) = \frac{1}{2}$$

Since the Max fraction is same for both X₁ and X₂ rows, we choose X₁ row as the source row arbitrarily. From this row we have,

$$7/2 = X_1 - 1/2 S_1 + 1S_2$$

On expressing the negative fraction as a sum of negative integer and a positive fraction, we have,

$$3 + 1/2 = X_1 + (-1 + 1/2) S_1 + 1S_2$$

The Gomorian constraint is given by,

$$1/2 S_1 \geq 1/2$$

i.e.,

$$-1/2 S_1 \leq -1/2 \Rightarrow -1/2 S_1 + G_1 = -1/2$$

Here, G₁ is the Gomory slack variable. Adding this new constraint at the bottom of the

above optimal simplex table, we get a new table.

We apply dual simplex method. Since G₁ = -1/2, G₁ leaves the basis.

Entering variable is given by,

$$\text{MAX} \left\{ \frac{\text{coefficients in the Z - row}}{a_{ij}}, a_{ij} < 0 \right\} = \text{Max} \left\{ \frac{1/2}{-1/2} \right\}$$

This gives the non-basic variable S_1 to enter into the basis. Remove G_1 from the basis and introduce S_1 into the basis.

-1 / 2	S_2	S_1	Basic Variable Value (RHS)	Replacement Ratio
X_2	0	1 / 2	7 / 2	←
X_1	1	-1 / 2	7 / 2	
S_3	-2	1	4	
G_1	0	-1 / 2	-1 / 2	
Z	1	1 / 2	21 / 2	

-1 / 2	S_2	G_1	Basic Variable Value (RHS)	Replacement Ratio
X_2	0	1	3	
X_1	1	-1	4	
S_3	-2	2	3	
S_1	0	-2	1	
Z	1	1	10	

Since all coefficients in the Z-row are non-negative, an optimum solution has been obtained in integers. Hence,

the integer optimum solution is given by,

$\text{Max } Z = 10, x_1 = 4, x_2 = 3$

Chapter 7: Transportation Problems

Learning Objectives

After completing this chapter, students should be able to:

1. Recognize and formulate a transportation problem.
2. Derive an initial feasible solution using various methods.
3. Obtain the optimal solution using the Modified Distribution (MODI) method.
4. Handle degenerate and unbalanced transportation problems.
5. Solve profit-maximization transportation problems by applying appropriate modifications to the transportation algorithm.

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7-1- Introduction

The transportation problem (TP) is a special type of linear programming problem that deals with the distribution of a single product from several sources (such as factories or warehouses) to several destinations (such as markets or retail stores) while minimizing the total transportation cost. This type of problem is particularly important in economics, logistics, and supply chain management, as it allows decision-makers to allocate resources efficiently and reduce operational costs.¹

This model has significant applications in economics and finance, particularly in areas related to supply chain management, distribution planning, resource allocation, and even cash distribution in banking systems.²

There are various types of transportation models and the simplest of them was first presented by F L Hitchcock (1941). It was further developed by T C Koopmans (1949) and G B Dantzig (1951). Several extensions of transportation models and methods have been subsequently developed.³

7-2 General Mathematical Model of Transportation Problem:

The transportation model considers three key factors:

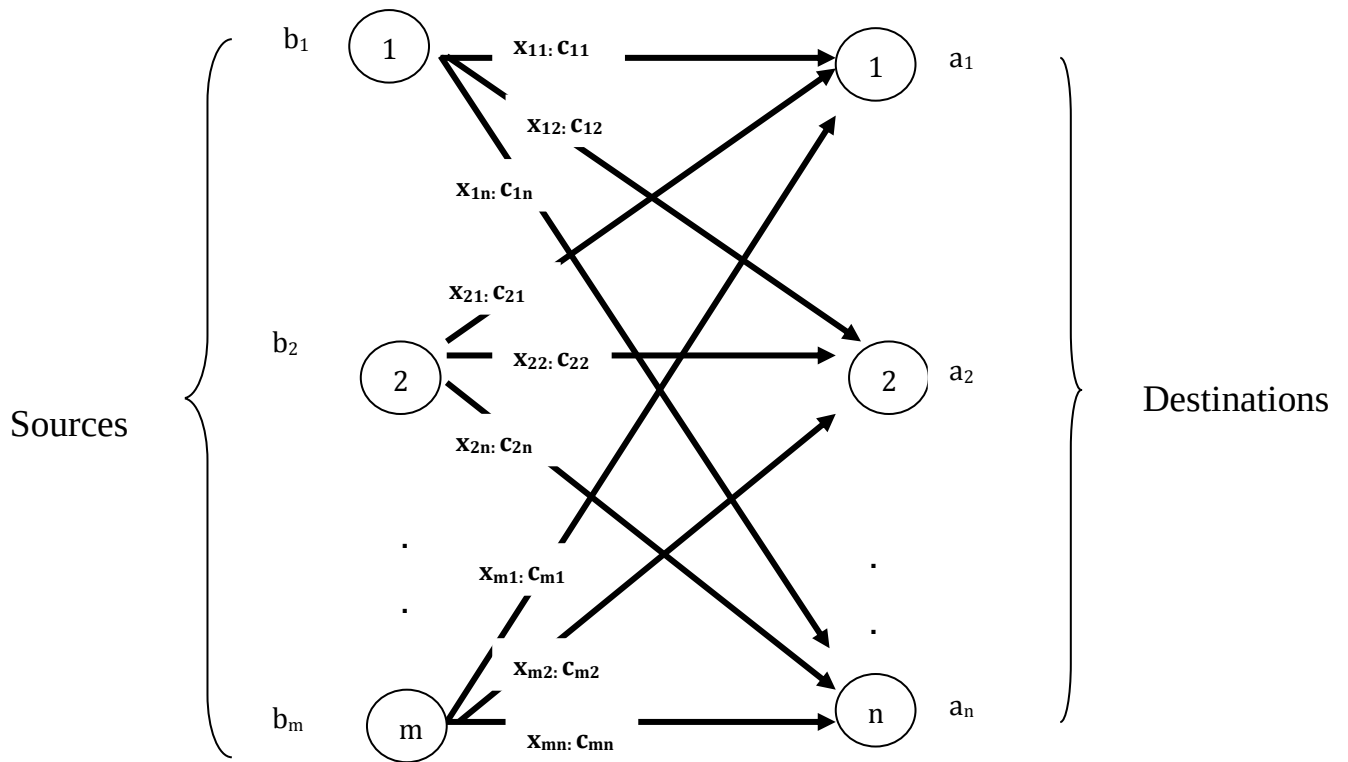
- ✓ The available supply at each source.
- ✓ The demand required at each destination.
- ✓ The unit cost of transportation from each source to each destination

The transport model can be visualized as a network consisting of m group of supply centers (sources) $i=1, \dots, m$ and n group of receiving centers (destinations) $j=1, \dots, n$ As follows:

¹ Taha, H. A. (2017). *Operations Research: An Introduction* (10th ed.). Pearson

² Chopra, S. & Meindl, P. *Supply Chain Management: Strategy, Planning, and Operation*. 7th ed, Pearson, 2019

³ Sharma, J. K, *op. cit.*, p. 257



Let there be m sources of supply, $S_1, S_2, \dots, S_m$ having a_i ($i = 1, 2, \dots, m$) units of supply (or capacity), respectively to be transported to n destinations, $D_1, D_2, \dots, D_n$ with b_j ($j = 1, 2, \dots, n$) units of demand (or requirement), respectively. Let c_{ij} be the cost of shipping one unit of the commodity from source i to destination j . If x_{ij} represents number of units shipped from source i to destination j , the problem is to determine the transportation schedule so as to minimize the total transportation cost while satisfying the supply and demand conditions. Mathematically, the transportation problem, in general, may be stated as follows:¹

$$\text{Min}Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij}$$

Subject to:

¹Sharma, J. K, Op.cit, p. 258

The sum of the amounts transferred from a given source. (*supply constraints*)

$$\sum_{j=1}^n x_{ij} \leq b_i \quad i = 1, 2 \dots m$$

Total quantities transported to the i place of request. (*demand constraints*)

$$\sum_{i=1}^m x_{ij} \geq a_j \quad j = 1, 2 \dots n$$

$$x_{ij} > 0 \text{ for all } i \text{ and } j \text{ (non negativity constraints)}$$

The first set of restrictions requires that the sum of the amounts transferred from the given source cannot exceed the available supply. Similarly, the second set of constraints requires that the sum of the quantities transported to a particular demand location must satisfy the needs of that demand.

When total supply equals total demand:

$$\sum_{i=1}^m b_i = \sum_{j=1}^n a_j$$

then total supply equals total demand, and the problem is said to be a balanced transportation problem.¹

$$\sum_{j=1}^n x_{ij} = b_i \quad \sum_{i=1}^m x_{ij} = a_j$$

¹ Winston, W.L, *op. cit*, p.363

7-2- Matrix Representation for the transportation problem

The relevant data for any transportation problem can be summarized in a matrix format using a tableau called the transportation costs tableau .The tableau displays the origins with their supply, the destinations with their demand and the transportation per-unit costs.

Source	Destination				Supply
	1	2	...	n	
1	c_{11} x_{11}	c_{12} x_{12}	...	c_{1n} x_{1n}	b_1
2	c_{21} x_{21}	c_{22} x_{22}	...	c_{2n} x_{2n}	b_2
...	...	...	...	...	...
m	c_{m1} x_{m1}	c_{m2} x_{m2}	...	c_{mn} x_{mn}	b_m
Demand	a_1	a_2	...	a_n	TC =

x_{11} : Number of units transferred from source 1 to destination 1

x_{12} : Number of units transferred from source 1 to destination 2

x_{1n} : Number of units transferred from source 1 to destination n

x_{21} : Number of units transferred from the source 2to destination 1

x_{22} : Number of units transferred from the source 2 to destination 2

x_{2n} : Number of units transferred from the source 2 to destination n

x_{m1} : Number of units transferred from the source m to destination 1

x_{m2} : Number of units transferred from the source m to destination 2

x_{mn} : Number of units transferred from the source m to destination n

7-3- Solving Balanced Transportation Problems (Optimality Test)

The basic steps of the transportation algorithm are exactly those of the simplex method. However, instead of using the regular simplex tableau, we take advantage of the special structure of the transportation model to carry out the algorithmic computations more conveniently.¹

If:

$$\sum_{i=1}^m b_i = \sum_{j=1}^n a_j \quad (\text{Supply} = \text{Demand})$$

It follows that the transport model includes an independent and effective excess equation, so the initial solution must include $n+m-1$ is a dependent equation with $n+m-1$ basic variables as in the simplex method.

7-3-1 Basic Feasible Solution of a Transportation Problem:

The initial basic solution (obtaining a first practical table) can be obtained by following one of two methods:

a) North West – Corner Method:

The North-West Corner Method (NWCN) is one of the simplest techniques for finding an initial basic feasible solution (IBFS) to a transportation problem. The method starts allocations from the "north-west" (top-left) corner of the cost matrix and proceeds systematically to satisfy supply and demand constraints. Although the method does not guarantee an optimal solution, it provides a straightforward and quick initial.²

The steps of the North-West Corner Method are as follows:

¹ Taha, H. A. *Op .cit*, p.211

² Ibid, pp. 325–326

Step 1: Choose the cell in the north-west corner of the transportation problem table and allocate as much as possible in this cell so that either the capacity of first row (supply) is exhausted or the destination requirement of the first column.

(demand) is exhausted. (i.e) $x_{11} = \min(a_1, b_1)$

Step 2: If the demand is exhausted ($b_1 < a_1$), move one cell right horizontally to the second column and allocate as much as possible. (i.e) $x_{12} = \min(a_1 - x_{11}, b_2)$.

If the supply is exhausted ($b_1 > a_1$), move one cell down vertically to the second row and allocate as much as possible. (i.e) $x_{21} = \min(a_2, b_1 - x_{11})$.

If both supply and demand are exhausted move one cell diagonally and allocate as much as possible.

Step 3: Continue the above procedure until all the allocations are made.

In this way, move horizontally until a supply source is exhausted, then vertically down until a destination demand is satisfied and diagonally, when the demand at the destination matches exactly the supply available, until the south east corner is reached

Remark; If during the process of making allocation at a particular cell, the supply equals demand, then the next allocation of magnitude zero can be made in a cell either in the next row or column. This condition is known as degeneracy.

Example1:

The Algerian National Grain Office has three (3) silos for storing durum wheat (W, X, Y). These are allocated to cover the needs of four (4) mills (A, B, C, and D), as the quantities available in the silos per day were estimated at 900, 900, and 1300 tons, respectively, while the mills' wheat needs per day were estimated at 1000, 1000, 700, and 400 tons, respectively. The costs of transporting one ton of wheat from a silo to a mill are shown in the following table:

	A	B	C	D
W	6	4	7	8
X	4	2	3	6
Y	3	5	5	6

Unit: €

Production centers	warehouses				Supply
	A	B	C	D	
W	6 900	4	7	8	900
X	4 100	2 800	3	6	900
Y	3	5 200	5 700	6 400	1300
Demand	1000	1000	700	400	

$$TC = 900(6) + 100(4) + 800(2) + 200(5) + 700(5) + 400(6) = \text{€}14300$$

b) Least Cost Method (LCM) or Minimum Cost Method :

Since the main objective is to minimize the total transportation cost, transport as much as possible through those routes (cells) where the unit transportation cost is lowest. This method takes into account the minimum unit cost of transportation for obtaining the initial solution and can be summarized as follows:¹

Step 1: Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row or column in which either the supply or demand is fulfilled. If a row and a column are both satisfied simultaneously then crossed off either a row or a column.

In case the smallest unit cost cell is not unique, then select the cell where the maximum allocation can be made.

Step 2: Cross out the row or column that has been satisfied (supply exhausted or demand fulfilled).

Step 3: After adjusting the supply and demand for all uncrossed rows and columns repeat the procedure to select a cell with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as

¹ Sharma, J. K, *Op.cit*, p. 261

possible to this cell. Then crossed off that row and column in which either supply or demand is exhausted.

Step 4: Repeat the procedure until the available supply at various sources and demand at various destinations is satisfied. The solution so obtained need not be non-degenerate.

Production centers	warehouses				Supply
	A	B	C	D	
W	6 100	4 400	7 400	8 400	900
X	4 900	2	3	6	900
Y	3 1000	5	5 300	6	1300
Demand	1000	1000	700	400	

$$TC = 100(4) + 400(7) + 400(8) + 900(2) + 1000(3) + 300(5) = \text{€}12700$$

7-3- 2 Finding the optimal solution to transportation problems (Optimality Test):

Once an initial solution is obtained, the next step is to check its optimality in terms of feasibility of the solution and total minimum transportation cost.

The test of optimality begins by calculating an opportunity cost associated with each unoccupied cell (represents unused route) in the transportation table. An unoccupied cell with the largest negative opportunity cost is selected to include in the new set of transportation routes (allocations). This value indicates the per unit cost reduction that can be achieved by making appropriate allocation in the unoccupied cell. This cell is also known as an incoming cell (or variable). The outgoing cell (or variable) from the current solution is the occupied cell (basic variable) where allocation will become zero as allocation is made in the unoccupied cell with the largest negative opportunity cost. Such an exchange reduces the total transportation

cost. The process is continued until there is no negative opportunity cost. That is, the current solution is an optimal solution.¹

The *Modified-distribution* (MODI) method (also called *u-v method* or *method of multipliers*) is used to calculate opportunity cost associated with each unoccupied cell and then improving the current solution leading to an optimal solution.

Steps of MODI Method (Transportation Algorithm)

To get the optimal solution, we follow the following steps:²

Step 1: For occupied cells; splitting the transportation cost C_{ij} into two parts:

Cost borne by the source (shipping cost): U_i

Cost borne by the destination (unloading cost): V_j

For an initial basic feasible solution with $m + n - 1$ occupied cells, calculate U_i and V_j for rows and columns. The initial solution can be obtained by any of the two methods discussed earlier.

Assuming the shipping cost for the first source = 0, The value of U_i and V_j for other rows and columns is calculated by using the relationship

$$C_{ij} = U_i + V_j \text{ for all occupied cells } (i, j).$$

Step 2: For unoccupied cells; calculate the opportunity cost (λ_{ij}) by using the relationship:

$$\lambda_{ij} = C_{ij} - (U_i + V_j) \text{ for all } i \text{ and } j.$$

Step 3: Examine sign of each λ_{ij} , If the opportunity cost is greater than or equal to zero ($\lambda_{ij} \geq 0$) for all unoccupied cells, we have obtained the optimal solution.

¹ Sharma, J. K, Op.cit, p. 266

² Ibid, p. 268

However, if at least one $\lambda_{ij} < 0$, the solution is not optimal and further saving in transportation cost is possible and we try to operate the cells with the Largest negative opportunity cost in absolute value.

Step 4: Construct a closed-path (or loop) for the unoccupied cell with largest negative value of λ_{ij} . Start the closed path with the selected unoccupied cell and mark a plus sign (+) in this cell. Trace a path along the rows (or columns) to an occupied cell, mark the corner with a minus sign (–) and continue down the column (or row) to an occupied cell. Then mark the corner with plus sign (+) and minus sign (–) alternatively. Close the path back to the selected unoccupied cell.

Step 5: Select the smallest quantity amongst the cells marked with minus sign on the corners of closed loop. Allocate this value to the selected unoccupied cell, add it to occupied cells marked with plus signs, and subtract it from the occupied cells marked with minus signs.

Step 6: Obtain a new improved solution by allocating units to the unoccupied cell according to Step 5 and calculate the new total transportation cost.

Step 7: Test optimality of the revised solution. The procedure terminates when all $\lambda_{ij} \geq 0$ for unoccupied.

Remarks: *The closed-loop (path) starts and ends at the selected unoccupied cell. It consists of successive horizontal and vertical (connected) lines whose end points must be occupied cells, except an end point associated with entering unoccupied cell. This means that every corner element of the loop must be an occupied cell.*

1. It is immaterial whether the loop is traced in a clockwise or anti-clockwise direction and whether it starts up, down, right or left (but never diagonally). However, for a given solution only one loop can be constructed for each unoccupied cell.

2. There can only be one plus (+) sign and only one minus (–) sign in any given row or column

Chapter 7: Transportation Problems

Applying the previous example, the equations associated with the basic variables (occupied cells) are as follows:

* Split the cost into two parts for occupied cells:

We always assume that the shipping cost to the first source = 0 ($U(W)=0$).

Thus, the values of the remaining costs become as follows:

$$W:B \quad U(W) + V(B) = 4 \Rightarrow V(B) = 4$$

$$W:C \quad U(W) + V(C) = 7 \Rightarrow V(C) = 7$$

$$W:D \quad U(W) + V(D) = 8 \Rightarrow V(D) = 8$$

$$X:B \quad U(X) + V(B) = 2 \Rightarrow U(X) = -2$$

$$Y:C \quad U(Y) + V(C) = 5 \Rightarrow U(Y) = -2$$

$$Y:A \quad U(Y) + V(A) = 3 \Rightarrow V(A) = 5$$

$$X_{12} = U_1 + V_2 = C_{11} = 4 \Rightarrow V_2 = 4$$

$$X_{13} = U_1 + V_3 = C_{13} = 7 \Rightarrow V_3 = 7$$

$$X_{14} = U_1 + V_4 = C_{14} = 8 \Rightarrow V_4 = 8$$

$$X_{22} = U_2 + V_2 = C_{22} = 2 \Rightarrow U_2 = -2$$

$$X_{33} = U_3 + V_3 = C_{33} = 5 \Rightarrow U_3 = -2$$

$$X_{31} = U_3 + V_1 = C_{33} = 3 \Rightarrow V_1 = 5$$



Stage (2) unoccupied cells can be evaluated and find the opportunity cost λ_{ij} as follows:

$$\lambda_{11} = C_{11} - (U_1 + V_1) = 6 - 5 = 1$$

$$\lambda_{21} = C_{21} - (U_2 + V_1) = 4 - 3 = 1$$

$$\lambda_{23} = C_{23} - (U_2 + V_3) = 3 - 5 = -2$$

$$\lambda_{24} = C_{24} - (U_2 + V_4) = 6 - 6 = 0$$

$$\lambda_{32} = C_{32} - (U_3 + V_2) = 5 - 2 = 3$$

$$\lambda_{34} = C_{34} - (U_3 + V_4) = 6 - 6 = 0$$

$$W:A \quad \lambda_{11} = 6 - 5 = 1$$

$$X:A \quad \lambda_{21} = 4 - 3 = 1$$

$$X:C \quad \lambda_{23} = 3 - 5 = -2$$

$$X:D \quad \lambda_{24} = 6 - 6 = 0$$

$$Y:B \quad \lambda_{32} = 5 - 2 = 3$$

$$Y:C \quad \lambda_{34} = 6 - 6 = 0$$



We start searching for the optimal solution from the first practical table obtained using the northwest corner method.

T1	5 A	4 B	7 C	8 D	Supply
0 W	6 5 1	4 100 +	7 400 -	8 400	900
-2 X	4 3 1	2 900 -	3 5 -2 +	6 6 0	900
-2 Y	3 1000	5 2 3	5 300	6 6 0	1300
Demand	1000	1000	700	400	TC1=12700

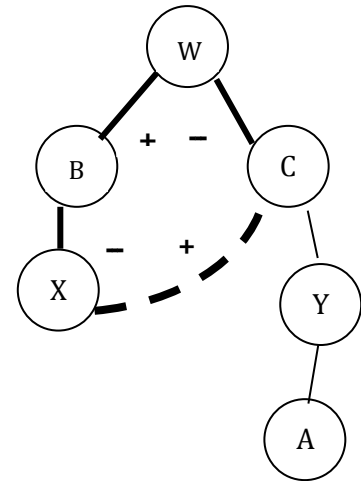


Figure 1

At this stage, we observe that cell X–C has the largest negative opportunity cost in absolute value. Therefore, to reduce the total transportation cost (TC), this cell should be introduced into the solution by allocating a certain quantity from source X to destination C.

This leads to two important questions. First, which cells will be affected when cell X–C enters the solution? Second, what is the maximum quantity that can be allocated to this cell?

To answer the first question, a closed loop is constructed by connecting the occupied cells. As shown in Figure 1, the allocations in cells marked with a positive (+) sign will increase, while those in cells marked with a negative (–) sign will decrease.

As for the second question, the quantity to be reallocated along the closed loop is determined by identifying the smallest allocation among the cells marked with a negative (–) sign. According to Figure 1, there are two such cells: W–C and X–B. Their current allocations are 400 and 900 units, respectively. Since the smallest allocation is 400 units, the quantity to be reallocated is 400 units. Consequently, each cell marked with a positive (+) sign will increase by 400 units, while each cell marked with a negative (–) sign will decrease by 400 units.

The total cost will decrease by $(400 \times (-2))$, or €800. We obtain the second table, T2, below. Since the MODI method is iterative, we will go back through the previous three steps to determine whether Table T2 represents the optimal solution.

T2	3 A	4 B	5 C	8 D	Supply
0 W	6 3 3	4 500 +	7 5 2	8 400 -	900
-2 X	4 1 3	2 500 -	3 400 +	6 6 0	900
0 Y	3 1000 4 1	5 4 1	5 300 -	6 8 -2 +	1300
Demand	1000	1000	700	400	TC2=11900

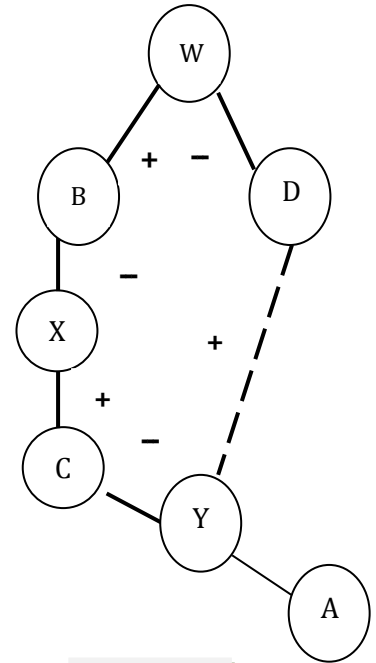


Figure 2

$$TC2 = TC1 - 2 (400) = \text{€}11900$$

The opportunity cost of cell Y–D is –2, which is the largest negative opportunity cost among all unoccupied cells. Therefore, this cell is selected to enter the basis. This means that for every unit reallocated through the corresponding closed loop, the total transportation cost will be reduced by €2.

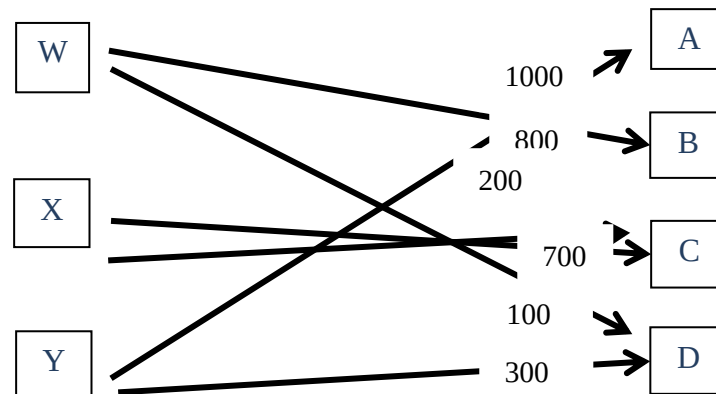
Following the same procedure, successive transportation tables are generated and evaluated until the optimal solution is reached.

Opt T	5 A	4 B	5 C	8 D	Supply
0 W	6 5 1	4 800	7 5 2	8 100	900
-2 X	4 3 1	2 200	3 700	6 6 0	900
-2 Y	3 1000	5 2 3	5 3 2	6 300	1300
Demand	1000	1000	700	400	TC2=11300

$$TC3 = TC2 - 2 (300) = \text{€}11300$$

The solution is optimal because no unoccupied cell has a negative opportunity cost.

Optimal transportation plan:



Remark : The cell Y: A has an opportunity cost of zero ($\lambda_{24}=0$), which means that there are alternative optimal solutions.

7-4-Unbalanced Transportation Problems

An unbalanced transportation problem occurs when the total supply is not equal to the total demand in a transportation problem. In other words, the sum of the supplies at all sources does not match the sum of the demands at all destinations:¹

(a) If the total supply exceeds the total demand, then an additional column (called a dummy demand center) can be added to the transportation table in order to absorb the excess supply. The unit transportation cost for the cells in this column is set equal to zero because these represent product items that are neither made nor sent.

(b) If the total demand exceeds the total supply, a dummy row (called a dummy supply center) can be added to the transportation table to account for the excess demand quantity. The unit transportation cost in such a case also, for the cells in the dummy row is set equal to zero.

¹ Sharma, J. K, Op.cit, p. 281

Example 2: An Unbalanced Transportation Problem (Supply Deficit)

T1	4	10	13	Supply
A	B	C		
0 W	4 60 -	10 10 -2	12 13 -1	60
6 X	10 40 +	17 16 1	19 40 -	80
2 Y	13 6 7	12 100 -	15 10 +	110
-13 Z*	0 -9 9	0 -3 3	0 50	50
Demand	100	100	100	TC1 = 2750

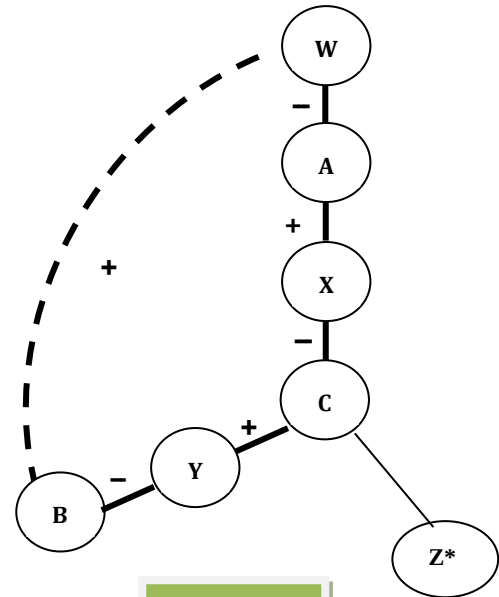


Figure 4

Here, the total supply is $60 + 80 + 110 = 250$, and the total demand is:

$100 + 100 + 100 = 300$. Since demand exceeds supply,

To balance the problem, we need to add a dummy source to absorb the excess demand. The excess demand is calculated as:

$$300(\text{total demand}) - 250(\text{total supply}) = 50$$

Remark: When starting the solution with the least cost method, it is preferable that the cells in the dummy source that have zero cost remain the last to be allocated the transferred quantities.

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T2	4	8	11	Supply
	A	B	C	
0 W	4 20	8 40	12 11 1	60
6 X	10 80	17 14 3	19 17 2	80
4 Y	13 8 5	12 60	15 50	110
-11 Z*	0 -7 7	0 -3 3	0 50	50
Demand	100	100	100	TC2=2670

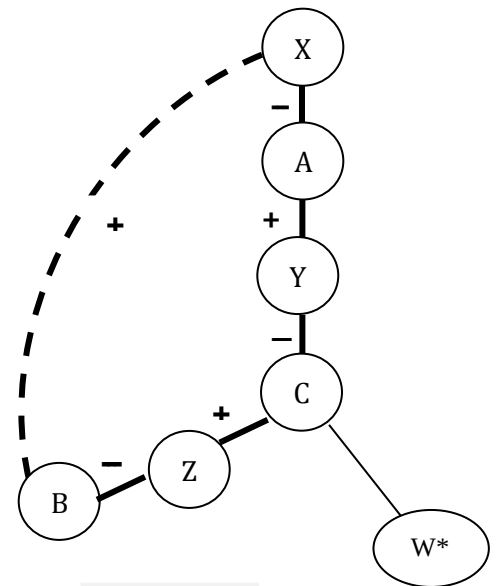
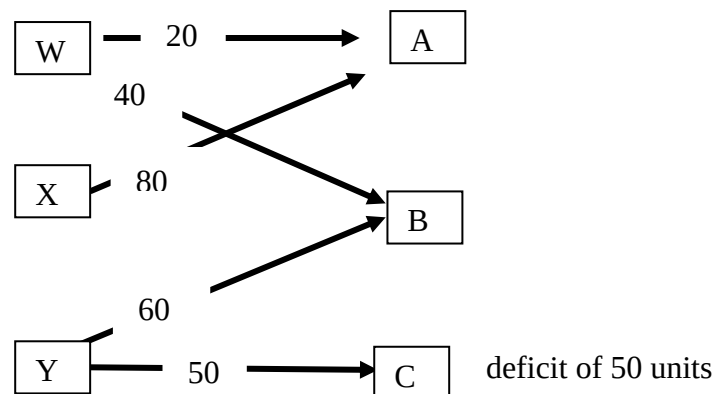


Figure 5

$$TC2 = TC1 - (2 \times 40) = 2670$$

Best transportation plan :



Example 3: An Unbalanced Transportation Problem (Excess Supply)

T1	4 A	8 B	11 C	-8 D*	Supply
0 W	4 60 -	8 40 +	12 11 1	0 -8 8	100
8 X	10 12 -2	17 16 1	19 50 -	0 50	100
4 Y	13 8 5	12 40 -	15 60 +	0 -4 4	100
Demand	60	80	110	50	TC1 = 2890

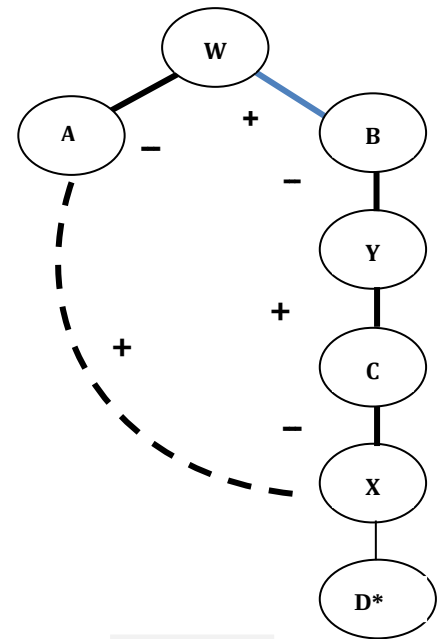


Figure 6

Here, the total supply is $100 + 100 + 100 = 300$, and the total demand is:

$60 + 80 + 110 = 250$. Since supply exceeds demand,

To balance the problem, we need to add a dummy destination to absorb the excess supply. The excess supply is calculated as: $300(\text{total supply}) - 250(\text{total demand}) = 50$

T2	4 A	8 B	13 C	-6 D*	Supply
0 W	4 20 -	8 80	12 13 -1	0 -6 8	100
6 X	10 40 +	17 14 3	19 10 -	0 50	100
2 Y	13 6 7	12 10 2	15 100	0 -4 4	100
Demand	60	80	110	50	TC2 = 2810

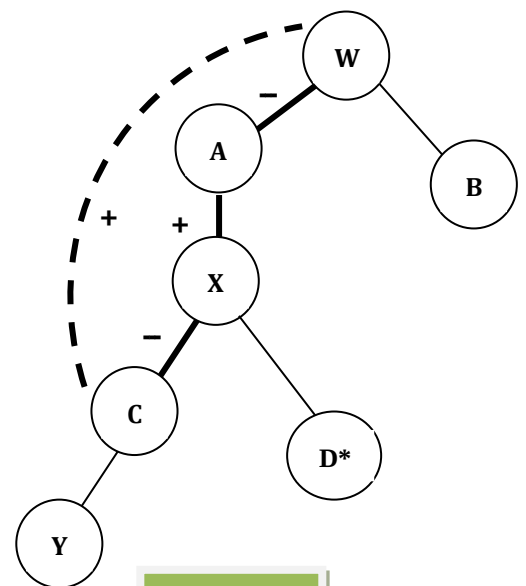


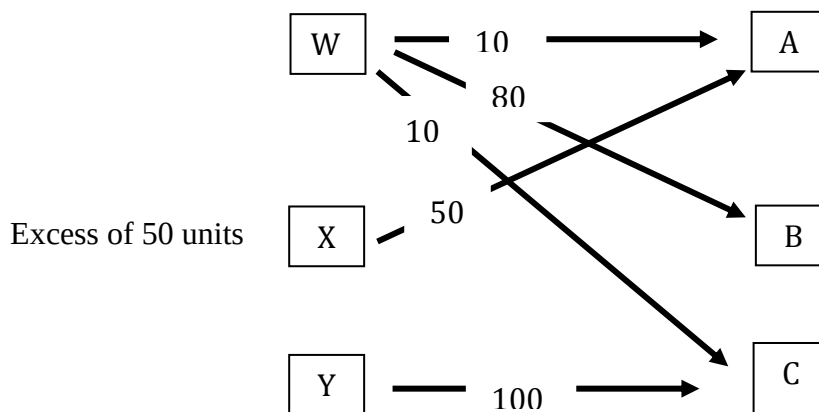
Figure 7

$$TC2 = TC1 - (2 \times 40) = 2810$$

Opt T	4 A	8 B	12 C	-6 D*	Supply
0 W	4 10	8 80	12 10	0 -6 6	100
6 X	10 50	17 14 3	19 18 1	0 50	100
3 Y	13 7 8	12 11 1	15 100	0 -3 3	100
Demand	60	80	110	50	TC3= 2800

$$TC3 = TC2 - (1 \times 10) = 2800$$

Best transportation plan :



7-5-Degeneracy and Its Resolution

A basic feasible solution for the general transportation problem must have exactly $m + n - 1$ (number of rows + number of columns – 1) positive allocations in the transportation table. If the number of occupied cells is less than the required number, $m + n - 1$, then such a solution is called a degenerate solution. In such cases, the current solution cannot be improved further because it is not possible to draw a closed path for every occupied cell. Also, the values of dual variables u_i and v_j that are used to test the optimality cannot be computed. Thus, degeneracy needs to be

removed in order to improve the given solution. The degeneracy in the transportation problems may occur at two stages:¹

- ✓ At initial basic feasible solution the number of occupied cells may be less than $m + n - 1$ allocations.
- ✓ At any stage while moving towards optimal solution two or more occupied cells may become simultaneously unoccupied.

To resolve degeneracy at the initial solution, allocate a very small quantity close to zero to one or more (if needed) unoccupied cells so as to get $m + n - 1$ number of occupied cells. This amount is denoted by a Greek letter ϵ (epsilon). This quantity would neither affect the total cost nor the supply and demand values. In a minimization transportation problem it is better to allocate ϵ to unoccupied cells that have lowest transportation costs, whereas in maximization problems it should be allocated to a cell that has a high payoff value. In some cases, ϵ must be added in one of those unoccupied cells that may help in the determination of u_i and v_j values.²

The quantity ϵ is considered to be so small that if it is transferred to an occupied cell it does not change the quantity of allocation.

The value of ϵ does not affect either the transportation cost or the supply and demand constraints. Hence, the quantity ϵ is used to evaluate unoccupied cells and to reduce the number of improvement cycles necessary to reach an optimal solution. Once the purpose is over, ϵ can be removed from the transportation table.³

Remark: *Epsilon is so small so that if it is added or subtracted from any number, it does not change the numerical value of the number for which it added or from which it is subtracted.*⁴

¹ Sharma, J. K, Op.cit, p. 283

² Ibid, P.184

³ Ibid

⁴ Rama Murthy, P. *Operations Research*. 2nd edn. New Age International Publishers, 2007., P.165

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Example 4: Consider the following transportation problem where the origins are plants and destinations are depots.

	A	B	C	D	Supply
W	12	15	11	0	300
X	15	17	10	0	
Y	10	8	14	0	300
Z	13	10	15	0	
Demand	500	300	400	100	

Solution:

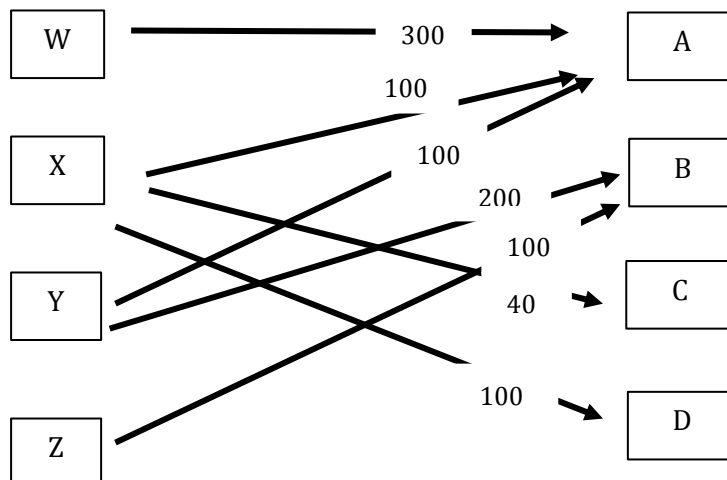
T1	12 A	10 B	7 C	-3 D	Supply
0 W	12 300	15 10 5	11 7 4	0 -3 3	300
3 X	15 100	17 13 4	10 400	0 100	
-2 Y	10 + E	8 300 -	14 5 9	0 -5 5	300
1 Z	13 100 -	10 11 -1 +	15 8 7	0 -2 2	
Demand	500	300	400	100	TC1=12800

T2	12 A	10 B	7 C	-3 D	Supply
0 W	12 300	15 10 5	11 7 4	0 -3 3	300
3 X	15 100	17 13 4	10 400	0 100	600
-2 Y	10 100	8 200	14 5 9	0 -5 5	300
0 Z	13 12 1	10 100	15 7 8	0 -3 3	100
Demand	500	300	400	100	TC2=12700

Lowest total shipping cost

$$300(12)+100(15)+400(10)+100(10+200(8)+100(100)=12700$$

Best transportation plan :



7-6-Maximization Transportation Problems

In general, the transportation model is used for cost minimization problems. However, it may also be used to solve problems in which the objective is to maximize total profit. That is, instead of unit cost c_{ij} , the unit profit or payoff p_{ij} associated with each route, (i, j) is given. The objective function in terms of total profit (or payoff) is then stated as follows:

$$\text{Maximize } Z = \sum_{i=1}^m \sum_{j=1}^n P_{ij} x_{ij}$$

These kinds of problems can be solved by converting the maximization problem into minimization problem. The conversion of maximization into minimization is done by subtract each profit value from the largest profit value in the table. The maximization of transportation problem is illustrated with the following Example:

Example 5: A company operates three factories located in cities X, Y, and Z. The factories Supply products to four dealers, A, B, C, and D, located throughout the country. The monthly production capacities of the factories are 1000, 600, and 1000 units, respectively, while the monthly demands of the dealers are 900, 800, 500, and 400 units, respectively. The net return per unit shipped from each factory to each dealer is given in the following table:

:

	Dealers				Supply
	A	B	C	D	
	6	6	6	4	
	4	2	4	5	
	5	6	7	8	
Demand	900	800	500	400	

- ✓ Determine a suitable allocation to maximize the total return.

This is a maximization problem. Hence first we have to convert this in to minimization problem.

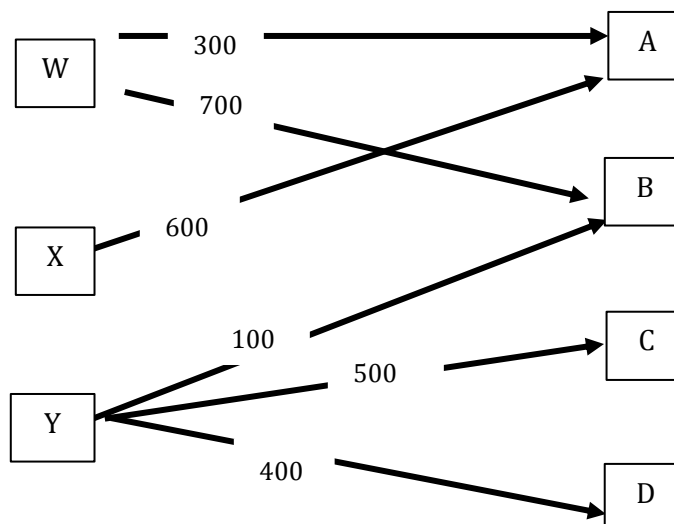
To convert the maximization transportation problem into a minimization problem, each unit profit is subtracted from the highest unit profit in the table. Since 8 is the highest unit profit, all profit values are subtracted from 8 to obtain the revised minimization transportation table shown below.

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T1	2 A	2 B	0 C	-1 D	Supply
0 W	2 200 +	2 800 -	2 0 2	4 -1 5	1000
2 X	4 600	6 4 2	4 2 2	3 -1 4	600
1 Y	3 100 -	2 3 -1 +	1 500	0 400	1000
Demand	900	800	500	400	TC1=3200

T2	2 A	2 B	1 C	0 D	Supply
0 W	2 300	2 700	2 1 1	4 0 4	1000
2 X	4 600	6 4 2	4 3 1	3 2 1	600
0 Y	3 2 1	2 100	1 500	0 400	1000
Demand	900	800	500	400	TC2=3100

Best transportation plan :



:

Remark; we may encounter situations where transportation routes are prohibited, as Sometimes in a given transportation problem, some routes may not be available. There could be several reasons for this such as bad road conditions or strike etc. In

such situations, there is a restriction on the route available for transportation. To handle such type of a situation, a very large cost (or a negative profit for the maximization problem) represented by M is assigned to each of such routes which are not available. Due to assignment of very large cost, such routes would automatically be eliminated in the final solution. The problem is then solved in the usual way.¹

7-8-Questions and Exercises

Part 1: Theoretical Questions

A. Short Questions

1. What is the main objective of the transportation problem?
2. Define the terms Supply, demand, and transportation cost in the context of the transportation model.
3. What is meant by a balanced transportation problem?
4. Why do we represent the transportation problem in matrix (tableau) format?

B. Multiple Choice Questions

1. In a balanced transportation problem:

- a) Total Supply = Total demand
- b) Total Supply > Total demand
- c) Total demand > Total Supply
- d) Supply and demand are ignored

2. The North-West Corner Method:

- a) Always gives the optimal solution
- b) Starts with the lowest cost cell
- c) Starts allocating from the top-left cell of the tableau
- d) Ignores Supply and demand values

¹ Operations research: The Transportation Problem. (2011) [PDF] Available at: [https://nscpolteksby.ac.id/ebook/files/Ebook/Accounting/Advanced%20Management%20Accounting%20Vo1.%201%20\(2011\)/13.%20Chapter%2012%20-%20The%20Transportation%20Problem.pdf](https://nscpolteksby.ac.id/ebook/files/Ebook/Accounting/Advanced%20Management%20Accounting%20Vo1.%201%20(2011)/13.%20Chapter%2012%20-%20The%20Transportation%20Problem.pdf) (Accessed: 10 - 4- 2025)

3. The Least Cost Method (LCM) improves the initial solution because:

- a) It considers the largest Supply values first
- b) It considers the lowest cost values first
- c) It allocates randomly
- d) It ignores demand values

C. True / False

1. The transportation problem is a type of linear programming problem. (T / F)
2. In the North-West Corner Method, allocation always starts with the cell that has the minimum cost. (T / F)
3. A feasible solution in the transportation problem must satisfy both Supply and demand constraints. (T / F)

Part 2: Practice Exercises

Exercise 1: A production company has four (4) production units (W, X, Y, Z) and three (3) distribution units (A, B, C). The maximum capacity of the production units was estimated at: 1000, 5000, 3000 and 2000 units per month, respectively, while the orders for the distribution units were estimated at: 4000, 3500 and 3500 per month, respectively. The costs of transporting a unit from a production point to a distribution point are shown in the following table:

	A	B	C
W	8	3	7
X	2	6	8
Y	3	5	9
Z	5	4	1

- ✓ Formulate the problem in the form of a transportation table.
- ✓ Find the initial basic solution using the northwest corner method and the least cost cell method.
- ✓ What is the best way to start the solution?

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Exercise 2: A private commercial company receives shipments of imported commercial vehicles from abroad at four (4) major seaports (W, X, Y, Z). The imported vehicles are shipped to three (3) major commercial agencies (A, B, C) located in different geographical areas. The number of vehicles waiting at the ports and the commercial agencies' vehicle needs for the coming period, along with the transportation costs per vehicle on each route, are shown in the following transportation table:

	A	B	C	Supply
W	10	7	4	6000
X	6	8	2	4000
Y	5	11	3	3000
Z	8	4	1	2000
Demand	5000	3000	7000	

- ✓ What is the optimal transportation plan for distributing cars from ports to commercial agencies?

Exercise 3: The Electricity Company has three (3) power plants (X, Y, Z) dedicated to cover the needs of four (4) population centers (A, B, C, D). The maximum production capacity in these plants was estimated at: 500, 400 and 200 megawatts respectively, while the electricity needs of the population centers were estimated at: 600, 150, 250 and 100 megawatts respectively. The costs of transporting one megawatt of electricity from a power plant to a population center are shown in the following transportation table:

	A	B	C	D	Supply
X	4	7	5	9	500
Y	3	1	4	2	400
Z	8	2	6	5	200
Demand	600	150	250	100	

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- ✓ What is the optimal plan for transporting electricity from power plants to population centers?

Exercise4: The cement company has three (3) limestone mines (W, X , Y) allocated to cover the needs of three (3) cement production plants (A, B, C) The maximum production capacity in the mines was estimated at: 400, 600 and 400 tons per day Respectively, while the daily needs of limestone for factories were estimated at: 500, 500 And 200 tons respectively. The costs of transporting one ton of limestone from a mine to a factory are shown in the following table:

	A	B	C	Supply
W	2	6	5	400
X	4	8	3	600
Y	6	11	9	400
Demand	500	500	200	

- ✓ What is the best plan to transport limestone from mines to factories?

Exercise 5: Almarai Dairy Company has three (3) dairy farms .(W, X, Y) and four (4) fresh milk distribution centers (A, B, C, D). The maximum milk production capacity of the farms was estimated at: 2000, 8000 and 2000 liters per day, respectively, while the distribution unit orders were estimated at: 3000, 5000, 2000 and 4000 liters per day, respectively. The costs of transporting one liter from a farm to a distribution center are shown in the following table:

	A	B	C	D	Supply
W	12	13	11	9	2000
X	15	17	10	7	8000
Y	10	8	14	14	2000
Demand	3000	5000	2000	4000	

- ✓ What is the best plan to transport milk from farms to distribution centers?

Exercise 6: A production company has three (3) production units (W, X, Y) and three (3) distribution units (A, B, C). The maximum capacity of the production units was estimated at: 500, 600 and 400 units per month, respectively, while the orders for the distribution units were estimated at: 500, 300 and 400 per month, respectively. The costs of transporting a unit from a production point to a distribution point are shown in the following table:

	A	B	C	Supply
W	2	6	5	500
X	4	8	3	600
Y	6	11	9	400
Demand	500	300	400	

- ✓ What is the best plan for distributing the company's products from production units to distribution units?
- ✓ The company has received an offer from a transportation company to transport its products from the production units to the distribution centers at a flat rate of €4 per unit, regardless of the destination. What would you recommend to the company's management?

7-9 Solutions and Answers

Part A: Short Questions

1. What is the main objective of the transportation problem?

Answer: To minimize the total transportation cost of shipping goods from Supply points to demand points.

2. Define the terms Supply, demand, and transportation cost in the context of the transportation model.

Answer:

- Supply: The quantity available at each source.
- Demand: The quantity required at each destination.

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- **Transportation cost:** The cost of transporting one unit of a product from a source to a destination.

3. What is meant by a balanced transportation problem?

Answer: A transportation problem is balanced when total Supply equals total demand.

4. Why do we represent the transportation problem in matrix (tableau) format?

Answer: Because it organizes costs, supplies, and demands clearly and helps in applying solution methods.

Part B: Multiple Choice Questions

1. Answer: (a)
2. Answer: (c)
3. Answer: (b)

Part C: True / False

1. True
2. False
3. True

Part 2: Practice Exercises

Exercise 1:

Northwest corner method:

	A	B	C	Supply
W	8 1000	3 12	7	1000 *
X	2 3000	6 2000	8	5000 *
Y	3	5 1500	9 1500	3000 *
Z	5	4	1 2000	2000 *
Demand	4000	3500	3500	

$$TC = 1000(8) + 3000(2) + 2000(6) + 1500(5) + 1500(9) + 2000(1) = \mathbf{49000}$$

Least Cost Method (LCM)

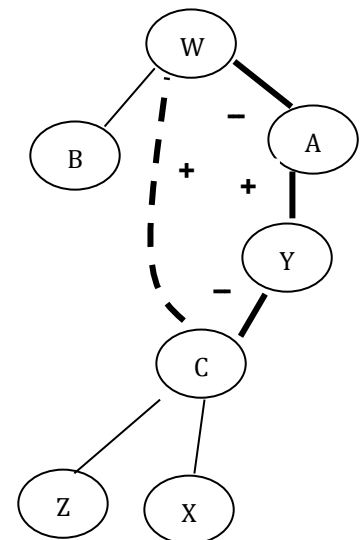
	A	B	C	Supply
W	8	3	7	1000 *
		1000		
X	2	6	8	5000 *
	4000		1000	
Y	3	5	9	3000 *
		2500	500	
Z	5	4	1	2000 *
			2000	
Demand	4000 *	3500 *	3500 *	

$$TC = 4000(2) + 1000(3) + 2500(5) + 1000(8) + 500(9) + 2000(1) = \mathbf{38000}$$

The best way to start the solution is with the **Least Cost Method** because it is an improved initial solution

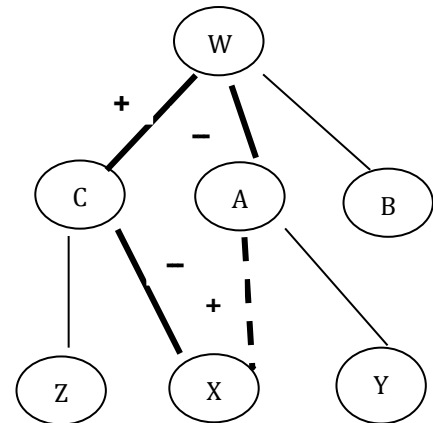
Exercise 2:

T1	10 A	7 B	8 C	Supply
0 W	10 3000 -	7 3000	4 8 -4 +	6000
-6 X	6 4 2	8 1 7	2 4000	4000
-5 Y	5 2000 +	11 2 9	3 1000 -	3000
-7 Z	8 3 5	4 0 4	1 2000	2000
Demand	5000	3000	7000	TC1= 74000



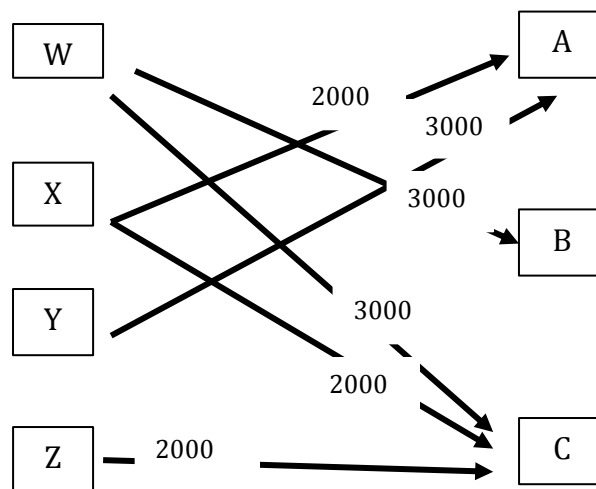
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T2	10 A	7 B	4 C	Supply
0 W	10 2000 -	7 3000	4 1000 +	6000
-2 X	6 8 -2 +	8 5 3	2 4000 -	4000
-5 Y	5 3000	11 2 9	3 -1 4	3000
-3 Z	8 7 1	4 4 0	1 2000	2000
Demand	5000	3000	7000	TC1=70000



Opt T	8 A	7 B	4 C	Supply
0 W	10 8 2	7 3000	4 3000	6000
-2 X	6 2000	8 5 3	2 2000	4000
-3 Y	5 3000	11 4 7	3 1 2	3000
-3 Z	8 5 3	4 4 0	1 2000	2000
Demand	5000	3000	7000	TC2=66000

Best transportation plan:

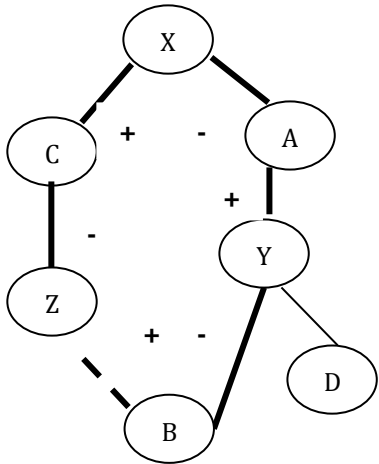


$$TC = 3000(7) + 3000(4) + 2000(6) + 2000(2) + 2000(6) + 2000(2) + 3000(5) + 2000(1) = \mathbf{66000}$$

Chapter 7: Transportation Problems

Exercise 3:

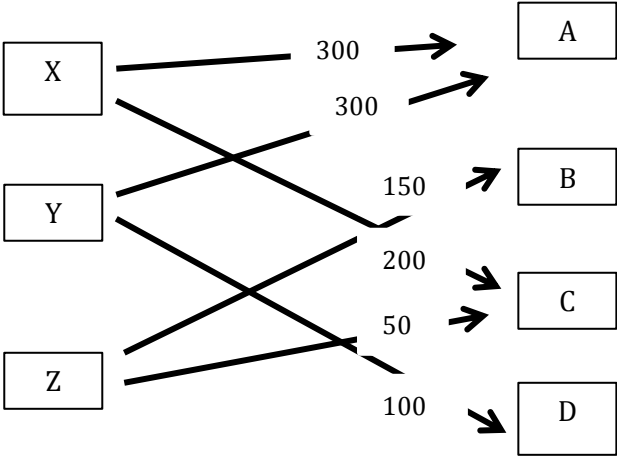
T1	4 A	2 B	5 C	3 D	Supply
0 X	4 450	7 2 5	5 50	9 3 6	500 *
-1 Y	3 150	1 150	4 4 0	2 100	400 *
1 Z	8 5 3	2 3 -1	6 200	5 4 1	200 *
demand	600 *	150 *	250 *	100 *	TC1=4050



Opt T	4 A	1 B	5 C	3 D	Supply
0 X	4 300	7 1 6	5 200	9 3 6	500 *
-1 Y	3 300	1 0 1	4 4 0	2 100	400 *
1 Z	8 5 3	2 150	6 50	5 4 1	200 *
Demand	600 *	150 *	250 *	100 *	TC2=3900

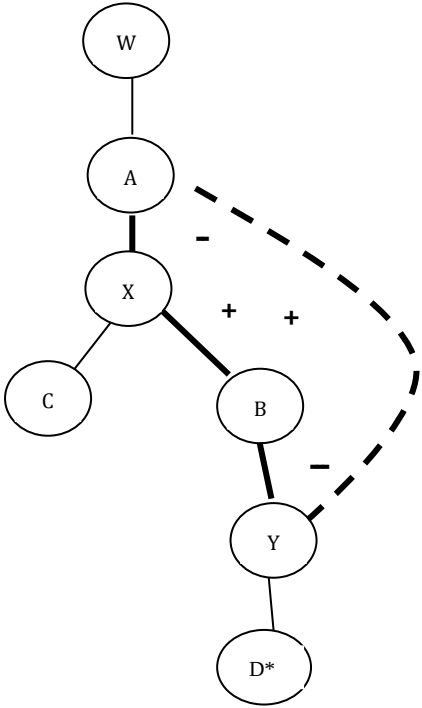
TC2=TC1-(1×150)=3900

Best transportation plan:

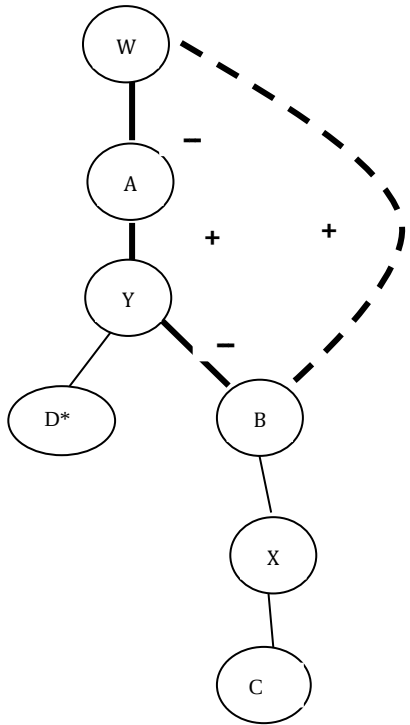


Exercise 4:

T1	2	6	1	-5	Supply
	A	B	C	D*	
0 W	2	6	5	0	
	400	6 0	1 4	-5 5	400
2 X	4	8	3	0	
	100	300	200	-3 3	600
	-	+			
5 Y	6	11	9	0	
	7 -1	200	6 3	200	400
	+	-			
demand	500	500	200	200	TC1=6400

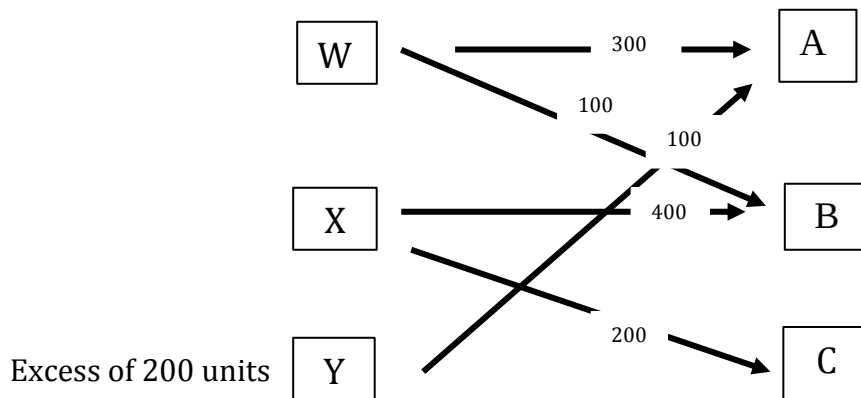


T2	2	7	2	-4	Supply
	A	B	C	D*	
0 W	2	6	5	0	
	400	7 -1	2 3	-4 4	400
	-	+			
1 X	4	8	3	0	
	3 1	400	200	-3 3	600
4 Y	6	11	9	0	
	100	100	6 3	200	400
	+	-			
demand	500	500	200	200	TC2=6300



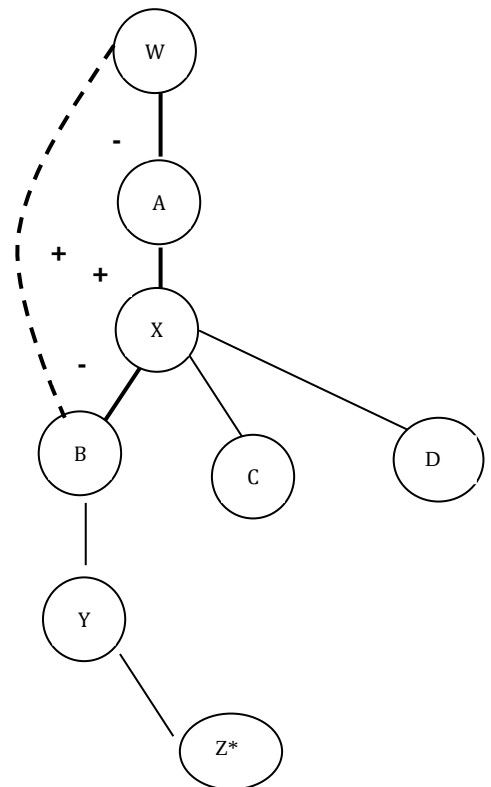
Opt T	2	6	1	-4	Supply
	A	B	C	D*	
0 W	2	6	5	0	
	300	100	1 4	-4 4	400
2 X	4	8	3	0	
	4 0	400	200	-2 2	600
4 Y	6	11	9	0	
	200	10 1	5 4	200	400
demand	500	500	200	200	TC3=6200

Best transportation plan:



Exercise 5:

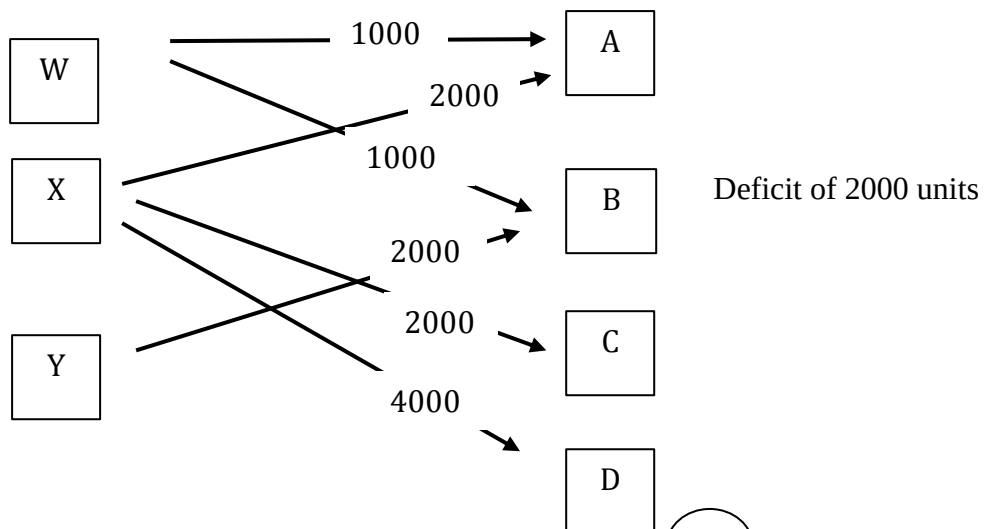
T1	12 A	14 B	7 C	4 D	Supply
0 W	12 2000	13 14 + -1	11 7 4	9 4 5	2000
3 X	15 1000	17 1000	10 2000	7 4000	8000
-6 Y	10 6 4	8 2000	14 1 13	14 -2 16	2000
-14 Z*	0 -2 2	0 2000	0 -7 7	0 -10 10	2000
Demand	3000	5000	2000	4000	TC1=120000



Opt T	12 A	13 B	7 C	4 D	Supply
0 W	12 1000	13 1000	11 7 4	9 4 5	2000
3 X	15 2000	17 16 1	10 2000	7 4000	8000
-5 Y	10 7 3	8 2000	14 2 12	14 -1 15	2000
-13 Z*	0 -1 1	0 2000	0 -6 6	0 -9 9	2000
Demand	3000	5000	2000	4000	TC2=119000

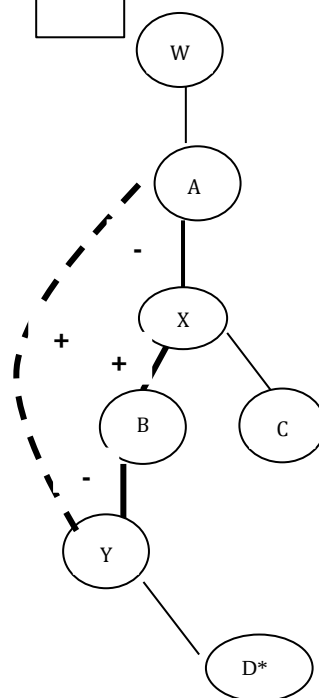
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Best transportation plan :

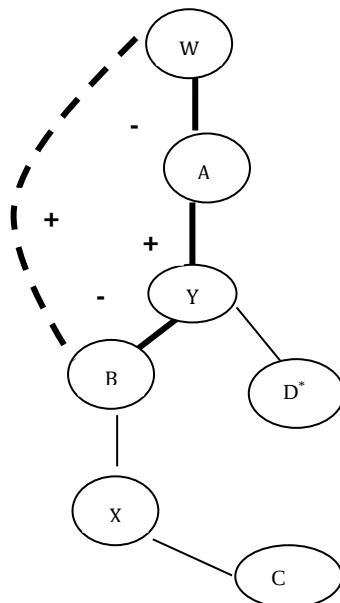


Exercise 6:

T1	2 A	6 B	1 C	-5 D*	Supply
0 W	2 500	6 0	5 1	0 -5	500
2 X	4 ε	8 200	3 400	0 -3	600
5 Y	6 7	11 100	9 6	0 3	400
Demand	500	300	400	300	TC1=4900



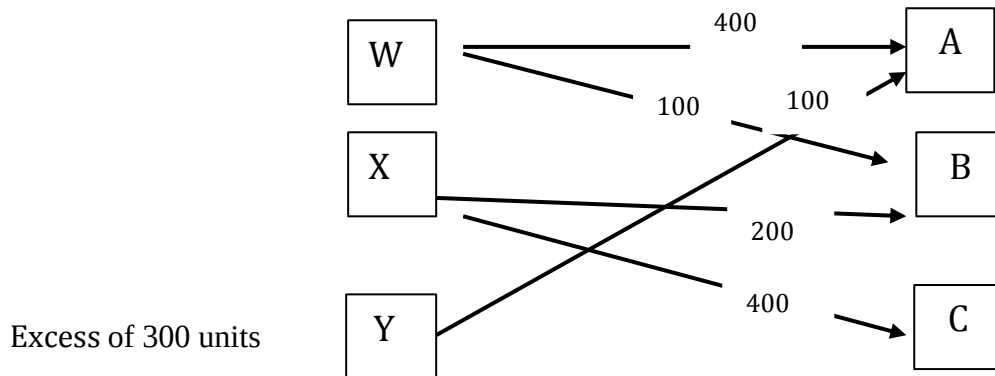
T2	2 A	7 B	2 C	-4 D*	Supply
0 W	2 500	6 7	5 2	0 -4	500
1 X	4 3	8 200	3 400	0 -3	600
4 Y	6 ε	11 100	9 6	0 3	400
Demand	500	300	400	300	TC2=4900



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Opt T	2 A	6 B	1 C	-4 D*	Supply
0 W	2 400	6 100	5 1 4	0 -4 4	500
2 X	4 4 0	8 200	3 400	0 -2 2	600
4 Y	6 100	11 10 1	9 5 4	0 300	400
Demand	500	300	400	300	TC3=4800

Best transportation plan :



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