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COMBINATORIAL STUDY AND LOG-CONCAVITY OF NUMERICAL SEQUENCES

Presented by

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DEDICATION

To my dear parents.

To my beloved wife.

*To my two sons, **Mohamed** and **Adem**.*

To my brother and his family.

To my sisters and their families.

To my mother-in-law.

In loving memory of my father-in-law,
may God grant him His mercy
and welcome him into His vast paradise.

*To all my family, the **KESSOURI** and **CHIBANE** families.*

*To all my friends and colleagues at **ENSTA**.*

To everyone who has helped me, directly or indirectly.

I dedicate this thesis to you all, with all my heart.

Abstract This thesis lies in the field of enumerative and algebraic combinatorics. It is devoted to the combinatorial and algebraic study of the sequences of Over-inversion numbers $i_{B'}(n,k)$ and Mahonian numbers of type B $i_B(n,k)$, together with their q-analogue $i_{B_q}(n,k)$, with a particular emphasis on their log-concavity and unimodality properties. We first introduce the concept of overlined permutations and define the associated Over-inversion numbers, for which we establish recurrence relations, fundamental identities, and a generating function. Several combinatorial interpretations are then provided in terms of lattice paths, overpartitions, and tilings, highlighting the deep combinatorial structure of these numbers. Next, we investigate Mahonian numbers of type B, providing recurrence relations, the Knuth–Netto formula, and generating functions for their subdiagonals on or below the main diagonal, as well as their combinatorial interpretations in terms of lattice paths, partitions, and tilings. These results are complemented by bijective interpretations involving lattice paths and signed permutations of the hyperoctahedral group B_n, showing how such permutations can be directly constructed from paths. A major contribution of this work is the introduction and analysis of a q-analogue of Mahonian numbers of type B, defined through a new inversion statistic on signed permutations. We explore its algebraic properties and provide combinatorial interpretations in terms of paths, partitions, and tilings. Finally, we establish combinatorial proofs of the log-concavity and unimodality of the Over-inversion numbers, and we prove that the q-analogue of Mahonian numbers of type B forms a strongly q-log-concave sequence of polynomials in k, which implies that the Mahonian numbers of type B form a log-concave sequence in k and therefore are unimodal.

Key words: Over-inversion numbers, Mahonian numbers of type B, q-analogue, inversions, permutations, signed permutations, lattice paths, partitions, overpartitions, tilings, log-concavity, unimodality, q-log-concavity.

Résumé Cette thèse s'inscrit dans le domaine de la combinatoire énumérative et algébrique. Elle est consacrée à l'étude combinatoire et algébrique des suites des nombres d'Over-inversion $i_{B'}(n,k)$ et des nombres Mahoniens de type B $i_B(n,k)$, ainsi que de leur q-analogue $i_{B_q}(n,k)$, avec une attention particulière portée à leurs propriétés de log-concavité et d'unimodalité. Nous introduisons d'abord la notion de permutations surlignées et définissons les nombres d'Over-inversion associés, pour lesquels nous établissons des relations de récurrence, des identités fondamentales et une fonction génératrice. Plusieurs interprétations combinatoires sont ensuite proposées en termes de chemins dans un réseau, de surpartitions et de pavages, mettant en évidence la structure combinatoire profonde de ces nombres. Nous étudions ensuite les nombres Mahoniens de type B, pour lesquels nous fournissons des relations de récurrence, la formule de Knuth–Netto, ainsi que des fonctions génératrices pour leurs sous-diagonales situées sur ou sous la diagonale principale, accompagnées d'interprétations combinatoires en termes de chemins, de partitions et de pavages. Ces résultats sont complétés par des interprétations bijectives faisant intervenir les chemins dans un réseau et les permutations signées du groupe hyperoctaédral B_n, montrant comment de telles permutations peuvent être construites directement à partir des chemins. Une contribution majeure de ce travail réside dans l'introduction et l'analyse d'un q-analogue des nombres Mahoniens de type B, défini à partir d'une nouvelle statistique d'inversion sur les permutations signées. Nous explorons ses propriétés algébriques et proposons des interprétations combinatoires en termes de chemins, de partitions et de pavages. Enfin, nous établissons des preuves combinatoires de la log-concavité et de l'unimodalité des nombres d'Over-inversion, et nous démontrons que le q-analogue des nombres Mahoniens de type B forme une suite fortement q-log-concave de polynômes en k, ce qui implique que les nombres Mahoniens de type B forment une suite log-concave et donc unimodale en k.

Mots-clés : Nombres d'Over-inversion, nombres Mahoniens de type B, q-analogue, inversions, permutations, permutations signées, chemins dans un réseau, partitions, surpartitions, pavages, log-concavité, unimodalité, q-log-concavité.

الملخّص

تندرج هذه الأطروحة في مجال التوافقيات الإحصائية والجبرية ، وهي مكرّسة للدراسة التوافقية والجبرية للمتتالية المكوّنة من أعداد الانعكاس المسطّر $i_{B'}(n,k)$ و الأعداد الماهونية من النمط B $i_B(n,k)$، إلى جانب q-التماثلي لها $i_{B_q}(n,k)$، مع تركيز خاص على خصائصها المتعلقة بالتقعّر اللوغاريتمي وأحادِيّة القمة.

نقدّم أولاً مفهوم التباديل المسطّرة ، ونعرّف أعداد الانعكاس المسطّر المرتبطة بها، حيث نثبت لها علاقات تراجعية ، ومتطابقات أساسية ، ودالّة مولّدة. ثم نعرض عدداً من التفسيرات التوافقية لهذه الأعداد من خلال المسارات الشبكية ، والتقسيمات الجزئية والتبليطات ، مبرزين البنية التوافقية العميقة لهذه المتتالية.

بعد ذلك، نتناول دراسة الأعداد الماهونية من النمط B حيث نقدّم علاقات تراجعية، وصيغة كنوث-نيتو وكذلك الدوال المولّدة الخاصة بالعناصر الواقعة على القطر الرئيسي أو تحته، إلى جانب تفسيرات توافقية لها من خلال المسارات الشبكية، والتقسيمات، والتبليطات. وتُستكمل هذه النتائج بتفسيرات تقابلية تربط بين المسارات الشبكية والتباديل ذات الإشارة لمجموعة التباديل ذات الإشارة B_n، مبيّنةً كيفية بناء مثل هذه التباديل مباشرة انطلاقاً من المسارات.

وتمثل إحدى المساهمات الرئيسية في هذا العمل في تقديم وتحليل q-التماثلي للأعداد الماهونية من النمط B والمُعرَّف من خلال إحصاء انعكاس جديد على التباديل ذات الإشارة. ندرس الخصائص الجبرية لهذا النظير ونقدّم تفسيرات توافقية له في سياق المسارات، والتقسيمات، والتبليطات.

وختامًا، نُقدِّم براهين توافقية تُثبت خاصيتي التقهّر اللوغاريتمي وأحادية القمة لمتتالية أعداد الانعكاس المسطّر، كما نُثبت أن q-التماثلي للأعداد الماهونية من النمط B يُشكّل متتالية من كثيرات الحدود قويّة q-التقعّر اللوغاريتمي بالنسبة إلى k، مما يعني أن الأعداد الماهونية من النمط B تُكوّن متتالية تقعّرية لوغاريتمّياً بالنسبة إلى k، وبالتالي فهي أحادِيّة القمة.

الكلمات المفتاحية: أعداد الانعكاس المسطّر، الأعداد الماهونية من النمطB ،q-التماثلي الانعكاسات، التباديل، التباديل ذات الإشارة، المسارات الشبكية، التقسيمات، التقسيمات الجزئية، التبليطات، التقعّر اللوغاريتمي، أحادِيّة القمة، التقعّر q-اللوغاريتمي.

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1) A. Kessouri, M. Ahmia and S. Mesbahi, Over-inversion numbers: Basic properties, log-concavity and unimodality, J. Combin. Math. Combin. Comput. **124**, 389-407 (2025).

2) A. Kessouri, M. Ahmia, H. Arslan, and S. Mesbahi, Combinatorics of q-Mahonian numbers of type B and log-concavity, published online in the **Indian Journal of Pure and Applied Mathematics** (2025).

Introduction

Combinatorics is an essential branch of mathematics concerned with the study of finite or discrete sets, as well as with the counting, arrangement, and structuring of objects according to specific rules. It does not merely aim to enumerate these objects, but also seeks to understand their properties, establish recurrence relations, determine generating functions, and provide combinatorial interpretations that explain, justify, or visualize the obtained results. Through its constructive and analytical approaches, combinatorics offers a rich framework for exploring discrete structures and their numerous connections with other areas of mathematics.

It is based on fundamental objects such as permutations, partitions, combinations, paths, graphs, and trees, and finds numerous applications in computer science, graph theory, algebra, and mathematical physics.

This thesis falls within the field of enumerative and algebraic combinatorics and focuses on two important properties: the log-concavity and unimodality of certain sequences and polynomials. The study of these properties often proves challenging when approached directly. This is why bijective combinatorics plays a central role: it allows the analysis of these properties to be transferred to other combinatorial structures in bijection with the studied objects. This approach forms the main methodological framework adopted in this thesis.

This thesis is organized into four chapters, structured as follows:

In the first chapter, we present the essential notions and fundamental concepts of combinatorics that form the theoretical foundation of the subsequent chapters of this thesis. We begin with the study of generating functions, a particularly powerful mathematical tool used to solve numerous combinatorial problems. We then discuss the notion of permutations and the algebraic framework of symmetric groups, before extending this study to signed permutations and hyperoctahedral groups. Special attention is devoted to the various statistics associated with permutations, whether signed or unsigned.

Subsequently, we introduce several fundamental combinatorial objects such as partitions, Ferrers and Young diagrams, overpartitions, and lattice paths. We also present the concepts related to Mahonian numbers, their analogues, and the Mahonian numbers of type B. Finally, this introductory chapter concludes with the presentation of the central concepts of log-concavity, q-log-concavity, and unimodality, which will play an important role in the remainder of this work.

In the second chapter, we introduce the concept of an overlined permutation σ of n elements, where the first elements associated with inversions can be independently overlined or not. In such a permutation, each element $\sigma(i)$, for $1 \leq i \leq n-1$, can be independently overlined or not, provided that there exists an index $j > i$ such that $\sigma(i) > \sigma(j)$. These permutations form a subset of the hyperoctahedral group B_n, denoted by B'_n. We focus on counting such permutations with exactly k inversions, denoted by $i_{B'}(n,k)$, which we refer to as Over-inversion numbers. This chapter is devoted to the combinatorial and algebraic study of these numbers: we establish several recurrence relations and identities, present their generating function, and derive a fundamental identity involving the double factorial. We also provide various combinatorial interpretations of these numbers through lattice paths, overpartitions, and tilings.

In the third chapter, we focus on studying new combinatorial properties and interpretations of the Mahonian numbers of type B, as well as an analogue of these numbers together with their combinatorial interpretations. We present the Knuth–Netto formula and the generating function for the subdiagonals on or below the main diagonal of the Mahonian numbers of type B, followed by their combinatorial interpretations in terms of lattice paths, partitions, and tilings. We also introduce a q-analogue of the Mahonian numbers of type B, based on a new statistic on permutations of the hyperoctahedral group B_n that we define, and we study its fundamental properties and combinatorial interpretations using lattice paths, partitions, and tilings.

In the last chapter, we provide a combinatorial proof showing that the Over-inversion numbers form a log-concave and unimodal sequence. Moreover, we prove combinatorially that the q-analogue of the Mahonian numbers of type B forms a strongly q-log-concave sequence of polynomials in k, which implies that the Mahonian numbers of type B form a log-concave and therefore unimodal sequence in k.

CHAPTER 1

Preliminaries

In this chapter, we introduce the fundamental notions and concepts of combinatorics that form the foundation of the remaining chapters of the thesis. We begin with generating functions, a powerful mathematical tool used to solve a wide range of combinatorial problems. Next, we present permutations and symmetric groups, followed by signed permutations and hyperoctahedral groups. We then discuss various statistics associated with both signed and unsigned permutations. Subsequently, we introduce several basic combinatorial tools and objects, including partitions, Ferrers diagrams, Young diagrams, overpartitions, and lattice paths. We also present key notions related to Mahonian numbers, q-Mahonian numbers, and Mahonian numbers of type B. Finally, we conclude this introductory chapter by defining the fundamental concepts of log-concavity, q-log-concavity, and unimodality.

1.1 Generating functions

Generating functions are powerful tools in mathematics that facilitate the manipulation of sequences and the resolution of problems in combinatorics, algebra, and probability theory. They were first introduced by Abraham de Moivre in 1730 [Chat19, Land03, Detem14, Nive69, Tuke12] to derive explicit formulas for sequences defined by linear recurrence relations. These functions were originally referred to as generating series.

In fact, there are several types of generating series, such as exponential generating series, ordinary generating series, Lambert series, Dirichlet series, and others. In this thesis, however, we will only need two types: ordinary generating functions and exponential generating functions.

1.1.1 Ordinary generating function

The ordinary generating function (OGF) is a valuable method for analyzing and manipulating sequences. It transforms combinatorial problems into questions of series analysis, which often simplifies computation. OGFs are particularly powerful for solving problems involving partitions and combinations.

Definition 1.1.1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers. The* ordinary generating function *(OGF) of this sequence is defined by*

$$G(z) = \sum_{n\geq 0} u_n z^n.$$

Example 1.1.2 *1. The ordinary generating function of the sequence of binomial coefficients is given by:*

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n .$$

2. The ordinary generating function of the Fibonacci sequence, denoted $F(x)$, is given by:

$$F(x) = \sum_{n\geq 0} F_n x^n = \frac{1}{1-x-x^2}.$$

1.1.2 Exponential generating function

The exponential generating function (EGF) is a powerful tool for manipulating sequences. EGFs are particularly useful for solving differential equations, analyzing series, and addressing counting problems in combinatorics. They also enable the transformation of sequence problems into analytical ones.

Definition 1.1.3 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers. The* exponential generating *(EGF) of this sequence is defined by*

$$E(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

Example 1.1.4 *The exponential generating function of the sequence of Euler numbers $\{E_n\}$ is given by:*

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + 61\frac{x^6}{6!} + \dots$$
$$= \frac{1}{\cos x}.$$

1.1.3 Algebraic Operations on Generating functions

A) Addition in OGF

The sum of two ordinary generating functions $A(x)$ and $B(x)$, corresponding to the sequences a_n and b_n, respectively [Wilf89], is obtained by adding each of the coefficients of like powers of x:

$$A(x) + B(x) = \sum_{n\geq 0}(a_n + b_n)x^n.$$

B) Product in OGF

The convolution product (also known as the **Cauchy product**) of two ordinary generating functions $A(x)$ and $B(x)$, corresponding to the sequences a_n and b_n, is defined by:

$$R(x) = A(x) \cdot B(x) = \sum_{n\geq 0} r_n x^n, \quad \forall n \geq 0,$$

where the coefficients r_n are given by:

$$r_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

C) Sum in EGF

The sum of two exponential generating functions $A(x)$ and $B(x)$, associated with the sequences a_n and b_n, corresponds to the term-by-term sum of their coefficients:

$$A(x) + B(x) = \sum_{n\geq 0}(a_n + b_n)\frac{x^n}{n!}.$$

D) Product in EGF

The product of two exponential generating functions $A(x)$ and $B(x)$ (also called the binomial convolution product), associated respectively with the sequences of numbers a_n and b_n, is defined as:

$$R(x) = A(x)B(x) = \sum_{n\geq 0} r_n \frac{x^n}{n!},$$

where the coefficients r_n are are given by the binomial convolution:

$$r_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}, \quad n \geq 0.$$

1.2 Permutations and symmetric groups

Permutations are central to many areas of mathematics. In combinatorics, they represent the different arrangements of a given set of objects. For instance, the order in which runners finish a race forms a permutation. A permutation of size n is typically represented as an arrangement of the numbers from 1 to n, and there are $n!$ such permutations. Permutations are fundamental in combinatorics because many problems involve counting those that satisfy specific properties.

Permutations can also be interpreted as bijections from the set $\{1, \ldots, n\}$ to itself, which naturally endows this set with a group structure known as the symmetric group. This perspective offers new tools for addressing enumerative problems involving permutations. In **1854**, Cayley showed that every finite group is isomorphic to a subgroup of a symmetric group, thereby renewing interest in its study. In this context, the irreducible representations of the symmetric group attracted the attention of prominent mathematicians such as Dedekind and Frobenius.

Furthermore, symmetric groups can be regarded as the simplest examples of Coxeter groups, specifically those of type **A**. Much of the theory developed for symmetric groups generalizes to other Coxeter groups, motivating our interest in their combinatorics. For more details, see [Saga01, Stanl11, Cayl54, Bona12, Knut05].

We first begin this section by defining a permutation.

Definition 1.2.1 *A permutation σ of a set E is a bijection from E to itself.*

Remark 1.2.2 *The set $S(E)$ of all permutations of a set E, equipped with the operation of composition $\circ$, forms a group called the* symmetric group *on E. In the case where $E = \{1, 2, \dots, n\}$, this group is denoted by S_n, and its order is $|S_n| = n!$.*

Example 1.2.3 *Let $E = \{1, 2, 3, 4\}$.*

$$S_4 = \{1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143,$$
$$2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241,$$
$$3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321\}.$$

The number of permutations of S_4 is $|S_4| = 4! = 24$.

Every permutation σ of S_n is also represented as follows:

$$\sigma = \begin{pmatrix} 1 & \cdots & i & \cdots & n \\ \sigma(1) & \cdots & \sigma(i) & \cdots & \sigma(n) \end{pmatrix}.$$

Example 1.2.4 *The permutation $\sigma = 3421$ of S_4 is written as:*

$$\sigma = \begin{pmatrix} 1\ 2\ 3\ 4 \\ 3\ 4\ 2\ 1 \end{pmatrix}.$$

Definition 1.2.5 *The permutation that maps each element to itself is called the* identity permutation, *denoted by* Id. *It serves as the identity element of the symmetric group S_n.*

Definition 1.2.6 *Let σ and τ be two permutations in the symmetric group S_n. The composition of σ and τ, denoted $\tau \circ \sigma$, is the permutation π defined by:*

$$\pi(x) = (\tau \circ \sigma)(x) = \tau(\sigma(x)) \quad \textit{for all } x \in \{1, 2, \dots, n\}.$$

Definition 1.2.7 *The inverse of a permutation σ is the permutation σ^{-1} such that $\sigma \circ \sigma^{-1} = \mathrm{Id}$ and $\sigma^{-1} \circ \sigma = \mathrm{Id}$.*

Definition 1.2.8 *The* backward permutation *of a permutation σ in S_n is the permutation σ' defined by $\sigma'_i = \sigma_{n+1-i}$ for all $1 \leq i \leq n$.*

Example 1.2.9 *Let $\sigma = 2314 \in S_4$ and $\pi = 4312 \in S_4$, we have*

1. *The inverse of the permutation σ is $\sigma^{-1} = 3124$.*

2. *The compositions of $\sigma \circ \pi$ and $\pi \circ \sigma$ are: $\sigma \circ \pi = 4123$ and $\pi \circ \sigma = 3142$.*

3. *The backward permutation of σ is $\sigma' = 4132$.*

Definition 1.2.10 *An* inversion *of a permutation σ is a pair $(i, j) \in \mathbb{N}^2$ such that $i < j$ and $\sigma(i) > \sigma(j)$.*

Notation 1.2.11 *The number of inversions of a permutation σ is denoted by $inv(\sigma)$.*

Example 1.2.12 *The inversions of the permutation $\sigma = 2341$ are the pairs: $(1, 4), (2, 4)$ and $(3, 4)$. Therefore, the number of inversions of σ is $inv(\sigma) = 3$.*

Remark 1.2.13

1. *The number of inversions of a permutation is equal to the number of inversions of its inverse, that is, $inv(\sigma) = inv(\sigma^{-1})$.*

2. *Since the identity permutation $\sigma = \mathrm{Id}$ maps each element to itself, it contains no inversions, and thus $inv(\sigma) = 0$.*

3. *The following permutation π:*

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$$

has the maximum number of inversions with $inv(\pi) = \binom{n}{2}$.

Definition 1.2.14 *A pair (σ_i, σ_j) is called a* backward inversion *of the permutation σ if $i < j$ and $\sigma_i < \sigma_j$.*

Remark 1.2.15 *For a permutation of length n with k inversions, the number of backward inversions is $\binom{n}{2} - k$ [Indo08].*

Example 1.2.16 *The backward inversions of the permutation $\sigma = 2341$ are the pairs: $(1, 2), (1, 3)$ and $(2, 3)$. Therefore, the number of backward inversions of σ is 3.*

Definition 1.2.17 *Let $\pi \in S_n$. The **signature** (or **sign**) of π, denoted by $\varepsilon(\pi)$, is defined as*

$$\varepsilon(\pi) = (-1)^{inv(\pi)}.$$

Example 1.2.18 *The signature of the permutation $\pi = 2341$ is $\varepsilon(\pi) = (-1)^3 = -1$.*

Definition 1.2.19 *Let $\pi \in S_n$. The* support *of π, denoted $supp(\pi)$, is the set*

$$supp(\pi) = \{k \in \{1, \ldots, n\} \mid \pi(k) \neq k\}.$$

Example 1.2.20 *The support of $\pi = 2341$ is $supp(\pi) = \{1, 2, 3, 4\}$.*

Definition 1.2.21 *A* transposition *is a permutation whose support consists of exactly two elements. In this case, if $\operatorname{supp}(\pi) = \{i, j\}$, then $\pi(i) = j$ and $\pi(j) = i$, and we write $\pi = (ij)$. Moreover, $\pi^2 = \mathrm{Id}$.*

Example 1.2.22 *The permutation $\pi = 2134$ is the transposition $(1\ 2)$, since $supp(\pi) = \{1, 2\}$.*

Theorem 1.2.23 *[Hers75] Every permutation can be written as a product of transpositions. The symmetric group S_n is generated by transpositions.*

1.3 Signed permutations and hyperoctahedral groups

Signed permutations, which generalize ordinary permutations by allowing elements to carry a sign, are fundamental objects in algebraic combinatorics and appear naturally in the study of reflection groups and Coxeter groups. The group of all signed permutations on n elements forms the hyperoctahedral group, often denoted by B_n, which serves as the symmetry group of the n-dimensional hypercube and cross-polytope. This group plays a central role in various branches of mathematics including Lie theory, geometry, and combinatorics [Bjor05, Hump90, Kane01, Brent96, Bagn07, Stanl11].

In this section, we explore the structure and representation of signed permutations and hyperoctahedral groups.

Definition 1.3.1 *A* signed permutation *of $\{1, 2, \ldots, n\}$ is a bijection*

$$\sigma : \{\pm 1, \pm 2, \ldots, \pm n\} \to \{\pm 1, \pm 2, \ldots, \pm n\}$$

such that $\sigma(-i) = -\sigma(i)$ for all $i \in \{1, 2, \ldots, n\}$.

Equivalently, a signed permutation can be described as a pair (π, ϵ), where $\pi \in S_n$ is a permutation and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with each $\epsilon_i \in \{\pm 1\}$ assigns a sign to each element.

Example 1.3.2 *Let $n = 3$. One example of a signed permutation is $\sigma = -23-1$, which maps:*

$$\sigma(1) = -2, \quad \sigma(2) = 3, \quad \sigma(3) = -1.$$

Then by the property $\sigma(-i) = -\sigma(i)$, we also have:

$$\sigma(-1) = 2, \quad \sigma(-2) = -3, \quad \sigma(-3) = 1.$$

Definition 1.3.3 *The* hyperoctahedral group *B_n is the group of all signed permutations of $\{1, 2, \ldots, n\}$ under composition. It can be described as the wreath product*

$$B_n \cong \mathbb{Z}_2 \wr S_n = \mathbb{Z}_2^n \rtimes S_n.$$

Its order is $|B_n| = 2^n n!$.

Example 1.3.4 *The group B_2 consists of the 8 signed permutations of 2 elements:*

$$12,\ -12,\ 1-2, 1-2,\ 21,\ -21,\ 2-1,\ -2-1.$$

1.4 Signed and unsigned permutation Statistics

Permutation statistics, whether signed or unsigned, play a fundamental role in combinatorics. These tools allow us to examine the structures and properties of permutations, particularly through concepts such as inversions. Their significance extends to various fields including combinatorics, algebra, geometry, group theory, and statistics. Formally, a permutation statistic is defined as a function that assigns to each permutation of a given set an integer or numerical value, thus characterizing certain of its properties. For further details on the statistics studied in combinatorics literature, the reader is referred to the following bibliography [Stanl11, Saga01, Bjor05, Bagn07, Foat90].

1.4.1 Unsigned permutation Statistics

In this subsection, we present the most commonly studied statistics for unsigned permutations.

Definition 1.4.1 *Let π be an unsigned permutation. The following statistics are defined:*

1. ***The descent set of π:***

$$Des(\pi) = \{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}.$$

2. ***The number of descents of π:*** *The number of positions i for which $\pi(i) > \pi(i+1)$, that is:*

$$des(\pi) = |Des(\pi)|.$$

3. ***Major index:*** *The major index of a permutation π is the sum of the indices i that belong to the descent set of π, that is:*

$$maj(\pi) = \sum_{i \in Des(\pi)} i.$$

4. ***Fixed points:*** *Indices i such that $\pi(i) = i$.*

5. ***Cycle structure:*** *Permutations can be decomposed into disjoint cycles. The number of these cycles and their lengths are essential statistics of permutations.*

6. ***Number of excedances:*** *The number of excedances of π is defined as:*

$$exc(\pi) = |\{1 \leq i \leq n \mid \pi(i) > i\}|.$$

Example 1.4.2 *Consider the unsigned permutation $\pi = 42513 \in S_5$. Let us compute the statistics defined above for π:*

$$inv(\pi) = 6, \quad Des(\pi) = \{1, 3\}, \quad des(\pi) = 2, \quad maj(\pi) = 4, \quad \text{and} \quad exc(\pi) = 2.$$

- The disjoint cycle decomposition of permutation π is: (1 4)(2)(3 5).

- The number of cycles is 3 and the lengths of the cycles are: 2, 1, and 2 respectively.

- There are no fixed points in π.

1.4.2 Signed permutation Statistics

In this subsection, we present the most commonly studied statistics for signed permutations.

Definition 1.4.3 *Let $\pi \in B_n$, define the following statistics on π:*

1. ***The type B descent number of π:*** $\mathrm{des}_B(\pi) = |\{0 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}|$, *where* $\pi(0) = 0$.

2. ***The major index of*** π***:*** $\mathrm{maj}(\pi) = \sum_{i\in \mathrm{Des}(\pi)} i$.

3. ***The negative set of*** π***:*** $\mathrm{Neg}(\pi) = \{i \in \{1, \ldots, n\} \mid \pi(i) < 0\}$.

4. ***The negative number of*** π***:*** $\mathrm{neg}(\pi) = |\,\mathrm{Neg}(\pi)|$.

5. ***The negative number sum of*** π***:*** $\mathrm{nsum}(\pi) = -\sum_{i\in \mathrm{Neg}(\pi)} \pi(i)$.

Example 1.4.4 *Consider the signed permutation* $\pi = -1 -24 -3 \in B_4$*. Let us compute the statistics defined above for* π*:*

$$des_B(\pi) = 3, \quad maj(\pi) = 4, \quad Neg(\pi) = \{1, 2, 4\}, \quad neg(\pi) = 3 \ \textit{and} \quad nsum(\pi) = 6.$$

1.5 Combinatorial objects

Combinatorial objects play an essential role in studying of structural properties of sequences, such as log-concavity and unimodality (see the last section of this chapter). They allow these properties to be interpreted through visual or algebraic constructs, that make these properties easier to analyze and prove. In this section, we present some of these fundamental objects notably partitions, overpartitions, Ferrers diagrams, Young diagrams and lattice paths which will be used throughout the remainder of this thesis.

1.5.1 Partitions

In 1674, Leibniz asked Bernoulli about expressing a positive integer as a sum of smaller ones, referring to this as the "number of divulsions," now known as partitions. Euler later explored this in depth, solving in 1741 a problem on partitioning 50 into 7 distinct parts, which he published in 1751 [Eule51]. In the 20^{th} century, the topic advanced with contributions from Rogers, Hardy, MacMahon, Ramanujan, and Rademacher [Dick71, Andr13]. Partition theory has since become a central area in combinatorics, with applications to q-binomial coefficients (also known as Gaussian polynomials) [Knut71] and symmetric functions [Macd98].

Definition 1.5.1 *A partition of the integer* n *is a sequence of positive integers written in the form :* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ *or* $\lambda = \lambda_1\lambda_2 \ldots \lambda_k$*, with the following two conditions :*

1. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$,

2. $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

For $1 \leq i \leq k$, λ_i is referred to as a part of λ. While partitions typically consist of positive integers, we may sometimes allow "zero" as a part for specific purposes.

Notation: The **length** of λ, denoted by $l(\lambda)$, is the number of parts in λ, and the **weight of** λ, denoted by $|\lambda|$, is the total sum of its parts.

The number of partitions of n is denoted by $p(n)$. For simplicity, we define $p(0) = 1$.

Example 1.5.2 *The partitions of* 5 *are as follows :*

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1,$$

so $p(5) = 7$.

The number of partitions $p(n)$ of n has the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

A partition into K parts, where the largest part is at most N, can be expressed in the following forms

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K) \qquad \text{or} \qquad \lambda = \lambda_1\lambda_2 \cdots \lambda_K,$$

where $N \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K \geq 0$.

Notation: Let $\mathcal{P}(N, K)$ denote the set of integer partitions into K parts, each at most N, and let $p(N, K) = |\mathcal{P}(N, K)|$ be the number of such partitions. We write $\lambda \subset (N)^K$ to indicate that $\lambda \in \mathcal{P}(N, K)$.

The number of partitions of n into k parts $P(n, k)$ satisfies the following recurrence relation

$$P(n, k) = P(n-1, k-1) + P(n-k, k),$$

with $P(0, 0) = 1$.

After defining a partition, it is useful to represent it graphically to better understand its structure. **Ferrers and Young diagrams** provide such visualizations, facilitating their study and application in combinatorics.

1.5.2 Ferrers diagram and Young diagram

Ferrers diagrams and Young diagrams [Fult97, Stan99, Saga01] are graphical tools commonly used to represent integer partitions, playing an important role in combinatorics and representation theory.

Definition 1.5.3 *The Ferrers diagram of a partition $\lambda \in p(N, K)$ is the set of points $(i_1, i_2) \in \mathbb{N}^2$ where $1 \leq i_2 \leq K$ and $1 \leq i_1 \leq \lambda_{i_2} \leq N$.*

Example 1.5.4 *Figure 1.1 illustrates the Ferrers diagram of the partition $\lambda = (4, 3, 2, 1)$.*

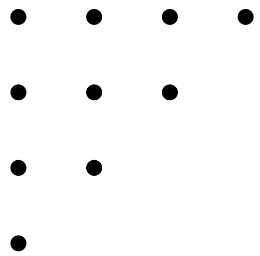

Figure 1.1: Diagram of Ferrers of $\lambda = (4, 3, 2, 1)$.

Remark 1.5.5 *A Young diagram is obtained from a Ferrers diagram by replacing the dots (or nodes) with boxes (or squares). Both diagrams represent the same partition, differing only in style.*

Example 1.5.6 *Figure 1.2 shows the Young diagram corresponding to the partition $\lambda = (4, 3, 2, 1)$.*

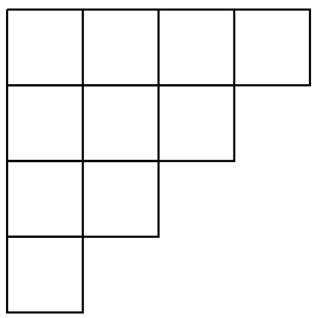

Figure 1.2: Young diagram of $\lambda = (4, 3, 2, 1)$.

Remark 1.5.7 *In the Young diagram of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K)$, the number of boxes in the i-th row corresponds to the i-th part λ_i of the partition. The total number of rows in the diagram equals the number of parts in λ.*

1.5.3 Overpartitions

Overpartitions arise in combinatorics and number theory as a natural generalization of classical partitions. The term **overpartition** for this generalization was first introduced by Corteel and Lovejoy [Cort04].

Definition 1.5.8 *[Cort04] An **overpartition** of a non-negative integer n is a non-increasing sequence of positive integers whose sum is n, in which the first occurrence of each distinct number (or equivalently, the last occurrence) may be overlined. The function $\overline{p}(n)$ denotes the number of overpartitions of n, with the convention that $\overline{p}(0) = 1$.*

Example 1.5.9 *There are 8 overpartitions of 3, enumerated as follows:*

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

The generating function for the overpartition numbers $\overline{p}(n)$ is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}.$$

Definition 1.5.10 *The Young diagram of an overpartition is an extension of the classical Young diagram. It represents overpartitions by introducing an additional feature: the last box in each row may be colored. When this box is indeed colored, it indicates that the part corresponding to that row is overlined in the overpartition. For instance, see Figure 1.3.*

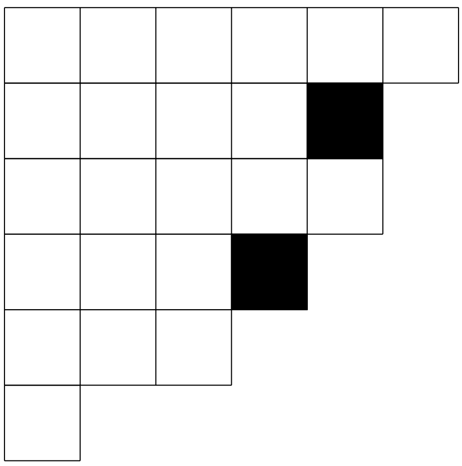

Figure 1.3: The Young diagram of $\lambda = (6, \overline{5}, 5, \overline{4}, 3, 1)$.

1.5.4 Lattice paths

A **lattice path** is a discrete trajectory consisting of a sequence of segments, called steps, which connect a sequence of points in $\mathbb{Z} \times \mathbb{Z}$. The first point in this sequence is called the initial point, while the last point is called the final point. These paths, which can be seen as displacements on a discrete grid, occupy a central place in combinatorics and in the study of discrete structures. In this section, we are interested in certain types of paths, such as **North-East paths** and **Delannoy paths**. The latter are powerful tools for solving enumeration, optimisation, and modelling problems in various areas of discrete mathematics and theoretical computer science. For more information, we refer the reader to [Hump10, Moha79, Bost17].

Definition 1.5.11 *A* **North-East path** $P = (p_0, p_1, \ldots, p_n)$ *in the discrete plane* $\mathbb{N} \times \mathbb{N}$ *is a sequence of sequence of North and East steps that starts at the origin* $(0,0)$ *and ends at the point* (n,k)*, such that:*

$$(i,j) = \begin{cases} (i, j-1) + (0,1), & \textit{vertical (North) step;} \\ (i-1, j) + (1,0), & \textit{horizontal (East) step.} \end{cases}$$

Example 1.5.12 *This Figure 1.4 show an example of North-East path.*

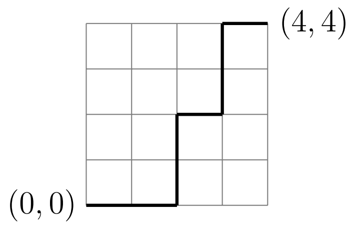

Figure 1.4: A North-East path in $(0,0)$ to $(4,4)$.

In combinatorics, the solution to many problems depends on the surface above the North-East path, known as the weight.

Definition 1.5.13 *The weight of a North-East path* P*, denoted* $w(P)$*, corresponds to the number of grid squares located above and to the left of* P*.*

Remark 1.5.14 *1. There is a relationship between North-East paths and partitions: each East step corresponds to a part equal to the number of squares above it, and the collection of these parts forms a partition.*

2. *The binomial coefficients count the number of possible paths in a grid from* $(0,0)$ *to* (n,k).

Example 1.5.15 *The weight of the path in Figure 1.4 is* 10*, corresponding to the partition* $\lambda = (4,4,2,0)$.

Definition 1.5.16 *A* **Delannoy path** $P = (p_0, p_1, \ldots, p_n)$ *in the discrete plane* $\mathbb{N} \times \mathbb{N}$ *is a sequence of (North,East,North-East) steps that starts at the origin* $(0,0)$ *and ends at the point* (n,k)*, such that:*

$$(i,j) = \begin{cases} (i, j-1) + (0,1), & \textit{North step}; \\ (i-1, j) + (1,0), & \textit{East step}; \\ (i-1, j-1) + (1,1), & \textit{North-East step}. \end{cases}$$

Example 1.5.17 *The Figure 1.5 shows an example of Delannoy path.*

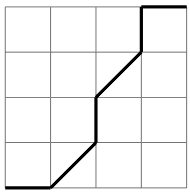

Figure 1.5: A Delannoy path in $(0,0)$ to $(4,4)$.

Remark 1.5.18 *The number of Delannoy paths from* $(0,0)$ *to* (n,k) *equals the Delannoy number* $D(n,k)$*, given by* $D(n,k) = \sum_{i=0}^{\min(n,k)} 2^i \binom{n}{i}\binom{k}{i}$.

Definition 1.5.19 *[Dous17] The weight of a Delannoy path* P*, denoted* $\omega t(P)$*, is defined as the sum of the weights of its steps. For a step* p_k *from* (i,j)*, the weight* $\omega t(p_k)$ *is given by:*

$$\omega t(p_k) = \begin{cases} 0, & \textit{if it moves from } (i,j) \textit{ to } (i+1,j), \\ i, & \textit{if it moves from } (i,j) \textit{ to } (i,j+1), \\ i+1, & \textit{if it moves from } (i,j) \textit{ to } (i+1,j+1). \end{cases}$$

Example 1.5.20 *The weight of the path in Figure 1.5 is* $\omega t(P) = 0+2+2+3+3+0 = 10$.

1.6 Mahonian numbers and it analogues

1.6.1 Mahonian numbers

The Mahonian numbers were first introduced by Rodrigues in the 19th century in the context of permutation statistics [Rodr39]. In the early 20th century, the British mathematician Percy Alexander MacMahon brought significant attention to these numbers through his extensive work on permutation statistics and integer partitions. His contributions laid the foundations of algebraic combinatorics, a field that studies discrete structures using methods and concepts from algebra. Moreover, MacMahon introduced the major index statistic, which has become one of the most widely studied statistics in combinatorics. Both the major index and the Mahonian numbers have applications in various areas, including representation theory and mathematical physics.

Definition 1.6.1 *Let $n \geq 1$ and $k \geq 0$. The number of permutations of length n with exactly k inversions, denoted by $i(n,k)$, is called the **Mahonian number**.*

The set of permutations of length n with exactly k inversions is denoted by $I(n,k)$, and its cardinality is the Mahonian number, i.e., $i(n,k) = |I(n,k)|$.

Remark 1.6.2 *MacMahon [Maho13] proved that $i(n,k)$ equals the number of permutations of length n whose major index is k.*

Now, we give a combinatorial interpretation of the Mahonian numbers.

Combinatorial interpretation : $i(n,k)$ counts the number of different ways to distribute k balls into $n-1$ urns where the j^{th} urn contains at most j balls.

The Mahonian numbers $i(n,k)$ have the following generating function.

Theorem 1.6.3 *[Muir98, Bona04] For positive integer n,*

$$\sum_{k=0}^{\binom{n}{2}} i(n,k)x^k = (1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}). \tag{1.6.1}$$

The Mahonian numbers $i(n,k)$ satisfy the following symmetry and longitudinal recurrence relations.

Theorem 1.6.4 *For $n > 0$ and $0 \leq k \leq \binom{n}{2}$, we have the symmetry relation*

$$i(n,k) = i\left(n, \binom{n}{2} - k\right) \tag{1.6.2}$$

and the longitudinal recurrence relation

$$i(n,k) = \sum_{j=0}^{n-1} i(n-1, k-j). \tag{1.6.3}$$

Indong and Peralta [Indo08] proved the two previous recurrence relations.

These numbers form a triangle known as the "**Mahonian triangle** ". The first values of the Mahonian triangle appear in OEIS [Sloa64] as A008302. Table 1.1 illustrates this triangle.

n/k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	3	5	6	5	3	1									
5	1	4	9	15	20	22	20	15	9	4	1					
6	1	5	14	29	49	71	90	101	101	90	71	49	29	14	5	1

Table 1.1: The Mahonian triangle.

1.6.2 q-Analogue of Mahonian numbers

In this section, we present the q-analogue of Mahonian numbers introduced by Ghemit and Ahmia in [Gmah23], their properties, and some combinatorial interpretations.

Let σ be a permutation of length n. m_i^σ is the number of appearances of the entry $i+1$ as first element of the inversions of σ, for $0 \leq i \leq n-1$. The total number of inversions of σ is exactly $\sum_{i=0}^{n-1} m_i^\sigma$.

The weight w_σ of the permutation σ is defined as follows,

$$w_\sigma := \sum_{i=0}^{n-1} i \times m_i^\sigma \tag{1.6.4}$$

From the previous definition of weight, the q-analogue of the Mahonian numbers $i_q(n,k)$ satisfy the following definitions:

Definition 1.6.5 *For $n \geq 1$ and $0 \leq k \leq \binom{n}{2}$, the q-analogue of the Mahonian numbers $i_q(n,k)$ is given by :*

$$i_q(n,k) = \sum_{\sigma \in I(n,k)} q^{w_\sigma}, \tag{1.6.5}$$

where $i_1(n,k) = i(n,k)$ is the classical Mahonian number.

These numbers $i_q(n,k)$ also have the recurrence relation,

$$i_q(n,k) = \sum_{j=0}^{n-1} q^{j(n-1)} i_q(n-1,k-j), \quad n > 1. \tag{1.6.6}$$

From the recurrence relation (1.6.6), Table 1.2 has been constructed for some initial values of the q-analogue of the Mahonian numbers.

n/k	0	1	2	3
1	1			
2	1	q		
3	1	$q+q^2$	q^3+q^4	q^5
4	1	$q+q^2+q^3$	$q^3+2q^4+q^5+q^6$	$\cdots$
5	1	$q+q^2+q^3+q^4$	$q^3+2q^4+2q^5+2q^6+q^7+q^8$	$\cdots$

Table 1.2: Table of the q-analogue of the Mahonian numbers.

The q-analogue of the Mahonian numbers satisfy the symmetry relation,

$$i_q(n,k) = q^{\frac{n(n-1)(2n-1)}{6}} i_{\frac{1}{q}}\left(n, \binom{n}{2} - k\right), n > 1. \tag{1.6.7}$$

And they have as generating function :

$$\sum_{k=0}^{\binom{n}{2}} i_q(n,k) z^k = \prod_{j=0}^{n-1} \left(1 + q^j z + \cdots + (q^j z)^j\right).$$

These numbers satisfy the following paths interpretation.

Theorem 1.6.6 *[Gmah23] Let $n \geq 1$, $0 \leq k \leq \binom{n}{2}$, then*

$$i_q(n,k) = \sum_{Q \in \mathcal{P}^I_{n,k}} q^{wt(Q)},$$

where $\mathcal{P}^I_{n,k}$ the subset of $\mathcal{P}(n,k)$ (the set of North-East paths starting from $(0,0)$ to $(n-1,k)$), and $wt(Q)$is the weight of Q wich counts the number of the boxes above and to the left of the path Q.

The link between the q-analogue of Mahonian numbers and Gaussian polynomial is given as follows.

Theorem 1.6.7 *[Gmah23] For $1 \leq k \leq \binom{n}{2}$,*

$$i_q(n,k) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q + \sum_{i=1}^{h} \begin{bmatrix} n+k-i-1 \\ k-i \end{bmatrix}_q \sum_{\lambda \in \mathcal{P}_D(j,n)} (-1)^{\ell(\lambda)} q^{\sum_{i=1}^{\ell(\lambda)} \lambda_i^2 - j},$$

where $\mathcal{P}_D(j,n)$ is the set of partitions of j into distinct parts where the largest part is $\leq n$.

In next section, we give a new generalization of Mahonian numbers and their combinatorial properties.

1.7 Mahonian number of type B

As a generalization of Mahonian numbers, Arslan [Arsl25] introduced the Mahonian numbers of type B, denoted by $i_B(n,k)$ [Bjor05, Peter15], which are based on a statistic on the hyperoctahedral group B_n, called the inversion of type B and denoted by $inv_B(\pi)$ [Foat78, Rein93, Bjor05, Peter15]. This number counts the signed permutations of length n with exactly k inversions of type B.

Definition 1.7.1 *Let $\pi \in B_n$. An* **inversion of type** *B of π is the sum of i-inversions $inv_i(\pi)$ of the permutation π, i.e.,*

$$inv_B(\pi) = \sum_{i=1}^{n} inv_i(\pi),$$

where $inv_i(\pi)$ satisfies the following relation [Arsl25]:

$$inv_i(\pi) = \begin{cases} inv_i(\sigma) = |\{(j, n+1-i) : j < n+1-i, \sigma_j > \sigma_{n+1-i}\}|, & \text{if } r_{n+1-i} = 0, \\ 1 + 2|\{(j, n+1-i) : j < n+1-i, \sigma_j < \sigma_{n+1-i}\}| + inv_i(\sigma), & \text{if } r_{n+1-i} = 1. \end{cases} \tag{1.7.1}$$

And each signed permutation $\pi \in \mathcal{B}_n$ has a unique representation of the form

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ (-1)^{r_1}\sigma_1 & (-1)^{r_2}\sigma_2 & \cdots & (-1)^{r_n}\sigma_n \end{pmatrix},$$

where $r_k = 0$ or 1 and $\sigma \in \mathcal{S}_n$.

Definition 1.7.2 *$Inv_B(\pi) = \{inv_1(\pi), \ldots, inv_n(\pi)\}$ is the* **inversion table** *of the permutation $\pi \in \mathcal{B}_n$.*

Example 1.7.3 *Let* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & -1 & 2 & -4 & 6 & 3 \end{pmatrix} \in \mathcal{B}_6$.
The inversion table of π *is* $Inv_B(\pi) = \{3, 0, 6, 1, 2, 0\}$, *and so* $inv_B(\pi) = 12$.

Remark 1.7.4 *The permutation of* B_n *having maximum number of inversions of type* B *is:*

$$\pi_0 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ -1 & -2 & \cdots & -(n-1) & -n \end{pmatrix},$$

which has $inv_B(\pi_0) = n^2$.

The total number of inversions of all permutations $\pi \in B_n$, namely $\mathcal{B}_n = \sum_{\pi \in B_n} inv_B(\pi)$, satisfies the following identities [Arsl25]:

$$\mathcal{B}_n = 2^{n-1} n^2 n!, \tag{1.7.2}$$

$$\mathcal{B}_n = 2^{n-1} n!(2n-1) + 2nB_{n-1} \text{ for } n \geq 2, \tag{1.7.3}$$

with $\mathcal{B}_1 = 1$.

Definition 1.7.5 *The number of signed permutations of length* n *with exactly* k *inversions of type* B *is called the Mahonian numbers of type* B, *and is denoted by* $i_B(n, k)$, *such that*

$$i_B(n, k) = |I_B(n, k)| = |\{\pi \in B_n : inv_B(\pi) = k\}|.$$

Similar to the combinatorial interpretation established by Ghemit and Ahmia [Gmah23] for classical Mahonian numbers $i(n, k)$, Arslan [Arsl25] gave the following combinatorial interpretation for Mahonian numbers of type B:

Combinatorial interpretation: $i_B(n, k)$ counts the number of ways to place "k" balls into "n" boxes such that the jth box contains at most "$2j - 1$" balls.

The Mahonian numbers of type B $i_B(n, k)$ have the following generating function [Arsl25].

Theorem 1.7.6

$$\sum_{k=0}^{n^2} i_B(n, k) q^k = (1 + q)(1 + q + q^2 + q^3) \cdots (1 + q + \cdots + q^{2n-1}). \tag{1.7.4}$$

And it satisfies the following recurrence relations [Arsl25, Theorems 4.2 and 4.3]:

Theorem 1.7.7 *For $n \geq 1$ and $0 \leq k \leq n^2$,*

$$i_B(n,k) = i_B(n, n^2 - k) \tag{1.7.5}$$

and

$$i_B(n,k) = \sum_{j=0}^{2n-1} i_B(n-1, k-j) \text{ for } n \geq 2 \tag{1.7.6}$$

where $i_B(1,0) = i_B(1,1) = 1$ and $i_B(n,k) = 0$ unless $0 \leq k \leq n^2$.

According to all properties of $i_B(n,k)$ and $\mathcal{B}_n$, Arslan [Arsl25] concluded that $\mathcal{B}_n = \sum_{k=0}^{n^2} i_B(n,k)k$.

These numbers form a triangle known as the "**Mahonian triangle of type** B". This triangle also appears in Sloane [Sloa64] as **A128084**. See, Table 1.3.

n/k	$\mathcal{B}_n$	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
1	**1**	1	1										
2	**16**	1	2	2	2	1							
3	**216**	1	3	5	7	8	8	7	5	3	1		
4	**3072**	1	4	9	16	24	32	39	44	46	44	39	$\cdots$

Table 1.3: The Mahonian triangle of type B.

1.8 Log-concavity, unimodality and q-log-concavity properties

Log-concave and unimodal sequences play an important role in combinatorics, algebra, geometry, statistics, probability and other areas of applied mathematics. Although various tools and technics have been developed to prove these properties, their demonstration generally remains difficult. In recent years, the development of q-analogues has enriched this field by introducing the concept of *q-log-concavity*. This notion extends the classical definition of log-concavity and offers a refined approach to studying polynomial sequences and their generating functions, revealing new perspectives and connections between combinatorics, algebra, and analysis. These advances have inspired considerable research activity in recent years. For more details, see [Bran97, Stanl89, Saga92].

Definition 1.8.1 *A sequence of nonnegative numbers $\{x_k\}_k$ is log-concave if*

$$x_{i-1}x_{i+1} \leq x_i^2 \quad for\ all \quad i > 0.$$

An equivalent definition can also be given as follows.

Definition 1.8.2 *A sequence of nonnegative numbers $\{x_k\}_k$ is log-concave if*

$$x_{i-1}x_{j+1} \leq x_i x_j \quad for \quad j \geq i \geq 1.$$

Example 1.8.3 *For a fixed n, the sequence of binomial coefficients is log-concave, since,*

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k+1)(n-k+1)}{k(n-k)} > 1.$$

Definition 1.8.4 *A finite sequence of real numbers $a_0, \ldots, a_m$ is said to be unimodal if there exist an index $0 \leq m^* \leq m$, called the mode of the sequence, such that a_k increases up to $k = m^*$ and decreases from then on, that is, $a_0 \leq a_1 \leq \cdots \leq a_{m^*}$ and $a_{m^*} \geq a_{m^*+1} \geq \cdots \geq a_m$.*

Remark 1.8.5 *A log-concave sequence $\{a_k\}_{0\leq k\leq n}$ is unimodal because, if it were not, it would have three consecutive values such that $a_{i-1} > a_i < a_{i+1}$, which contradicts the definition of log-concavity.*

Definition 1.8.6 *A polynomial is called log-concave (resp. unimodal) if its coefficients form a log-concave (resp. an unimodal) sequence.*

Example 1.8.7 *The q-binomial coefficients, also known as the Gaussian polynomials , are defined by*

$$\binom{n}{k}_q = \prod_{i=1}^{k} \frac{1-q^{n-i+1}}{1-q^i}, \quad 1 \leq k \leq n.$$

For a fixed n, the sequence of coefficients of $\left\{\binom{n}{k}_q\right\}_{1\leq k\leq n}$ forms a unimodal sequence in k [Zeil89].

Remark 1.8.8 *The product of log-concave (resp. unimodal) polynomials is log-concave [Stanl89] (resp. unimodal).*

Example 1.8.9 *The polynomial $\sum_{k=0}^{\binom{n}{2}} i(n,k)\, x^k$ is log-concave (respectively, unimodal) because it admits the following factorized form:*

$$\sum_{k=0}^{\binom{n}{2}} i(n,k)\, x^k = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1}).$$

It follows that the sequence of coefficients $i(n,0), i(n,1), \ldots, i(n, \binom{n}{2})$ is log-concave (respectively, unimodal).

Below, we give some examples of log-concave sequences.

Example 1.8.10 *1. The sequence of Stirling numbers of the second kind $\{S(n,k)\}_{n,k\geq 0}$ [Kurt72].*

2. *The sequences corresponding to transversals in Pascal's triangle $\left\{\binom{n-k}{k}\right\}_{k\geq 0}$ and $\left\{\binom{n-ak}{k}\right\}_{k\geq 0}$ [Tann74, Tann76].*

3. *The sequence of Eulerian numbers $\{A(n,k)\}_{k\geq 0}$ [Wang05].*

4. *The sequence of the unsigned Stirling numbers of the first kind $\{s(n,k)\}_{k\geq 0}$ [Wang05].*

5. *The sequences corresponding to transversals of the ordinary and generalized Pascal triangles $\left\{\binom{u+\alpha k}{v+\beta k}\right\}_{k\geq 0}$ and $\left\{\binom{u+\alpha k}{v+\beta k}_s\right\}_{k\geq 0}$, under certain conditions on the parameters α, β, u, and v [Belb08].*

Let $(A_n(q))_n$ be a sequence of polynomials where q is an indeterminate.

Definition 1.8.11 *1. We say that $\{A_n(q)\}_n$ is q-log-concave, if, for each $n \geq 1$, $A_n(q)^2 - A_{n-1}(q)A_{n+1}(q)$ has nonnegative coefficients as polynomial in q (for short, $A_n(q)^2 - A_{n-1}(q)$ $A_{n+1}(q) \geq_q 0$).*

2. *We say that $\{A_n(q)\}_n$ is strongly q-log-concave, if $A_m(q)A_n(q) - A_{m-1}(q)A_{n+1}(q) \geq_q 0$, for all $n \geq m \geq 1$.*

For more details, see Stanley [Stanl89].

Remark 1.8.12 *The strong q-log-concavity implies the q-log-concavity. But the converse is not true in general (see, Sagan [Saga92]). Furthermore, the q-log-concavity implies the log concavity and therefore the unimodality for each fixed positive number q.*

Below, we give some examples of q-log-concave sequences.

Example 1.8.13 *1. The sequence of the q-binomial coefficients [Butl90].*

2. *The sequence of the q-bi^{s}nomial coefficients [Baze18].*

3. *The sequence of q-Mahonian numbers $i_q(n,k)$ for n and k [Gmah23].*

4. *The sequence of Over q-bi^{s}nomial coefficients [Gahm23].*

CHAPTER 2

Combinatorics of Over-inversion Numbers

2.1 Introduction

The recent generalization of Mahonian numbers via generalized symmetric groups has attracted considerable attention from researchers in combinatorics. This generalization draws on both classical and modern developments in permutation statistics. A prominent example is the Mahonian numbers of type B, which extend the classical Mahonian numbers originally defined for the symmetric group S_n to the **hyperoctahedral group** B_n by introducing inversions of type B . These inversions account for both the order of element pairs and their signs. The group B_n consists of signed permutations of n elements, where each element can independently be positive or negative. For further details on Mahonian numbers of type B, see [Arsl25, Keah25].

Considering overlined permutations, where each element may independently be overlined or not, rather than signed permutations, still places us entirely within the hyperoctahedral group B_n. The hyperoctahedral groups have been the subject of extensive study, as demonstrated by works such as [Boro10, Cart72, Hump90, Kerb71, Kerb75, Yuon30].

In this chapter, we introduce the concept of an overlined permutation σ of n elements, in which the elements involved in inversions may be independently overlined or not. In such a permutation, each element $\sigma(i)$, for $1 \leq i \leq n-1$, can independently be overlined or not, provided there exists an index $j > i$ such that $\sigma(i) > \sigma(j)$. These permutations form a subset of the group of all overlined permutations $G_{2,n}$, denoted by B'_n. We study the enumeration of such permutations having exactly k inversions, denoted by $i_{B'}(n,k)$, and designate these numbers as the **Over-inversion numbers**. We also explore their combinatorial interpretations and the identities they satisfy.

2.2 Basic properties of Over-inversion numbers

This section is devoted to the study of the main properties of Over inversion numbers. We start by introducing their definition, then we present a recurrence relation that allows their computation. We also give some basic identities involving these numbers, as well as their generating function, which plays a key role in understanding their combinatorial behavior.

Definition 2.2.1 *Let* $\Sigma_{2,n} := \{1,\ldots,n,\overline{1},\ldots,\overline{n}\}$ *be the set of* $2n$ *symbols, where each* $\overline{i}$ *denotes an* overlined element *associated with* i. *An overlined permutation* π *is a permutation defined on the set* $\Sigma_{2,n}$ *that satisfies the property* $\pi(\overline{a}) = \overline{\pi(a)}$ *for all* $a \in \Sigma_{2,n}$.

Example 2.2.2 *The following is an overlined permutation of* $\Sigma_{2,4}$*:*

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \overline{4} & \overline{2} & 3 & 1 \end{pmatrix}.$$

By omitting the first row, we obtain the one-line notation $\overline{4}\,\overline{2}31$.

Let $G_{2,n}$ denote the group of all overlined permutations of $\Sigma_{2,n}$. This group is isomorphic to the hyperoctahedral group B_n, also known as the Coxeter group of type B, with cardinality $|G_{2,n}| = 2^n n!$. In algebraic combinatorics, $G_{2,n}$ can be expressed as the wreath product $C_2 \wr S_n$, combining the symmetric group S_n with the cyclic group C_2 on $\{0,1\}$. For the purposes of this thesis, however, the detailed group theoretic structure of $G_{2,n}$ will not be required.

We consider the subset $B'_n \subset G_{2,n}$ obtained by imposing specific constraints on the elements of the overlined permutation group $G_{2,n}$. In this subset, the first elements participating in inversions may be independently overlined or not. The enumeration of B'_n is then carried out with respect to the number of inversions.

Example 2.2.3 *The overlined permutations of* B'_3 *are*

$$123, 132, 1\overline{3}2, 213, \overline{2}13, 231, \overline{2}31, 2\overline{3}1, \overline{2}\,\overline{3}1, 312, \overline{3}12, 321, \overline{3}21, 3\overline{2}1, \overline{3}\,\overline{2}1.$$

Definition 2.2.4 *A permutation σ on S_n is an involution if $\sigma^2(i) = i$ for all $i = 1, \ldots, n$, and it is* **fixed-point-free** *if no element is mapped to itself, meaning $\sigma(i) \neq i$ for all i.*

Remark 2.2.5

1. *The cardinality $|B'_n|$ is equal to the number of fixed-point-free involutions in the symmetric group S_{2n}.*
2. *The cardinality $|B'_n|$ is equal also to the number of permutations in the symmetric group S_{2n} whose cycle decomposition is a product of n disjoint transpositions.*

Based on the definition of B'_n, in the next, we introduce the concept of Over-inversion numbers along with their fundamental properties.

Definition 2.2.6 *Let (σ_i, σ_j) be an inversion of the permutation σ. We say (σ_i, σ_j) is an* **overlined inversion** *if the entry σ_i is overlined or not.*

Definition 2.2.7 *Let $i_{B'}(n, k)$ represent the number of overlined permutations in B'_n of length n that contain exactly k inversions. We refer to this quantity as the* ***Over-inversion number****.*

Example 2.2.8 *Consider the overlined permutations of B'_3 with 2 inversions:*

$$312, \overline{\mathbf{3}}12, 231, \overline{\mathbf{2}}31, 2\overline{\mathbf{3}}1, \overline{\mathbf{2}}\,\overline{\mathbf{3}}1.$$

Therefore, $i_{B'}(3, 2) = 6$.

From Definition 2.2.7, we can derive the recurrence relation for the Over-inversion number $i_{B'}(n, k)$ as follows:

Theorem 2.2.9 *For positive integers n and k, the Over-inversion number satisfies the recurrence relation*

$$i_{B'}(n, k) = i_{B'}(n-1, k) + 2\sum_{j=1}^{n-1} i_{B'}(n-1, k-j), \tag{2.2.1}$$

with the initial conditions $i_{B'}(n, 0) = 1$ and $i_{B'}(n, k) = 0$ unless $0 \leq k \leq \binom{n}{2}$.

Proof We prove this theorem combinatorially using the inversion combinatorial interpretation given in Definition 2.2.7.

Consider a permutation $\sigma = \sigma_1 \cdots \sigma_n$ of length n with k overlined inversions. Removing the first element σ_1 form σ results in either:

1. A permutation of length $n-1$ with k overlined inversions, if σ_1 does not form an overlined inversion with any other element. This corresponds to $i_{B'}(n-1,k)$, or

2. A permutation of length $n-1$ with $k-j$ overlined inversions, if σ_1 either forms j overlined inversions (where $1 \leq j \leq n-1$) with other elements or is itself overlined and forms j overlined inversions with the others. In both cases, the corresponding interpretation is $i_{B'}(n-1,k-j)$.

Summing these possibilities yields $2\sum_{j=1}^{n-1} i_{B'}(n-1,k-j)$, which leads to the equality (2.2.1). ∎

From Theorem 2.2.9, we derive the following recurrence relation for the Over-inversion numbers $i_{B'}(n,k)$:

Proposition 2.2.10 *For $0 \leq k \leq \binom{n}{2}$, the Over-inversion numbers satisfy the recurrence*

$$i_{B'}(n,k) = i_{B'}(n,k-1) + i_{B'}(n-1,k) + i_{B'}(n-1,k-1) - 2i_{B'}(n-1,k-n). \tag{2.2.2}$$

Proof From relation (2.2.1), we have the following expressions:

$$i_{B'}(n,k) = i_{B'}(n-1,k) + 2i_{B'}(n-1,k-1) + \cdots + 2i_{B'}(n-1,k-n+1) \tag{2.2.3}$$

and

$$i_{B'}(n,k-1) = i_{B'}(n-1,k-1) + 2i_{B'}(n-1,k-2) + \cdots + 2i_{B'}(n-1,k-n). \tag{2.2.4}$$

Using relation (2.2.4), we substitute into the equation for $i_{B'}(n,k)$:

$$\begin{aligned} i_{B'}(n,k) &= i_{B'}(n-1,k) + i_{B'}(n-1,k-1) + [i_{B'}(n-1,k-1) + 2i_{B'}(n-1,k-2) \\ &\quad + \cdots + 2i_{B'}(n-1,k-n)] - 2i_{B'}(n-1,k-n) \\ &= i_{B'}(n-1,k) + i_{B'}(n-1,k-1) + i_{B'}(n,k-1) - 2i_{B'}(n-1,k-n) \\ &= i_{B'}(n-1,k) + i_{B'}(n-1,k) + i_{B'}(n-1,k-1) - 2i_{B'}(n-1,k-n). \end{aligned}$$

This completes the proof. ∎

Based on Definition 2.2.7, we can also derive the following identities.

Proposition 2.2.11 *For any integer $n \geq 1$, the following hold:*

1. $i_{B'}(n, \binom{n}{2}) = 2^{n-1}$,

2. $i_{B'}(n, 1) = 2 \times (n-1)$,

3. $i_{B'}(n, k)$ *is even for* $n \geq 2$ *and* $k \geq 1$.

Proof 1. The expression $i_{B'}(n, \binom{n}{2})$ counts the number of permutations of length n that have $\binom{n}{2}$ overlined inversions. This implies that the permutation is in the form $\sigma_1 > \sigma_2 > \cdots > \sigma_{n-1} > \sigma_n$, where the entries $\sigma_1, \ldots, \sigma_{n-1}$ may or may not be overlined. Thus, there are 2^{n-1} possible permutations.

2. The expression $i_{B'}(n, 1)$ counts the number of permutations of length n that have exactly one overlined inversion. This corresponds to a permutation in the form $\sigma_1 < \sigma_2 < \cdots < \sigma_j > \sigma_{j+1} < \cdots < \sigma_n$, where σ_j may or may not be overlined, for each $1 \leq j \leq (n-1)$. Therefore, there are $2 \times (n-1)$ possible permutations.

3. For $n \geq 2$ and $k \geq 1$, we can have overlined inversions. For each inversion (σ_i, σ_j), there are two possibilities: either σ_i is overlined or it is not. Consequently, the total number of permutations of length n with exactly k overlined inversions is even.

The Over-inversion numbers $i_{B'}(n, k)$ satisfy the following row-generating function:

Theorem 2.2.12 *For any positive integer* n,

$$\sum_{k=0}^{\binom{n}{2}} i_{B'}(n,k)z^k = \begin{cases} 1, & \textit{if} \quad n = 1, \\ (1+2z)\cdots(1+2z+\cdots+2z^{n-1}), & \textit{if} \quad n > 1. \end{cases}$$

Proof Let $f_n(z) = \sum_{k=0}^{\binom{n}{2}} i_{B'}(n,k)z^k$ be the row generating function of the Over-inversion number.

For $n = 1$, the result is trivial. For $n \geq 2$, using equation (2.2.1), we have

$$f_n(z) = \sum_{k=0}^{\binom{n}{2}} i_{B'}(n,k)z^k = \sum_{k=0}^{\binom{n}{2}} \left(i_{B'}(n-1,k) + 2\sum_{j=1}^{n-1} i_{B'}(n-1,k-j) \right) z^k.$$

This expands to

$$f_n(z) = \sum_{k=0}^{\binom{n}{2}} i_{B'}(n-1,k)z^k + 2\sum_{k=0}^{\binom{n}{2}} \sum_{j=1}^{n-1} i_{B'}(n-1,k-j)z^k.$$

The first term is simply $f_{n-1}(z)$, and the second term can be written as

$$2\sum_{j=1}^{n-1} z^j \sum_{k=j}^{\binom{n}{2}-j} i_{B'}(n-1,k-j)z^{k-j} = 2\sum_{j=1}^{n-1} z^j \sum_{k=0}^{\binom{n-1}{2}} i_{B'}(n-1,k)z^k.$$

Thus, we have

$$f_n(z) = f_{n-1}(z)\left(1 + 2\sum_{j=1}^{n-1} z^j\right).$$

Iterating this process, we obtain the final result:

$$f_n(z) = (1+2z)\cdots(1+2z+\cdots+2z^{n-1}).$$

∎

By setting $z = 1$ in Theorem 2.2.12, we obtain the following interesting result:

Corollary 2.2.13 *For any positive integer n, the Over-inversion numbers satisfy the identity :*

$$|B_n'| = \sum_{k=0}^{\binom{n}{2}} i_{B'}(n,k) = (2n-1)!!,$$

where $(2n-1)!! = 1 \times 3 \times \cdots \times (2n-1)$.

Consider the set of $2n-1$ symbols

$$\Sigma_{2,n}' := \left\{1, 2, \overline{2}, \ldots, n, \overline{n}\right\}.$$

Let $C_j = \{\sigma \in B_n' : \sigma(1) = j\}$ for $j \in \Sigma_{2,n}'$. It is clear that

$$|C_j| = (2n-3)!!, \tag{2.2.5}$$

with $|C_1| = 1$ when $n = 1$.

The set C_j gives the decomposition

$$B_n' = \biguplus_{j\in\Sigma_{2,n}'} C_j.$$

Therefore, we have

$$\mathcal{B}_n' = \sum_{\omega\in B_n'} inv(\omega) = \sum_{j\in\Sigma_{2,n}'}\sum_{\sigma\in C_j} inv(\sigma), \tag{2.2.6}$$

where $\mathcal{B}_n'$ represents the total number of inversions of all overlined permutations in B_n'.

Let $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ j & \pi(2) & \cdots & \pi(n) \end{pmatrix} \in C_j$ and τ be is an overlined permutation in B'_{n-1} defined by

$$\tau = \begin{pmatrix} a_1 & \cdots & a_{n-1} \\ \pi(2) & \cdots & \pi(n) \end{pmatrix} \in B'\left([n]\backslash\{|j|\}\right),$$

where $a_1, \ldots, a_{n-1}$ are an arrangement of elements of $[n]\backslash\{|j|\}$ in increasing order, and $B'\left([n]\backslash\{|j|\}\right)$ is the group of all the overlined permutation of the set $[n]\backslash\{|j|\}$. So, if we set

$$\pi_{\tau,j} = \begin{pmatrix} 1 & 2 & \cdots & n \\ j & \tau(a_1) & \cdots & \tau(a_{n-1}) \end{pmatrix},$$

then we obtain $\pi = \pi_{\tau,j}$. Hence by the definition of the statistic of the classical inversion, we conclude

$$inv(\pi) = (j-1) + inv(\tau). \tag{2.2.7}$$

Equation (2.2.7) give us a recursive formula for $\mathcal{B}'_n$ and we state it as a proposition.

Proposition 2.2.14 *We have $\mathcal{B}'_1 = 0$, and*

$$\mathcal{B}'_n = (2n-3)!!n(n-1) + (2n-1)\mathcal{B}'_{n-1}, \text{ for } n \geq 2. \tag{2.2.8}$$

Proof Since $B'_1 = \{\iota\}$, then $\mathcal{B}'_1 = inv(\iota) = 0$. Now suppose $n \geq 2$. We have two cases:

Case 1. For $j = 1$, we obtain

$$\begin{aligned} \sum_{\pi \in C_j} inv(\pi) &= \sum_{\tau \in B'([n]\backslash\{|j|\})} inv(\pi_{\tau,j}) \\ &= \sum_{\tau \in B'([n]\backslash\{|j|\})} inv(\tau) \\ &= \mathcal{B}'_{n-1}. \end{aligned}$$

Case 2. For $j \in \Sigma'_{2,n}\backslash\{1\}$. From equation (4.2.2), we obtain

$$\begin{aligned} \sum_{\pi \in C_j} inv(\pi) &= \sum_{\tau \in B'([n]\backslash\{|j|\})} inv(\pi_{\tau,j}) \\ &= \sum_{\tau \in B'([n]\backslash\{|j|\})} [(j-1) + inv(\tau)] \\ &= (j-1)(2n-3)!! + \mathcal{B}'_{n-1}. \end{aligned}$$

From equation (2.2.6), we get

$$\begin{aligned}\mathcal{B}'_n &= \mathcal{B}'_{n-1} + 2\sum_{j=2}^{n}\left[(j-1)(2n-3)!! + \mathcal{B}'_{n-1}\right]\\ &= 2(2n-3)!!\sum_{j=2}^{n}(j-1) + (2n-1)\mathcal{B}'_{n-1}\\ &= (2n-3)!!n(n-1) + (2n-1)\mathcal{B}'_{n-1}\end{aligned}$$

as desired. ∎

Note that $\mathcal{B}'_n = \sum_{k=0}^{\binom{n}{2}} inv'_B(n,k)k$. The numbers $i_{B'}(n,k)$ form a triangle known as the "**Over-inversion triangle**", as shown in Table 2.1.

n/k	$\mathcal{B}'_n$	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
1	**0**	1											
2	**2**	1	2										
3	**28**	1	4	6	4								
4	**376**	1	6	16	26	28	20	8					
5	**5484**	1	8	30	72	126	172	188	164	112	56	16	$\cdots$

Table 2.1: The Over-inversion triangle.

2.3 Combinatorial interpretations of the Over-inversion numbers

In this section, we provide combinatorial interpretations of the Over-inversion numbers through lattice paths, overpartitions, and tilings.

2.3.1 A lattice path interpretation

Ghemit and Ahmia in [Gahm23] demonstrated that the Mahonian number $i(n,k)$ counts the number of lattice paths from $u_1 = (0,0)$ to $v_1 = (n-1,k)$ that take at most j North steps at the level j.

Following a similar approach, we provide a combinatorial interpretation for the Over-inversion numbers $i_{B'}(n,k)$ using lattice paths.

Let $\mathcal{P}_{n,k}^{B'}$ represent the set of lattice paths from the point $(0,0)$ to $(n-1,k)$, using only North steps (vertical steps $(0,1)$), East steps (horizontal steps $(1,0)$), and North-East steps (diagonal steps $(1,1)$). These paths must satisfy the condition that, for each vertical level j, the number of North steps is at most j unless a diagonal step occurs before level j, in which case the number of North steps is restricted to at most $j-1$ after the diagonal step. The levels correspond to the vertical lines ranging from 0 to $n-1$, as illustrated in Figure 2.1.

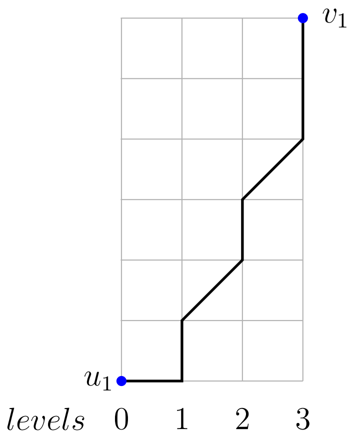

Figure 2.1: A path P in $\mathcal{P}_{4,6}^{B'}$.

Theorem 2.3.1 *The Over-inversion number $i_{B'}(n,k)$ represents the count of lattice paths from $u_1=(0,0)$ to $v_1=(n-1,k)$, where the path takes at most j North steps at level j before any diagonal step, and at most $(j-1)$ North steps at level j after a diagonal step. Specifically, we have*

$$i_{B'}(n,k) = \mid \mathcal{P}_{n,k}^{B'} \mid .$$

Proof Since $i_{B'}(n,k)$ counts the number of overlined permutations of length n with exactly k overlined inversions, it suffices to show a bijection between these permutations and the lattice paths in $\mathcal{P}_{n,k}^{B'}$. We do so as follows :

For each path $P \in \mathcal{P}_{n,k}^{B'}$, we can uniquely determine the associated permutation. This is done by assigning the entry 1 to the point $(0,0)$, and then following the steps of the path. The first step must be either an East step or a North-East step. If the step is East, we move to the point $(1,0)$ and place the entry 2 to the right of 1 (resulting in the partial permutation 12). If the step is North-East, we move to the point $(1, 1)$ and place the entry $\overline{2}$ to the left of 1, yielding the partial permutation $\overline{2}1$, which introduces an inversion.

At the point $(1, 0)$, there are three possible cases for the next step of P:

- If the next step is East, we add the entry 3 to the right of 2, resulting in the partial permutation 123.
- If the next step is North-East, we add the entry $\overline{3}$ to the right of 2 and shift $\overline{3}$ one position to the left, producing the permutation $1\overline{3}2$, which introduces another inversion.
- If the next step is North, we shift the entry 2 one position to the left, yielding the permutation 21, which also introduces an inversion.

At the point $(1, 1)$, there are two possible cases for the next step:

- If the next step is East, we add the entry 3 to the right of 1, producing the permutation $\overline{2}13$.
- If the next step is North-East, we add $\overline{3}$ to the right of 1 and shift $\overline{3}$ one position to the left, resulting in the permutation $\overline{2}\,\overline{3}1$ which introduces an inversion.

We continue applying these rules at each step of the path until we reach the point $(n-1, k)$, which gives us the desired permutation. An example of this process is shown in Figure 2.2. ∎

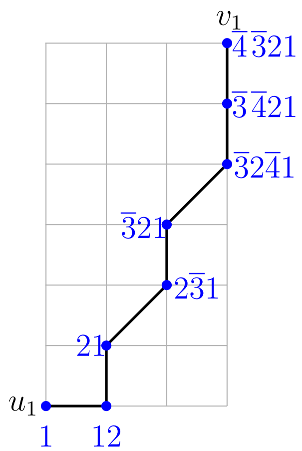

Figure 2.2: The path associated to the permutation $\sigma = \overline{4}\,\overline{3}21$.

2.3.2 An overpartition interpretation

In this subsection, we present combinatorial interpretations of the Over-inversion numbers using overpartitions. Observing that the bijection between lattice paths and tilings preserves weight, we derive the following interpretation of the Over-inversion numbers in terms of overpartitions :

Theorem 2.3.2 *The Over-inversion number $i_{B'}(n,k)$ represents the count of overpartitions into k parts, where each part j can appear at most j times if it is not overlined, and at most $j-1$ times if it is overlined. Additionally, the largest part must satisfy $\leq n-1$.*

Proof The term $i_{B'}(n,k)$ represents the count of lattice paths from $(0,0)$ to $(n-1,k)$, where the paths adhere to specific rules: at most j North steps are allowed at level j if no diagonal step occurs beforehand, and at most $j-1$ North steps are permitted at level j if a diagonal step has occurred. If we interpret the parts (resp. the overlined parts) as representing the cases above each North step (resp. North-East step) in the paths associated with $i_{B'}(n,k)$, we can deduce a direct connection to overpartitions. Specifically, $i_{B'}(n,k)$ enumerates the number of overpartitions of k into parts where each part j can appear at most j times if it is not overlined and at most $j-1$ times if it is overlined, with the additional condition that the largest part must be at most $n-1$. ∎

For instance, Figure 2.3 illustrates the 16 overpartitions with $k=2$ parts, where the largest part is $n-1=3$.

2.3.3 A tiling interpretation

Ghemit and Ahmia [Gmah23] demonstrated that the Mahonian number $i(n,k)$ represents the number of ways to tile a board of length $n+k-1$ using k red squares and $n-1$ blue squares, subject to the condition that no more than j consecutive red squares appear if preceded by j blue squares. Building on their findings, this subsection provides a combinatorial interpretation of the Over-inversion numbers using the same combinatorial structures.

Let $\mathcal{T}_{n,k}^{B'}$ denote the set of all tilings of a board of length $(n+k-1)$ that satisfy the following conditions:

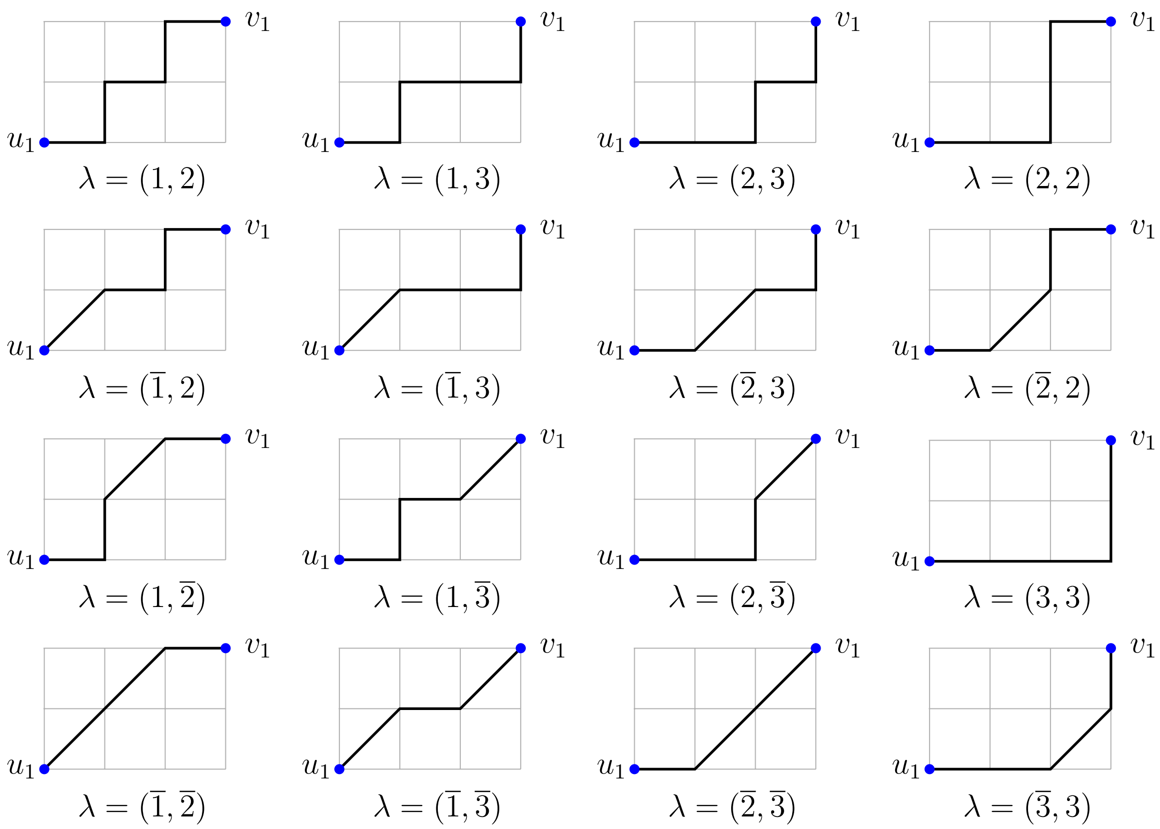

Figure 2.3: All lattice paths/overpartitions corresponding to $n = 4$ and $k = 2$.

1. The tiling uses at most $n-1$ blue squares, at most k red squares, and black rectangles (each covering an area of two squares, referred to as square pairs).

2. The number of consecutive red squares does not exceed j if preceded by j blue squares, and does not exceed $j-1$ if preceded by a black rectangle.

3. The total number of red squares and black rectangles is exactly k.

4. The total number of blue squares and black rectangles is exactly $n-1$.

5. Each black rectangle is treated as equivalent to one blue square followed by one red square in that order.

Remark 2.3.3 *There exists a bijection between the sets $\mathcal{P}^{B'}_{n,k}$ and $\mathcal{T}^{B'}_{n,k}$. Specifically, each East step corresponds to a blue square, each North step corresponds to a red square, and each diagonal step corresponds to a black rectangle. This correspondence is illustrated in Figure 2.4.*

Based on this observation, we can derive the following tiling interpretation for the Over-inversion numbers:

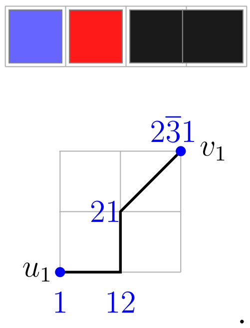

Figure 2.4: The tiling/lattice path corresponding to the permutation $\sigma = 2\bar{3}1$.

Corollary 2.3.4 *The Over-inversion number $i_{B'}(n,k)$ is precisely the cardinality of the set $\mathcal{T}^{B'}_{n,k}$. In other words, $i_{B'}(n,k) = \mid \mathcal{T}^{B'}_{n,k} \mid$.*

Example 2.3.5 *For $n = 3$ and $k = 2$, the $i_{B'}(3,2)$ tilings consist of 2 blue squares, 2 red squares, and a black rectangle, as illustrated in Figure 2.5.*

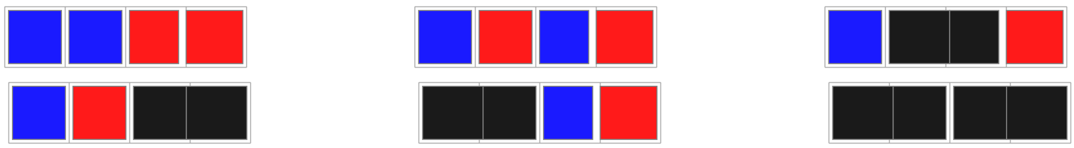

Figure 2.5: The tiling interpretation associated to $i_{B'}(3,2)$.

CHAPTER 3

Combinatorics of Mahonian numbers of type B and their analogue

This chapter is a continuation of the earlier work of Arslan et al.[Arsl25], who studied algebraic and combinatorial properties of the Mahonian numbers of type B, addressing questions raised in their paper "A combinatorial interpretation of Mahonian numbers of types B and D", published in the "Ramanujan Journal" (2025). We first give the Knuth-Netto formula and generating function for the subdiagonals on or below the main diagonal of the Mahonian numbers of type B, then its combinatorial interpretations by lattice path/partition and tiling. Next, we propose a q-analogue of Mahonian numbers of type B by using a new statistics on the permutations of the hyperoctahedral group B_n that we introduced, then we study their basic properties and their combinatorial interpretations by lattice path/partition and tiling.

3.1 Combinatorial formulas and interpretations

This section is devoted to some combinatorial formulas: a recurrence relation, the Knuth-Netto formula, and generating function for the subdiagonals on or below the main diagonal of Mahonian numbers of type B and its combinatorial interpretations using lattice paths and partitions/tilings. The lattice path interpretation allows us to say that there is a bijection between these lattice paths and the signed permutations of the hyperoctahedral group B_n, and this means that we can build these permutations through these lattice paths.

We can easily obtain from (1.7.6) that the following recurrence relation for the Mahonian numbers of type B holds.

Proposition 3.1.1 *For $n > 1$ and $0 \leq k \leq n^2$, we have*

$$i_B(n,k) = i_B(n,k-1) + i_B(n-1,k) - i_B(n-1,k-2n). \tag{3.1.1}$$

3.1.1 The Knuth-Netto formula of Mahonian numbers of type B

In this subsection, we give the analogue of the Knuth-Netto [Knut73, Nett27] formula for the case of Mahonian numbers $i_B(n,k)$ of type B, a question proposed by Arslan in his paper [Arsl25].

First of all, we give Knuth-Netto [Knut73, Nett27] formula for the kth Mahonian number $i(n,k)$ when $k \leq n$ as follows,

$$i(n,k) = \binom{n+k-1}{k} + \sum_{j=1}^{+\infty}(-1)^j \binom{n+k-u_j-j-1}{k-u_j-j} + \sum_{j=1}^{+\infty}(-1)^j \binom{n+k-u_j-1}{k-u_j}, \tag{3.1.2}$$

where $u_j = \frac{j(3j-1)}{2}$ is the jth pentagonal number, see Figure 3.1.

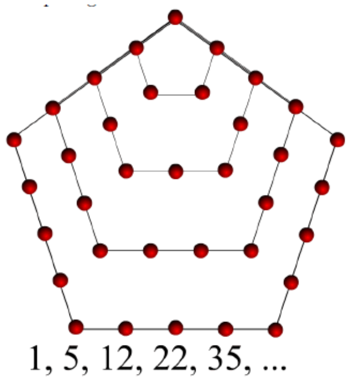

Figure 3.1: The pentagonal numbers.

This formula follows from the generating function and **Euler**'s pentagonal number theorem.

Theorem 3.1.2 *[Andr98, Hard54, Indo08] We have,*

$$\prod_{j=1}^{+\infty}(1-q^j) = \sum_{k=-\infty}^{+\infty}(-1)^k q^{u_k} = 1 + \sum_{k=1}^{+\infty}(-1)^k \left(q^{\frac{k(3k-1)}{2}} + q^{\frac{k(3k+1)}{2}}\right). \tag{3.1.3}$$

By letting $q \mapsto q^2$ in the previous theorem, we find an analogue of the generating function and Euler's pentagonal number theorem as follows.

Corollary 3.1.3 *We have,*

$$\prod_{j=1}^{+\infty}(1-q^{2j}) = \sum_{k=-\infty}^{+\infty}(-1)^k q^{2u_k} = 1+\sum_{k=1}^{+\infty}(-1)^k\left(q^{k(3k-1)}+q^{k(3k+1)}\right). \tag{3.1.4}$$

Using this corollary, we obtain the analogue of the Knuth-Netto formula associated to Mahonian numbers $i_B(n,k)$ of type B.

Theorem 3.1.4 *For $n \geq 1$ and $0 \leq k \leq n$, we have*

$$i_B(n,k) = \binom{n+k-1}{k} + \sum_{j=1}^{+\infty}(-1)^j\binom{n+k-2u_j-2j-1}{k-2u_j-2j} + \sum_{j=1}^{+\infty}(-1)^j\binom{n+k-2u_j-1}{k-2u_j}. \tag{3.1.5}$$

Proof From the generating function of $i_B(n,k)$, we have

$$\begin{aligned}\sum_{k=0}^{n} i_B(n,k)q^k &= \prod_{j=1}^{n}\frac{1-q^{2j}}{1-q} = \frac{1}{(1-q^n)}\prod_{j=1}^{n}(1-q^{2j}) \\ &= \prod_{j=1}^{n}(1-q^{2j})\sum_{l=0}^{+\infty}\binom{n+l-1}{l}q^l.\end{aligned}$$

The coefficients of $\prod_{j=1}^{n}(1-q^{2j})$ in the last equation will match those in the power series expansion of the infinite product of Corollary 3.1.3 given by the analogue of Euler's pentagonal number theorem up to the coefficient on q^n.

By Corollary 3.1.3, we consider the product

$$\prod_{j=1}^{+\infty}(1-q^{2j})\sum_{l=0}^{+\infty}\binom{n+l-1}{l}q^l = \left(1+\sum_{j=1}^{+\infty}(-1)^j\left(q^{j(3j-1)}+q^{j(3j+1)}\right)\right)\sum_{l=0}^{+\infty}\binom{n+l-1}{l}q^l.$$

From this equation, the coefficient of q^k, for $k \leq n$, is

$$\binom{n+k-1}{k} + \sum_{j=1}^{+\infty}(-1)^j\binom{n+k-2u_j-2j-1}{k-2u_j-2j} + \sum_{j=1}^{+\infty}(-1)^j\binom{n+k-2u_j-1}{k-2u_j},$$

which is exactly the right side of the desired identity of $i_B(n,k)$. ∎

Example 3.1.5 *We can illustrate the formula in Theorem 3.1.4 as follows:*

$$i_B(4,7) = \binom{9}{5} - \binom{8-2u_1}{5-2u_1} - \binom{10-2u_1}{7-2u_1} = 44$$

where $u_1 = 1$.

3.1.2 The generating functions for the subdiagonals of Mahonian triangle of type B

In this subsection, we shall derive generating functions for the subdiagonals on or below the main diagonal of **the Mahonian triangle of type** B. For $j \geq 0$, let $T_j(x) = \sum_{n\geq 0} i_B(n, n-j)x^n$ is the generating function for signed permutations of $\langle n\rangle = \{-n, \ldots, -1, 1, \ldots, n\}$, of the hyperoctahedral group $\mathcal{B}_n$, with exactly $n-j$ inversions.

Example 3.1.6 *The first four functions of $T_j(x)$ are*

$$T_0(x) = 1 + x + 2x^2 + 7x^3 + 24x^4 + 86x^5 + \cdots$$
$$T_1(x) = x + 2x^2 + 5x^3 + 16x^4 + 54x^5 + 190x^6 + \cdots$$
$$T_2(x) = x^2 + 3x^3 + 9x^4 + 30x^5 + 104x^6 + 371x^7 + \cdots$$
$$T_3(x) = x^3 + 4x^4 + 14x^5 + 50x^6 + 181x^7 + 664x^8 + \cdots$$

By using the concepts of inversion table in the symmetric group $\mathcal{S}_n$ and subdiagonal sequence, Calaesson et al. [Clae23] showed that the generating function for permutations of $[n]$ with exactly $n-j$ inversions, denoted by $S_j(x)$, satisfies the following relation:

$$S_j(x) = (xC(x))^j\, S_0(x) \tag{3.1.6}$$

where $S_0(x) = 1 + x^3 + 5x^4 + 22x^5 + \cdots$ is the generating function for permutations of $[n]$ with exactly n inversions, and $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, which are shown [Clae23, Lemma 1] equal to number of weakly increasing subdiagonal sequences of length n.

By the same concepts used by Calaesson et al. in [Clae23] but in the hyperoctahedral group $\mathcal{B}_n$, we establish in the following theorem a similar relation to (3.1.6) for the generating function of signed permutations of $\langle n\rangle$ with exactly $n-j$ inversions.

We start by the following notation: Let $\mathcal{C}_B(n)$ be the subset of $I_B(n, n-1)$ (i.e., the set of permutations of B_n having $n-1$ inversions of type B) consisting of those permutations whose every prefix of length $k \geq 1$ has fewer than k inversions of type B.

Calaesson et *al.* [Clae23, Lemma 1] proved that $|\mathcal{C}_n| = C_{n-1}$, where $\mathcal{C}_n$ is the subset of the set $I(n, n-1)$ (i.e., the set of classical permutations of n having $n-1$ inversions) consisting of those permutations whose every prefix of length $k \geq 1$ has fewer than k inversions.

From the definition of $\mathcal{C}_B(n)$ and [Clae23, Lemma 1], we can give the following lemma.

Lemma 3.1.7 *For $n \geq 1$, $|\mathcal{C}_B(n)| = |\mathcal{C}_n| = C_{n-1}$.*

Proof To prove this lemma, it suffices to show that the set $\mathcal{C}_B(n)$ is exactly the set $\mathcal{C}_n$.

The subset of $I_B(n, n-1)$ consisting of those permutations whose every k-prefix, $k \geq 1$, has fewer than k inversions of type B is the same $\mathcal{C}_n$, because the only permutations of B_n that have k-prefixes have fewer than k inversions of type B are, from Definition 1.7.1 and relation (1.7.1), those that have elements with a positive sign. This gives us the same permutations of the set $\mathcal{C}_n$. Consequently, the sets $\mathcal{C}_B(n)$ and $\mathcal{C}_n$ are the same, which gives us the result of the lemma, see Example 3.1.8. ∎

Example 3.1.8 *For $n = 4$, the set of signed permutations of length 4 having 3 inversions is $I_B(4,3) = \{4123, 1432, 2413, 3142, 2341, 3214, -3124, -2143, -1423, -2-134, -2314, -1342, 3-124, 2-143, 1-234, 23-14\}$. Then, the set of permutation of length 4 whose each prefix of length $k \geq 1$ has fewer than k inversions of type B is*

$$\mathcal{C}_B(4) = \{4123, 1432, 2413, 3142, 2341\}.$$

Thus, $|\mathcal{C}_B(4)| = C_3 = 5$.

Recall that $T_j(x)$ is the generating function for signed permutations of length n with $n-j$ inversions of type B:

$$T_j(x) = \sum_{n \geq 0} |I_B(n, n-j)| x^n.$$

We omit the proof of Theorem 3.1.9 for brevity since it can be easily seen using Lemma 3.1.7 and the same arguments in [Clae23, Theorem 3].

Theorem 3.1.9 *For $j \geq 1$, we have*

$$I_B(n, n-j-1) \simeq \bigsqcup_{i=0}^{n} I_B(i, i-j) \times \mathcal{C}_B(n-i)$$

and thus the generating function $T_{j+1}(x)$ and $T_j(x)$ satisfy the identity

$$T_{j+1}(x) = xC(x)T_j(x),$$

equivalently

$$T_j(x) = (xC(x))^j \, T_0(x). \tag{3.1.7}$$

Calaesson et al. [Clae23, Theorem 4] used the partition theory to prove that $S_0(x)$ satisfies the following relation:

$$S_0(x) = R\left(xC(x)\right), \tag{3.1.8}$$

where $R(x) = \frac{1-x}{1-2x}\prod_{n=1}^{+\infty}(1-x^n)$.

By the same approach given in [Clae23, Theorem 4], we establish the following theorem which generates the coefficients of the generating function $T_0(x)$. We omit the details of the proof for brevity.

Theorem 3.1.10 *We have,*

$$T_0(x) = L\left(xC(x)\right), \tag{3.1.9}$$

where $L(x) = \frac{1-x}{1-2x}\prod_{n=1}^{+\infty}(1-x^{2n})$.

3.1.3 Lattice path interpretation

Due to Ghemit and Ahmia [Gmah23], the Mahonian number $i(n,k)$ counts the number of lattice paths from $u=(0,0)$ to $v=(n-1,k)$ taking at most j North steps at the level j. This interpretation allows us to give in this subsection a combinatorial interpretation by lattice paths for the Mahonian numbers of type B. Let $\mathcal{P}^B_{n,k}$ denote the set of lattice paths from the point $u=(0,0)$ to the point $v=(n,k)$ for $n\geq 1$ and $0\leq k\leq n^2$, with only North steps (vertical steps $=(0,1)$) and East steps (horizontal steps $=(1,0)$), such that the number of North steps in each level $j\geq 1$ is at most $2j-1$, where the levels associated to vertical lines are from 1 to n, as shown in the example of Figure 3.2.

Using Definition 1.7.1 and the previous notations, we can interpret the Mahonian numbers of type B as follows.

Theorem 3.1.11 *The Mahonian numbers of type B counts the number of lattice paths from* $u=(0,0)$ *to* $v=(n,k)$ *taking at most* $2j-1$ *North steps at the level* j *for* $j\geq 1$*, that is,*

$$i_B(n,k) = \mid \mathcal{P}^B_{n,k} \mid.$$

Proof To prove this theorem, it suffices to show that there is a bijection between the set of signed permutations of the hyperoctahedral group B_n having k inversions of type B (i.e., the inversions that satisfy the relations of Definition 1.7.1) and the set of lattice paths $\mathcal{P}^B_{n,k}$. So we proceed as follows:

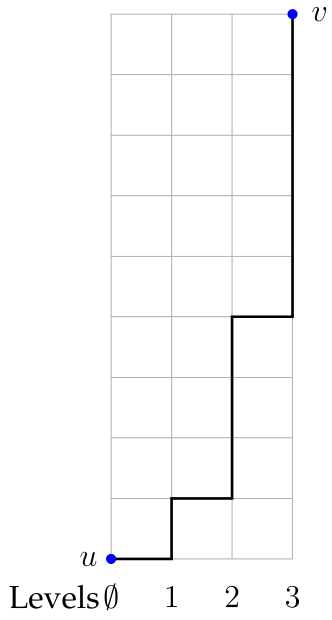

Figure 3.2: A path P in $\mathcal{P}^B_{3,9}$.

For each path $P \in \mathcal{P}^B_{n,k}$, we can easily find the signed permutation of length n having k inversions of type B associated to P, this permutation is obtained as follows: we associate to the point $(0, 0)$ the entry $\emptyset$, the first step is necessarily an East step, we move to the point $(1, 0)$ and we associate to this point the entry 1. **At the point (1,0)**, we have two cases: **if** the next step of P is an East step, we add the entry 2 to the right of 1 (i.e., 12), **and if** the next step of P is an North step, here we add the negative sign to the entry 1 (i.e.,-1, which gives us an inversion of type B from Definition 1.7.1). **At the point (1,1)**, here, the next step of P is necessarily an East step (since at level $j = 1$, the maximum number of North steps is $2j - 1 = 1$), we add the entry 2 to the right of -1 (i.e., -12). We move to **the point (2,1)**, here we have two cases : **if** the next step of P is an East step, we add the entry 3 to the right of -12 (i.e, -123), **else if** the next step of P is an North step, here we permute -1 with 2 and we permute the signs (i.e. -21, which gives us an inversion of type B).

In the general case, if at a given point we have the signed permutation $\pi = \pi_1 \cdots \pi_l$. **If** the next step of P is an East step, we add the entry $(l + 1)$ to the right of π (i.e., $\pi_1 \cdots \pi_l(l + 1)$), **if** the next step of P is an North step, here we have three cases : **if** $\pi_l = 1$ we add the negative sign to π_l (i.e., $\pi_1 \cdots \pi_{l-1}(-1)$, which gives us an inversion of type B), **if** $\pi_l > 0$ we search π_m such that $\pi_l =| \pi_m | +1$, here if $\pi_m > 0$, we permute π_m with π_l (i.e., $\pi_1 \cdots \pi_l \cdots \pi_m$, which gives us an inversion of type B), and if $\pi_m < 0$, we permute π_m with π_l and we permute their signs (i.e., $\pi_1 \cdots - \pi_l \cdots - \pi_m$, which gives us an inversion of type B), **if** $\pi_l < 0$ we search π_m such that $\mid \pi_l \mid = \mid \pi_m \mid -1$, here if $\pi_m < 0$, we permute π_m with π_l (i.e., $\pi_1 \cdots \pi_l \cdots \pi_m$, which gives us an inversion of type B), and if $\pi_m > 0$, we permute π_m with π_l and we permute their signs (i.e., $\pi_1 \cdots - \pi_l \cdots - \pi_m$, which gives us an inversion of type B). We proceed the same operations with next points until we arrive to the point (n, k), that gives us the desired permutation having k inversions of type B. An example is shown in Figure 3.3. ∎

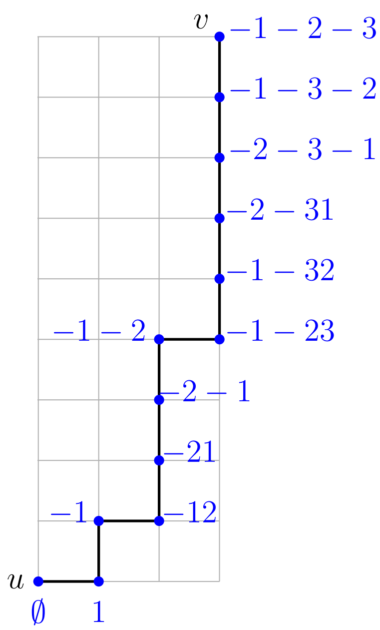

Figure 3.3: The path of $\pi = -1-2-3$ which has 9 inversions of type B.

Remark 3.1.12 *Conversely, to obtain such a path P from a signed permutation π, it suffices to take the value inv_{n-j+1} from the inversion table of π as the number of North steps at each level j of the path.*

As an illustration of the above theorem, the following figure shows the bijection between the elements of B_n with k inversions of type B and the set of lattice paths from $(0, 0)$ to (n, k), where at most $2j - 1$ North steps are taken at level j for $j \geq 1$.

3.1.4 Partition/tiling interpretation

In this subsection, we give a partition/tiling interpretation for the Mahonian numbers of type B. For the partition/tiling interpretation of the classical Mahonian number, we refer the readers to the paper of Ghemit and Ahmia [Gmah23].

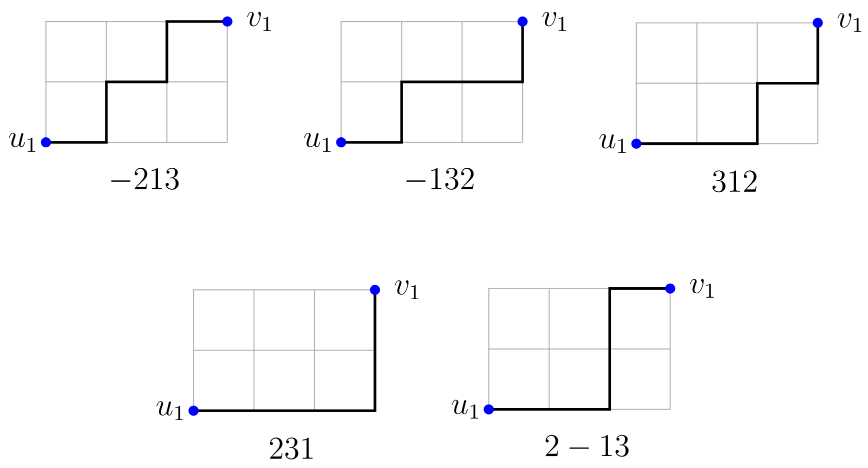

Figure 3.4: All $i_B(3,2) = |\mathcal{P}^B_{3,2}| = 5$ lattice paths from $(0,0)$ to $(3,2)$ and their corresponding signed permutations.

Recall to the path interpretation of Mahonian numbers of type B (i.e., Theorem 3.1.11), $i_B(n,k)$ counts the number of lattice paths from $(0,0)$ to (n,k) taking at most $2j-1$ North steps at the level j for $j \geq 1$. So, we can easily see the correspondence between North-East paths and partitions, just see that each number of boxes above and to the left of the path.

Let $\mathcal{P}r_{n,k}$ the set of partitions into k parts, and the largest part is less or equal to n, such that each part j can be repeated at most $2j-1$ times. Then, we obtain the following partition interpretation of Mahonian numbers of type B.

Theorem 3.1.13 *For $n \geq 1$ and $0 \leq k \leq n^2$, the Mahonian number of type B counts the number of partitions into k parts, and the largest part is less or equal to n, such that each part j can be repeated at most $2j-1$ times, that is,*

$$i_B(n,k) = \mid \mathcal{P}r_{n,k} \mid .$$

For example in Figure 3.5, each row represents a part, and then, these parts give us a partition.

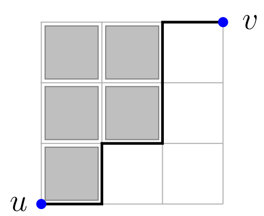

Figure 3.5: The path corresponds to the partition $\lambda = (2,2,1)$.

To establish the tiling interpretation of Mahonian numbers of type B, we give the following definition: let $\mathcal{T}^B_{n,k}$ be the set of weighted tilings of an $(n+k)\times 1$-board in which we use only n green squares and k orange squares where the number of successive orange squares is at most $2j-1$ if there are j green squares before. Thus, we have the following tiling interpretation.

Theorem 3.1.14 *The $i_B(n,k)$ counts the number of tilings of size $(n+k)\times 1$-board taking only n green squares and k orange squares, where the number of successive orange squares is at most $2j-1$ if there are j green squares before, that is,*

$$i_B(n,k) = \mid \mathcal{T}^B_{n,k} \mid .$$

Proof The set of tilings $\mathcal{T}^B_{n,k}$ is in bijection with set of paths $\mathcal{P}^B_{n,k}$. Each green square is corresponded to an East step, and each orange square to a North step, and vice versa. Moreover, this bijection is weight-preserving. ∎

As an example, in Figure 3.6 we have a tiling $T \in \mathcal{T}^B_{3,3}$ and its corresponded path.

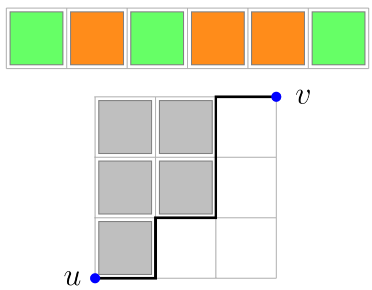

Figure 3.6: A tiling T and its corresponded path.

3.2 q-Analogue of Mahonian numbers of type B

In this section, we propose a q-analogue of Mahonian numbers of type B and we study some basic properties. To do this, we use a new statistics on signed permutations of hyperoctahedral group B_n, this statistics based on inversion $inv_i(\pi)$ defined by Arslan [Arsl25]. Besides that, we give some combinatorial interpretations by lattice paths/partitions and tilings for this analogue of Mahonian numbers of type B.

3.2.1 Definition and some identities

We now give the following statistic: For each signed permutation $\pi \in B_n$, let

$$\omega(\pi) = \sum_{i=1}^{n}(n-i+1)inv_i(\pi). \tag{3.2.1}$$

Using this statistic, we propose in the following definition a q-analogue of $i_B(n,k)$.

Definition 3.2.1 *For $n \geq 1$ and $0 \leq k \leq n^2$, we define the q-Mahonian number of type B as follows:*

$$i_{B_q}(n,k) = \sum_{\pi \in I_B(n,k)} q^{\omega(\pi)}, \tag{3.2.2}$$

where $I_B(n,k) = \{\pi \in B_n : inv_B(\pi) = k\}$.

We can write the hyperoctahedral group B_n as follows:

$$B_n = \biguplus_{j=-n}^{n} C_j$$

where $C_j = \{\pi \in B_n : \pi(n) = j\}$.

Let $\pi = \begin{pmatrix} 1 & \cdots & (n-1) & n \\ \pi(1) & \cdots & \pi(n-1) & j \end{pmatrix} \in C_j$ and τ be is a signed permutation defined by

$$\tau = \begin{pmatrix} a_1 & \cdots & a_{n-1} \\ \pi(1) & \cdots & \pi(n-1) \end{pmatrix} \in P\left([n]\backslash\{|j|\}\right)$$

where $a_1, \ldots, a_{n-1}$ are an arrangement of elements of $[n]\backslash\{|j|\}$ in increasing order, and $P\left([n]\backslash\{|j|\}\right)$ is the group of all the signed permutation of the set $[n]\backslash\{|j|\}$. So, if we set

$$\pi_{\tau,j} = \begin{pmatrix} 1 & \cdots & n-1 & n \\ \tau(a_1) & \cdots & \tau(a_{n-1}) & j \end{pmatrix}$$

then we obtain $\pi = \pi_{\tau,j}$. Hence by (1.7.1), we conclude that

$$inv_B(\pi) := \sum_{i=1}^{n} inv_i(\pi) = \begin{cases} n-j+inv_B(\tau), & \text{if } j > 0, \\ n-j-1+inv_B(\tau), & \text{if } j < 0. \end{cases} \tag{3.2.3}$$

If $\pi \in C_j$, for $j \in \{-n, \ldots, -1, 1, \ldots, n\}$, we can rewrite (3.2.1) from (3.2.3) as follows:

$$\omega(\pi) = \omega(\pi_{\tau,j}) = \begin{cases} n(n-j)+\omega(\tau), & \text{if } j > 0, \\ n(n-j-1)+\omega(\tau), & \text{if } j < 0, \end{cases} \tag{3.2.4}$$

where $\tau \in P\left([n]\backslash\{|j|\}\right)$.

By Definition 3.2.1 and relation (3.2.4), we can establish that the q-Mahonian numbers of type B satisfy the following recurrence relation which is the q-analogue of the relation (1.7.6).

Theorem 3.2.2 *For $n > 1$, we have*

$$i_{B_q}(n,k) = \sum_{j=0}^{2n-1} q^{nj} i_{B_q}(n-1,k-j), \tag{3.2.5}$$

where $i_{B_q}(1,0) = 1$ and $i_{B_q}(1,1) = q$.

Proof It is clear that $i_{B_q}(1,0) = 1$ and $i_{B_q}(1,1) = q$. Now from (3.2.1) and Definition 3.2.1, for $n \geq 2$, we have

$$i_{B_q}(n,k) = \sum_{\pi \in I_B(n,k)} q^{w(\pi)}.$$

Using (3.2.4), we obtain

$$\begin{aligned}
i_{B_q}(n,k) &= \sum_{j=-n}^{-1} q^{n(n-j-1)} \sum_{\substack{\tau \in P([n]\backslash\{|j|\}) \\ n-j-1+inv_B(\tau)=k}} q^{w(\tau)} + \sum_{j=1}^{n} q^{n(n-j)} \sum_{\substack{\tau \in P([n]\backslash\{|j|\}) \\ n-j+inv_B(\tau)=k}} q^{w(\tau)} \\
&= \sum_{j=n}^{2n-1} q^{nj} \sum_{\substack{\tau \in P([n]\backslash\{|n-j-1|\}) \\ j+inv_B(\tau)=k}} q^{w(\tau)} + \sum_{j=0}^{n-1} q^{nj} \sum_{\substack{\tau \in P([n]\backslash\{|n-j|\}) \\ j+inv_B(\tau)=k}} q^{w(\tau)} \\
&= \sum_{j=n}^{2n-1} q^{nj} \sum_{\substack{\tau \in P([n]\backslash\{|n-j-1|\}) \\ inv_B(\tau)=k-j}} q^{w(\tau)} + \sum_{j=0}^{n-1} q^{nj} \sum_{\substack{\tau \in P([n]\backslash\{|n-j|\}) \\ inv_B(\tau)=k-j}} q^{w(\tau)} \\
&= \sum_{j=0}^{2n-1} q^{nj} \sum_{\substack{\tau \in P([n]\backslash\{|n-j|\}) \\ inv_B(\tau)=k-j}} q^{w(\tau)} \\
&= \sum_{j=0}^{2n-1} q^{nj} i_{B_q}(n-1,k-j)
\end{aligned}$$

which gives the desired relation. ∎

We call the table values of $i_{B_q}(n,k)$ as the q-**Mahonian triangle of type** B. See Table 3.1.

From Theorem 3.2.2, we can deduce the following recurrence relation for the q-Mahonian numbers of type B.

n/k	0	1	2	3	4
1	1	q			
2	1	q^2+q	q^4+q^3	q^6+q^5	q^7
3	1	q^3+q^2+q	$q^6+q^5+2q^4+q^3$	$q^9+q^8+2q^7+2q^6+q^5$	$q^{12}+q^{11}+2q^{10}+\cdots$

Table 3.1: The q-Mahonian triangle of type B.

Corollary 3.2.3 *For $n > 1$ and $0 \leq k \leq n^2$, we have*

$$i_{B_q}(n,k) = i_{B_q}(n-1,k) + q^n i_{B_q}(n-1,k-1) - q^{2n^2} i_{B_q}(n-1,k-2n). \tag{3.2.6}$$

The following result is the q-analogue of the symmetry relation (1.7.5).

Theorem 3.2.4 *For $n \geq 1$ and $0 \leq k \leq n^2$, we have*

$$i_{B_q}(n,k) = q^{\frac{n(n+1)(4n-1)}{6}} i_{B_{\frac{1}{q}}}(n, n^2-k). \tag{3.2.7}$$

Proof Let $\pi_0 \in B_n$ be the signed permutation which has the maximum number of inversions of type B and then the largest value of the statistic ω. Let $\mathcal{F} : T_1 \to T_2$ be a mapping where $T_1 = \{\pi \in B_n : inv_B(\pi) = k\}$ and $T_2 = \{\pi \in B_n : inv_B(\pi) = n^2 - k\}$. Thus the mapping $\mathcal{F} : T_1 \to T_2$, $\mathcal{F}(\pi) = \pi_0\pi$, is bijective since $inv_B(\pi_0\pi) = inv_B(\pi_0) - inv_B(\pi) = n^2 - k$ and then $\omega(\pi_0\pi) = \omega(\pi_0) - \omega(\pi)$ by (3.2.1).

It is easy to see from (3.2.1) that $\omega(\pi_0) = \sum_{i=1}^{n}(n+1-i)(2n-2i+1)$. Hence, if $\omega(\pi) = M$, then we have

$$\omega(\pi_0\pi) = \omega(\pi_0) - M = \frac{n(n+1)(4n-1)}{6} - M.$$

Therefore, by relation (3.2.2) we get the relation (3.2.7). ∎

By the recurrence relation (3.2.5) of Theorem 3.2.2, we establish that the q-Mahonian numbers of type B have the following generating function.

Theorem 3.2.5 *The q-Mahonian numbers of type B verify, for $n \geq 1$, the following product:*

$$\sum_{k=0}^{n^2} i_{B_q}(n,k) z^k = \prod_{j=1}^{n} \left(1 + q^j z + \cdots + (q^j z)^{2j-1}\right). \tag{3.2.8}$$

Proof Let $B_n(z;q) := \sum_{k=0}^{n^2} i_{B_q}(n,k)z^k$ be the row generating function of the q-Mahonian numbers of type B. Then, it follows from (3.2.5) that

$$\begin{aligned} B_n(z;q) &= \sum_{k=0}^{n^2}\sum_{j=0}^{2n-1} q^{nj} i_{B_q}(n-1,k-j)z^k \\ &= \sum_{j=0}^{2n-1}(q^n z)^j \sum_{k=0}^{n^2} i_{B_q}(n-1,k-j)z^{k-j}. \end{aligned}$$

Since $i_{B_q}(n-1,k-j) = 0$ unless $0 \le k-j \le (n-1)^2$, we have

$$\begin{aligned} B_n(z;q) &= \sum_{j=0}^{2n-1}(q^n z)^j \sum_{k=j}^{(n-1)^2+j} i_{B_q}(n-1,k-j)z^{k-j} \\ &= \sum_{j=0}^{2n-1}(q^n z)^j \sum_{k=0}^{(n-1)^2} i_{B_q}(n-1,k)z^{k} \\ &= B_{n-1}(z;q)\sum_{j=0}^{2n-1}(q^n z)^j. \end{aligned}$$

Iterating, we obtain

$$\sum_{k=0}^{n^2} i_{B_q}(n,k)z^k = \prod_{j=1}^{n}\left(1 + q^j z + \cdots + (q^j z)^{2j-1}\right).$$

∎

3.2.2 Lattice path and partition/tiling interpretations

We know from Theorems 3.1.11, 3.1.13 and 3.1.14, that there is a bijection between the signed permutations of $I_B(n,k)$, the paths of $\mathcal{P}^B_{n,k}$, the partitions of $\mathcal{P}r_{n,k}$ and the tilings of $\mathcal{T}^B_{n,k}$. This allows us to give in this subsection the path and partition/tiling interpretations of q-Mahonian numbers of type B.

We start by the path interpretation of $i_{B_q}(n,k)$: for each path $P \in \mathcal{P}^B_{n,k}$, we denote by $\omega t(P)$ the weight associated to the path P counting the number of boxes above P. For example, the weight of the path of Figure 3.7 is $\omega t(P) = 1\times 1 + 2\times 2 + 3\times 3 = 14$.

Then, we can interpret $i_{B_q}(n,k)$ by paths as follows.

Theorem 3.2.6 *For $n \ge 1$ and $0 \le k \le n^2$, we have*

$$i_{B_q}(n,k) = \sum_{P\in\mathcal{P}^B_{n,k}} q^{\omega t(P)}. \tag{3.2.9}$$

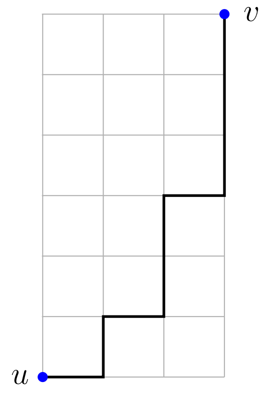

Figure 3.7: A path P in $\mathcal{P}^I_{n,k}$ of the weight $\omega t(P) = 14$.

Proof The weight $\omega t(P)$ can be seen as the sum of the product of the number of vertical steps in each level with the number associated to this level. It is not difficult to see that form each $\pi \in I_B(n,k)$ the weight $\omega(\pi)$ equals to the number of cases above the path $P \in \mathcal{P}^B_{n,k}$ associated to π, i.e., $\omega t(P) = \omega(\pi)$. ∎

From the previous theorem, we can give the partition interpretation of the q-Mahonian numbers of type B as follows.

Proposition 3.2.7 *For $n \geq 1$ and $0 \leq k \leq n^2$, the q-Mahonian number of type B is the generating function of the number of partitions into k parts in which each part j must be used at most $2j-1$ times and the largest part $\leq n$, that is,*

$$i_{B_q}(n,k) = \sum_{\lambda \in \mathcal{P}r_{n,k}} q^{|\lambda|} = \sum_{\lambda \sqsubset n^k} q^{|\lambda|}, \tag{3.2.10}$$

where $|\lambda| = \sum_{j=1}^{k} \lambda_j$.

Proof The exponent of q which is the weight of each path $P \in \mathcal{P}^B_{n,k}$ is given by counting the number of boxes that fit above and to the left of the lattice path, such that the number of boxes above and to the left of the path in each line represents a part of the partition as shown in Figure 3.8. Or, in other words, the number of vertical steps in level j represents the number of the appearances of the part j. ∎

For the tiling interpretation of $i_{B_q}(n,k)$, let me give the following notation: for each tiling $T \in \mathcal{T}^B_{n,k}$, let ω_T be the weight of T, which is the sum of all the weights of orange squares in

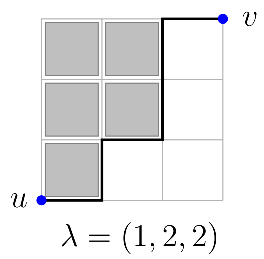

Figure 3.8: The lattice path/partition associated to $q^{|(1,2,2)|} = q^{1+2+2} = q^5$.

the tiling T, where the weight of an orange square is equal to the number of green squares to the left of that orange square.

Proposition 3.2.8 *For $n \geq 1$ and $0 \leq k \leq n^2$, we have*

$$i_{B_q}(n, k) = \sum_{T \in \mathcal{T}^B_{n,k}} q^{\omega_T}.$$

Proof Since the bijection between lattice paths and tiling is weight-preserving. Each green square in the tiling represents an East step of the path, and each orange square represents a North step, (see Figure 3.9). It remains to show that the weight of the tiling and its associated lattice path are the same. In fact, calculating the weight of the path is the sum of the number of boxes above and to the left of the path in each row, each North step gives us a row. Calculating the number of boxes in each row refers to calculate the number of East step before the North step associated to this row, which is exactly how to calculate the weight of a tiling. ∎

For example, the weight of the tiling 'gogoog' is $q^{1+2+2} = q^5$, as showed in Figure 3.9.

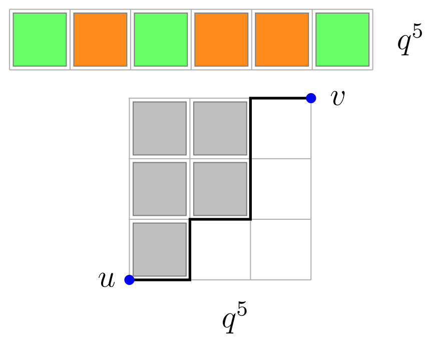

Figure 3.9: Tiling and lattice path interpretation of q^5.

CHAPTER 4

Study of log-concavity and unimodality properties

In this chapter, we provide a combinatorial proof demonstrating that the Over-inversion numbers form a log-concave and unimodal sequence. Then, we prove combinatorially that the q-analogue of Mahonian numbers of type B form a strongly q-log-concave sequence of polynomials in k, which implies that the Mahonian numbers of type B form a log-concave sequence in k and therefore unimodal.

4.1 Combinatorial proof of the log-concavity of the Over-inversion numbers

A polynomial is called log-concave (resp. unimodal) if its coefficients form a log-concave (resp. unimodal) sequence. Notably, Gaussian polynomials and q-multinomial coefficients are well-known examples of unimodal sequences, as established in [Sylv78, Theorem 3.11]. Additionally, it is a classic result, with proofs available in works such as [Bona04], that the product of log-concave (resp. unimodal) polynomials remains log-concave (resp. unimodal). For instance, the polynomial $\sum_{k=0}^{\binom{n}{2}} i(n,k)x^k$ is log-concave (resp. unimodal), meaning the sequence $i(n,0),\dots,i(n,\binom{n}{2})$ is log-concave (resp. unimodal).

Bóna [Bona05] provided the first non-generating function proof of the log-concavity of the Mahonian numbers $i(n,k)$ in k, using an injection property and an induction hypothesis over n, although the injection was non-constructive. To address this, Ghemit and Ahmia [Gmah23] offered a constructive proof that replaces Bóna's non-constructive injection.

The proof of the log-concavity (or unimodality) of the Over-inversion numbers $\{i_{B'}(n,k)\}_k$ in k can be easily derived from Theorem 2.2.12, since the generating function $\sum_{k=0}^{\binom{n}{2}} i_{B'}(n,k)x^k$ is the product of log-concave (or unimodal) polynomials. However, in this section, motivated by the work of Ghemit and Ahmia [Gmah23], we provide a new approach to demonstrate that the sequence of Over-inversion numbers $\{i_{B'}(n,k)\}_k$ is log-concave in k, and therefore unimodal, by constructing an appropriate injection.

We begin by introducing the following key definition.

Definition 4.1.1 *Let σ be a permutation of length n. For $0 \leq i \leq n-1$, let m_i^σ denote the number of times the entry $i+1$ appears as the first element of an overlined inversion of σ. The total number of overlined inversions of σ is then given by $\sum_{i=0}^{n-1} m_i^\sigma$.*

Based on this definition, we can derive the following result regarding the log-concavity of Over-inversion numbers:

Theorem 4.1.2 *The sequence of Over-inversion numbers $\{i_{B'}(n,k)\}_k$ is log-concave in k. Specifically, we have the inequality:*

$$(i_{B'}(n,k))^2 - i_{B'}(n,k-1)i_{B'}(n,k+1) \geq 0,$$

for all $0 < k \leq \binom{n}{2}$.

Proof Let $I_{B'}(n,k)$ denote the set of permutations of length n with k overlined inversions, and let $i_{B'}(n,k) = |I_{B'}(n,k)|$ represent the number of such permutations.

To prove the log-concavity of the sequence of Over-inversion numbers $i_{B'}(n,k)_k$, it is equivalent to find an injection $f_{n,k,k}$ from $I_{B'}(n,k+1) \times I_{B'}(n,k-1)$ to $I_{B'}(n,k) \times I_{B'}(n,k)$, where $f_{n,k,k}(\sigma,\tau) = (\theta,\pi)$ with $(\sigma,\tau) \in I_{B'}(n,k+1) \times I_{B'}(n,k-1)$ and $(\theta,\pi) \in I_{B'}(n,k) \times I_{B'}(n,k)$.

Let $(\sigma,\tau) \in I_{B'}(n,k+1) \times I_{B'}(n,k-1)$. We will define I as the largest integer which satisfies the following three conditions:

$$\sum_{j\geq I} m_j^\sigma \geq \sum_{j\geq I+1} m_j^\tau + 1, \quad \text{if the entry } (j+1) = (I+1) \quad \text{of the } \sigma \quad \text{not overlined}, \tag{4.1.1}$$

$$\sum_{j\geq I} m_j^\sigma > \sum_{j\geq I+1} m_j^\tau + 1, \quad \text{if the entry } (j+1) = (I+1) \quad \text{of the } \sigma \quad \text{is overlined} \tag{4.1.2}$$

and

$$\sum_{j\geq I} m_j^{\tau} + 1 - \sum_{j\geq I+1} m_j^{\sigma} \leq I, \quad \text{if} \sum_{j\geq I} m_j^{\sigma} < \sum_{j\geq I} m_j^{\tau} + 1. \tag{4.1.3}$$

Now, we define the map $f_{n,k,k}$ as follows:

$$f_{n,k,k}(\sigma,\tau) = (\theta,\pi),$$

where :

- θ is obtained by the following modifications on τ:

  - For the entry $(j+1)$, from $(I+2)$ to n in ascending order: if the entry $(j+1)$ from σ is overlined, then overlined the entry $(j+1)$ from τ, else not overlined the entry $(j+1)$ from τ.
  - For the entry $(j+1)$, from n to $(I+2)$ in decreasing order: the entry $j+1$ moves m_j^{τ} positions to the right and overlined, if the entry $j+1$ is overlined. Else, the entry $j+1$ only moves m_j^{τ} positions to the right.
  - For the entry $j+1 = I+1$: the entry $j+1$ moves $\left(\sum_{i=1}^{I} m_i^{\sigma} - \sum_{i=1}^{I} m_i^{\tau} - 1\right)$ positions to the left and overlined, if the entry $j+1$ is overlined. Else, the entry $j+1$ only moves $\left(\sum_{i=1}^{I} m_i^{\sigma} - \sum_{i=1}^{I} m_i^{\tau} - 1\right)$ positions to the left.
  - For the entry $(j+1)$, from $(I+2)$ to n in ascending order: the entry $j+1$ moves m_j^{σ} positions to the left and overlined, if the entry $j+1$ is overlined. Else, the entry $j+1$ only moves m_j^{σ} positions to the left.

- π is obtained by the following modifications on σ:

  - For the entry $(j+1)$, from $(I+2)$ to n in ascending order: if the entry $(j+1)$ from τ is overlined, then overlined the entry $(j+1)$ from σ, else not overlined the entry $(j+1)$ from σ.
  - For the entry $(j+1)$, from n to $(I+2)$ in decreasing order: the entry $j+1$ moves m_j^{σ} positions to the right and overlined, if the entry $j+1$ is overlined. Else, the entry $j+1$ only moves m_j^{σ} positions to the right.

- For the entry $j+1 = I+1$: the entry $j+1$ moves $\left(\sum_{i=1}^{I} m_i^\sigma - \sum_{i=1}^{I} m_i^\tau - 1\right)$ positions to the right and overlined, if the entry $j+1$ is overlined. Else, the entry $j+1$ only moves $\left(\sum_{i=1}^{I} m_i^\sigma - \sum_{i=1}^{I} m_i^\tau - 1\right)$ positions to the right.
- For the entry $(j+1)$, from $(I+2)$ to n in ascending order: the entry $j+1$ moves m_j^τ positions to the left and overlined, if the entry $j+1$ is overlined. Else, the entry $j+1$ only moves m_j^τ positions to the left.

To prove that $f_{n,k,k}$ is well defined we only need to check that:

$$0 \leq \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 \leq m_I^\sigma \tag{4.1.4}$$

and

$$m_I^\tau + \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 \leq I. \tag{4.1.5}$$

Now, we simplify the inequality (4.1.4) as follows:

$$\begin{aligned}
\sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 &= \sum_{j=1}^{n-1} m_j^\sigma - \sum_{j=I+1}^{n-1} m_j^\sigma - \sum_{j=1}^{n-1} m_j^\tau + \sum_{j=I+1}^{n-1} m_j^\tau - 1 \\
&= \left(k+1 - \sum_{j=I+1}^{n-1} m_j^\sigma\right) - \left(k-1 - \sum_{j=I+1}^{n-1} m_j^\tau\right) - 1 \\
&= 1 - \sum_{j=I+1}^{n-1} m_j^\sigma + \sum_{j=I+1}^{n-1} m_j^\tau.
\end{aligned}$$

Then inequality (4.1.5) becomes:

$$\begin{aligned}
m_I^\tau + \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 &= m_I^\tau + 1 - \sum_{j=I+1}^{n-1} m_j^\sigma + \sum_{j=I+1}^{n-1} m_j^\tau \\
&= 1 - \sum_{j=I+1}^{n-1} m_j^\sigma + \sum_{j=I}^{n-1} m_j^\tau.
\end{aligned}$$

Now, we show inequality (4.1.4), i.e.

$$0 \leq \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 \leq m_I^\sigma.$$

Here, we have two cases :

First case : If $\sum_{j\geq I} m_j^\sigma \geq \sum_{j\geq I+1} m_j^\tau + 1$, we have:

$$\begin{aligned}1-\sum_{j\geq I} m_j^\sigma + \sum_{j\geq I+1} m_j^\tau \leq 0 &\Leftrightarrow 1-\sum_{j\geq I+1} m_j^\sigma - m_I^\sigma + \sum_{j\geq I+1} m_j^\tau \leq 0\\ &\Leftrightarrow 1-\sum_{j\geq I+1} m_j^\sigma + \sum_{j\geq I+1} m_j^\tau \leq m_I^\sigma\\ &\Leftrightarrow \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 \leq m_I^\sigma.\end{aligned}$$

Second case : If $\sum_{j\geq I} m_j^\sigma > \sum_{j\geq I+1} m_j^\tau + 1$, we have :

$$\begin{aligned}1-\sum_{j\geq I} m_j^\sigma + \sum_{j\geq I+1} m_j^\tau < 0 &\Leftrightarrow 1-\sum_{j\geq I+1} m_j^\sigma - m_I^\sigma + \sum_{j\geq I+1} m_j^\tau < 0\\ &\Leftrightarrow 1-\sum_{j\geq I+1} m_j^\sigma + \sum_{j\geq I+1} m_j^\tau < m_I^\sigma\\ &\Leftrightarrow \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 < m_I^\sigma.\end{aligned}$$

It remains to show that :

$$\sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 \geq 0.$$

For this, we have two cases:

First case : If $\sum_{j=I+1}^{n-1} m_j^\sigma < \sum_{j=I+2}^{n-1} m_j^\tau + 1$, we have :

$$\begin{aligned}-\sum_{j=I+1}^{n-1} m_j^\sigma > -\sum_{j=I+2}^{n-1} m_j^\tau - 1 &\Leftrightarrow 1-\sum_{j=I+1}^{n-1} m_j^\sigma > -\sum_{j=I+2}^{n-1} m_j^\tau\\ &\Leftrightarrow 1-\sum_{j=I+1}^{n-1} m_j^\sigma + \sum_{j=I+2}^{n-1} m_j^\tau > 0\\ &\Leftrightarrow 1-\sum_{j=I+1}^{n-1} m_j^\sigma + \sum_{j=I+1}^{n-1} m_j^\tau > m_{I+1}^\tau \geq 0\\ &\Leftrightarrow 1-\sum_{j=I+1}^{n-1} m_j^\sigma + \sum_{j=I+1}^{n-1} m_j^\tau > 0\\ &\Leftrightarrow \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 > 0.\end{aligned}$$

Second case : If $\sum_{j=I+1}^{n-1} m_j^\sigma \geq \sum_{j=I+2}^{n-1} m_j^\tau + 1$ and $\sum_{j=I+1}^{n-1} m_j^\sigma < \sum_{j=I+1}^{n-1} m_j^\tau + 1$, we have

$$1-\sum_{j=I+1}^{n-1} m_j^\sigma + \sum_{j=I+1}^{n-1} m_j^\tau > 0 \Leftrightarrow \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 > 0.$$

We show the inequality (4.1.5): $m_I^\tau + \sum_{j=1}^I m_j^\sigma - \sum_{j=1}^I m_j^\tau - 1 \leq I.$

Here, we have two cases:

First case : If $\sum_{j\geq I} m_j^\sigma < \sum_{j\geq I} m_j^\tau + 1$, using the condition (4.1.3), we obtain:

$$\sum_{j\geq I} m_j^\tau + 1 - \sum_{j\geq I+1} m_j^\sigma \leq I \Leftrightarrow m_I^\tau + \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 \leq I.$$

Second case : If $\sum_{j\geq I} m_j^\sigma \geq \sum_{j\geq I} m_j^\tau + 1$, we have :

$$\begin{aligned}
1 - \sum_{j\geq I} m_j^\sigma + \sum_{j\geq I} m_j^\tau \leq 0 &\Leftrightarrow 1 - \sum_{j\geq I+1} m_j^\sigma + \sum_{j\geq I} m_j^\tau \leq m_I^\sigma \leq I \\
&\Leftrightarrow 1 - \sum_{j\geq I+1} m_j^\sigma + \sum_{j\geq I} m_j^\tau \leq I \\
&\Leftrightarrow m_I^\tau + \sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1 \leq I.
\end{aligned}$$

Now, we must check that $(\theta, \tau) \in I_{B'}(n,k) \times I_{B'}(n,k)$ which means that the number of inversions in θ is k and the number of inversions in π is k.

The number of inversions in θ is calculated as follows

$$\begin{aligned}
\sum_{j=1}^{I} m_j^\tau + \left(\sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1\right) + \sum_{j=I+1}^{n-1} m_j^\sigma &= \sum_{j=1}^{I} m_j^\sigma + \sum_{j=I+1}^{n-1} m_j^\sigma - 1 \\
&= \sum_{j=1}^{n-1} m_j^\sigma - 1 \\
&= (k+1) - 1 = k.
\end{aligned}$$

The number of inversions of π is calculated as follows

$$\begin{aligned}
\sum_{j=1}^{I} m_j^\sigma - \left(\sum_{j=1}^{I} m_j^\sigma - \sum_{j=1}^{I} m_j^\tau - 1)\right) + \sum_{j=I+1}^{n-1} m_j^\tau &= \sum_{j=1}^{n-1} m_j^\tau + 1 \\
&= (k-1) + 1 = k.
\end{aligned}$$

Then $(\theta, \tau) \in I_{B'}(n,k) \times I_{B'}(n,k)$. Furthermore, $f_{n,k,k}^{-1} = f_{n,k+1,k-1}$, which gives us the injectivity of $f_{n,k,k}$ and completes the proof. ∎

Here, we give an illustrative example.

Example 4.1.3 *Let $\sigma = \overline{3}\,\overline{2}\,\overline{4}\,\overline{5}1$ a permutation with 5 overlined inversions and let $\tau = 12\overline{5}\,\overline{4}3$ a permutation with 3 overlined inversions.*

$m_1^\sigma = 1$	$m_1^\tau = 0$
$m_2^\sigma = 2$	$m_2^\tau = 0$
$m_3^\sigma = 1$	$m_3^\tau = 1$
$m_4^\sigma = 1$	$m_4^\tau = 2$

First, we must find the index I, we have :

For $I = 4$, $I + 1 = \overline{5}$, so $m_4^\sigma = 1 = 0 + 1$ then $I = 4$ does not satisfy the second condition (4.1.2).

For $I = 3$, $I + 1 = \overline{4}$, so $m_3^\sigma + m_4^\sigma = 2 < m_4^\tau + 1 = 3$ then $I = 3$ does not satisfy the second condition (4.1.2).

For $I = 2$, $I + 1 = \overline{2}$, so $m_2^\sigma + m_3^\sigma + m_4^\sigma = 4 = m_3^\tau + m_4^\tau + 1 = 4$ then $I = 2$ does not satisfy the second condition (4.1.2).

For $I = 1$, $I + 1 = \overline{3}$, so $m_1^\sigma + m_2^\sigma + m_3^\sigma + m_4^\sigma = 5 > m_2^\tau + m_3^\tau + m_4^\tau + 1 = 4$ then $I = 1$ satisfies the second condition (4.1.2). Thus, $I = 1$ and our application gives :

$$\overline{3}\,\overline{2}\,\overline{4}\,\overline{5}1 \to 3\overline{2}\,\overline{4}\,\overline{5}1 \to 3\overline{2}\,\overline{4}1\,\overline{5} \to 3\overline{2}1\overline{4}\,\overline{5} \to \overline{2}13\overline{4}\,\overline{5} \to \overline{2}1\overline{4}3\overline{5} \to \overline{2}1\overline{5}\,\overline{4}3$$

$$12\overline{5}\,\overline{4}3 \to 12\overline{5}\,\overline{4}\,\overline{3} \to 12\overline{4}\,\overline{3}\,\overline{5} \to 12\overline{3}\,\overline{4}\,\overline{5} \to \overline{3}12\overline{4}\,\overline{5} \to \overline{3}1\overline{4}2\overline{5} \to \overline{3}1\overline{4}\,\overline{5}2$$

Hence, $f_{5,4,4}(\overline{3}\,\overline{2}\,\overline{4}\,\overline{5}1, 12\overline{5}\,\overline{4}3) = (\overline{2}1\overline{5}\,\overline{4}3, \overline{3}1\overline{4}\,\overline{5}2)$, where $\theta = \overline{2}1\overline{5}\,\overline{4}3$ has 4 overlined inversions, and $\pi = \overline{3}1\overline{4}\,\overline{5}2$ has 4 overlined inversions.

Since a log-concave sequence is unimodal [Brant89], the previous theorem immediately leads to the following corollary:

Corollary 4.1.4 *The sequence of Over-inversion numbers $\{i_{B'}(n,k)\}_k$ is unimodal in k.*

Remark 4.1.5 *Since there is a bijection between the permutations with overlined inversions and the paths as shown in Subsection 2.3.1, then the previous theorem can be proved using Sagan's paths approach [Saga92] by adding a new condition to his involution as follows. (See also the approaches proposed in [Gmah23, Baah22]).*

Let $P_1, P_2 \in P_{n,k}^{B'}$. And let $u \xrightarrow{P} v$ denote P has initial vertex u and final vertex v.

Definition 4.1.6 *Given $u_1 \xrightarrow{P_1} v_1$ and $u_2 \xrightarrow{P_2} v_2$. Then define the involution $\varphi_I(P_1, P_2) = (P_1', P_2')$ where :*

$$P_1' = u_1 \xrightarrow{P_1} v_0 \xrightarrow{P_2} v_2 \text{ and } P_2' = u_2 \xrightarrow{P_2} v_0 \xrightarrow{P_1} v_1,$$

i.e., switches the portions of P_1 and P_2 after v_0 (see, Figure 4.2), where v_0 is the last vertex of $P_1 \cap P_2$.

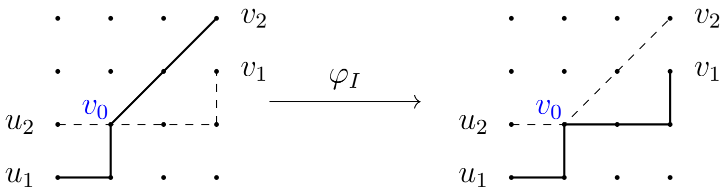

Figure 4.1: The involution φ_I.

4.2 q-Log-concavity of q-Mahonian numbers of type B

The first result dealing with the q-log-concavity property is due to Butler [Butl90] who proved the strong q-log-concavity of q-binomial coefficients using partitions. Sagan [Saga92] proved the same property for these coefficients using paths. Bazeniar et al. [Baah22] studied the same property for the q-bi^{s}nomial coefficients by a similar approach to that of Sagan. Dousse and Kim [Dous18] generalized this property for the over analogues of q-binomial coefficients by a similar approach to that of Butler. Recently, Ghemit and Ahmia in [Gmah23, Gahm23] established this property for the over analogues of q-bi^{s}nomial coefficients and the q-analogue of Mahonian numbers, using respectively paths/partitions and the injection property over the set of permutations.

The q-log-concavity (resp. log-concavity or unimodality) of the q-Mahonian numbers of type B $\{i_{B_q}(n,k)\}_k$ (resp. Mahonian numbers of type B $\{i_B(n,k)\}_k$) in k follows directly from Theorem 3.2.5, as the generating function $\sum_{k=0}^{n^2} i_{B_q}(n,k)z^k$ factors as a product of q-log-concave (resp. unimodal) polynomials.

Motivated by earlier works, we prove in this section that the q-Mahonian numbers of type B form a strongly q-log-concave sequence of polynomials in k, by using the path approach due to Sagan [Saga92], which implies that the Mahonian numbers of type B form a log-concave sequence in k, and therefore unimodal.

According to Theorem 3.2.6, we have

$$i_{B_q}(n,k) = \sum_{P \in \mathcal{P}^B_{n,k}} q^{\omega t(P)} \tag{4.2.1}$$

and the weight $wt(P)$ equals to the number of cases above the path $P \in \mathcal{P}^B_{n,k}$ with P starts from $u_1 = (0,0)$ to $v_1 = (n,k)$, then

$$(i_{B_q}(n,k))^2 = \sum_{P_1 \in \mathcal{P}^B_{n,k}} q^{\omega t(P_1)} \sum_{P_2 \in \mathcal{P}^B_{n,k}} q^{\omega t(P_2)}, \tag{4.2.2}$$

where P_1 starts from $u_1 = (0,0)$ to $v_1 = (n,k)$ and P_2 starts from $u_2 = (0,1)$ to $v_1 = (n,k+1)$. The previous equation is equivalent to

$$(i_{B_q}(n,k))^2 = \sum_{P_1,P_2 \in \mathcal{P}^B_{n,k}} q^{\omega t(P_1P_2)}, \tag{4.2.3}$$

where $\omega t(P_1P_2) = \omega t(P_1) + \omega t(P_2)$.

Let $P_1, P_2 \in P^B_{n,k}$. And let $u \xrightarrow{P} v$ denote P has initial vertex u and final vertex v.

Using similar technic of Sagan in [Saga92] to prove the strong q-log-concavity of the q-Mahonian numbers of type B, we recall then the following definition used by him.

Definition 4.2.1 *Given $u_1 \xrightarrow{P_1} v_1$ and $u_2 \xrightarrow{P_2} v_2$. Then define the involution $\varphi_I(P_1,P_2) = (P_1', P_2')$ where*

$$P_1' = u_1 \xrightarrow{P_1} v_0 \xrightarrow{P_2} v_2 \text{ and } P_2' = u_2 \xrightarrow{P_2} v_0 \xrightarrow{P_1} v_1,$$

i.e., switches the portions of P_1 and P_2 after v_0 (see, Figure 4.2), where v_0 is the last vertex of $P_1 \cap P_2$ such that the number of vertical steps, in the vertical level of v_0, of P_1 and P_2 all together does not exceed $2m-1$ if the vertical level of v_0 is m.

Now, we give the following theorem.

Theorem 4.2.2 *For n fixed, the sequence of polynomials $\left(i_{B_q}(n,k)\right)_{0 \le k \le n^2}$ is strongly q-log concave in k, that is,*

$$i_{B_q}(n,l) \times i_{B_q}(n,k) - i_{B_q}(n,l-1) \times i_{B_q}(n,k+1) \ge_q 0,$$

where $0 < l \le k$.

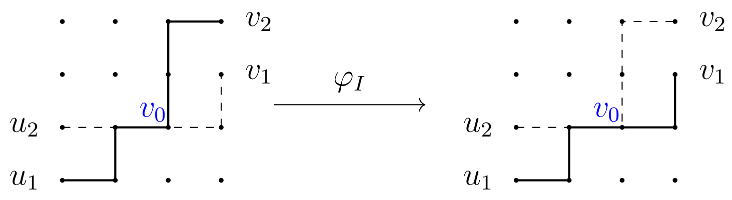

Figure 4.2: The involution φ_I.

Proof First, we will show that

$$\left(i_{B_q}(n,k)\right)^2 - i_{B_q}(n,k-1) \times i_{B_q}(n,k+1) \geq_q 0. \tag{4.2.4}$$

By relation (4.2.2), the left-hand side of the inequality (4.2.4) can be written as follows

$$\left(i_{B_q}(n,k)\right)^2 - i_{B_q}(n,k-1) \times i_{B_q}(n,k+1) = \sum_{P_1,P_2} (-1)^{P_1P_2} q^{\omega t(P_1P_2)}, \tag{4.2.5}$$

where the sum is over all pairs P_1, P_2 such that P_1 starts at $u_1 = (0,0)$, P_2 starts at $u_2 = (0,1)$, and

$$(-1)^{P_1P_2} = \begin{cases} +1 & \text{if } u_1 \xrightarrow{P_1} v_1 \quad \text{and} \quad u_2 \xrightarrow{P_2} v_2, \\ -1 & \text{if } u_1 \xrightarrow{P_1} v_2 \quad \text{and} \quad u_2 \xrightarrow{P_2} v_1, \end{cases}$$

with $v_1 = (n,k)$ and $v_2 = (n,k+1)$.

On each path pair (P_1, P_2) with sign -1 in equation (4.2.5), we apply the involution φ_I given in Definition 4.2.1, we obtain a path pair (P_1', P_2') with sign $+1$, and since P_1 and P_2 start on the same vertical line, the sum of the number of boxes above P_1' and the number of boxes above P_2' after switching remains the same i.e., $\omega t(P_1P_2) = \omega t(P_1'P_2')$ (see, Figure 4.2), because the boxes lost by the first are gained by the second. Thus the inequality (4.2.4) is verified.

To prove the general case

$$i_{B_q}(n,l) \times i_{B_q}(n,k) - i_{B_q}(n,l-1) \times i_{B_q}(n,k+1) \geq_q 0, \tag{4.2.6}$$

where $0 < l \leq k$, it suffices to use the same approach with $u_2 = (0, k-l+1)$ as new condition.

As an illustration, we give the following example.

Example 4.2.3 *For $n = 2$ and $k = 2$, we have*

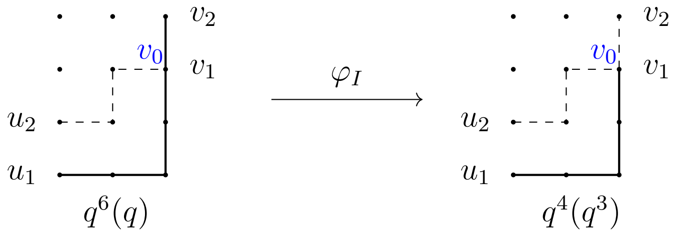

Figure 4.3: The application of the involution φ_I.

$$(i_{B_q}(2,2))^2 = (q^4 + q^3)^2 = q^8 + 2q^7 + q^6,$$
$$i_{B_q}(2,1).i_{B_q}(2,3) = (q^2 + q).(q^6 + q^5) = q^8 + 2q^7 + q^6.$$

Applying φ_I on the paths P_1 ($u_1 \xrightarrow{P_1} v_2$) and P_2 ($u_2 \xrightarrow{P_2} v_1$), as shown in Figure 4.3, associated to the term of $i_{B_q}(2,3) \times i_{B_q}(2,1)$: q^6 and q respectively, we obtain the paths P_1' ($u_1 \xrightarrow{P_1'} v_1$) and P_2' ($u_2 \xrightarrow{P_2'} v_2$) associated to the term of $(i_{B_q}(2,2))^2$: q^4 and q^3 respectively.

By setting $q = 1$ in Theorem 4.2.2, we can establish that the Mahonian numbers of type B satisfy the following results about the log-concavity and unimodality.

Corollary 4.2.4 *The Mahonian numbers $\{i_B(n,k)\}_{0\leq k\leq n^2}$ of type B form a log-concave sequence in k, and therefore unimodal.*

Conclusion

This thesis has been devoted to the study of Over-inversion numbers $i_{B'}(n,k)$ and Mahonian numbers of type B $i_B(n,k)$, together with the q-analogue of Mahonian numbers of type B $i_{B_q}(n,k)$, through a combinatorial and algebraic perspective including log-concavity property.

We began by introducing the Over-inversion numbers and examining their main properties. Recurrence relations were established for their computation, along with basic identities and a generating function, which turned out to play a central role in understanding their combinatorial behavior. Several combinatorial interpretations were then provided in terms of lattice paths, overpartitions, and tilings.

We then investigated combinatorial formulas such as recurrence relations, the Knuth–Netto formula, and generating functions for the subdiagonals of Mahonian numbers of type B. These results were further enriched with interpretations in terms of lattice paths and partitions/tilings, highlighting a bijection between lattice paths and signed permutations of the hyperoctahedral group B_n. This bijective approach shows that signed permutations of B_n can be constructed directly from lattice paths.

A significant contribution of this work is the introduction and analysis of a q-analogue of the Mahonian numbers of type B, defined through a new inversion statistic on signed permutations of B_n that we introduce. We studied its main properties and developed additional combinatorial interpretations in terms of lattice paths, partitions, and tilings.

We also established the unimodality and log-concavity of the sequence of Over-inversion numbers. However, determining the number and location of the modes remains an unresolved problem. Furthermore, we proved that the q-analogue of Mahonian numbers of type B forms a strongly q-log-concave sequence of polynomials in k. For the variable n, however, the same approach could not be applied due to technical constraints inherent in the involution method. Numerical evidence nevertheless suggests log-concavity (and strong q-log-concavity) with respect to n, although a rigorous proof is still missing.

These investigations lead to several unresolved problems:

- **Question 1.** Find the number and location of the modes of the unimodal sequence $\{i_{B'}(n,k)\}_k$.
- **Question 2.** For fixed k, is the sequence (resp. the polynomial sequence) $(i_B(n,k))_n$ (resp. $(i_{B_q}(n,k))_n$) log-concave (resp. strongly q-log-concave) in n?
- **Question 3.** Find the number and location of the modes of the unimodal sequence $(i_B(n,k))_k$.

Overall, this thesis establishes fundamental properties of Over-inversion numbers and Mahonian numbers of type B, provides new q-analogues and interpretations, and proves important results such as unimodality and (strong) log-concavity. At the same time, it opens promising directions for future research on the precise description of modes and the extension of log-concavity results, thereby strengthening the links between enumerative combinatorics, lattice path theory, and permutation statistics.

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